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THE GENERALIZED $\mathcal{A}_\infty$-ALGEBRA STRUCTURE ON BGG SEQUENCES AND GENERALIZED ASSOCIATIVE OPERAD

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ABSTRACT. We discuss from scratch the structure of a generalized $\mathcal{A}_\infty$-algebra on the set of pieces of Bernstein-Gelfand-Gelfand (BGG) sequences. This algebra is determined by a curvature endomorphism, BGG differential operators, cup product and higher multilinear differential operators. We compute explicit form of higher operations of this generalized $\mathcal{A}_\infty$-algebra, e.g. the pentagon condition.

In the second part, we show that these generalized $\mathcal{A}_\infty$-algebras are algebras over certain operad in the monoidal category of graded vector spaces with a distinguished, not necessary nilpotent, endomorphism.

1. INTRODUCTION

There are two main directions of interest in the study of $\mathcal{A}_\infty$-algebras arising in geometry. The first one is related to problems in the study of suitable subsequences (subcomplexes) of the twisted de Rham sequence (the de Rham complex) and its corresponding Hodge theory. The second one (in some sense related to the previous problem) originates in the problems of deformation theory of geometrical structures on manifolds, e.g. the structure of the formal moduli space of flat connections on a fixed vector bundle over base manifold.

We shall focus in this article on the first problem. The discussion of the second one can be found in [1].

In the first part of this article, we shall motivate the origin of our example of generalized $\mathcal{A}_\infty$-algebra in parabolic geometry by comparison with $\mathcal{A}_\infty$-algebra structure associated to (real, complex etc.) Hodge theory of Kähler manifolds and we shall indicate their similar and different features.

In the second part we discuss in more (than in [1]) detail the origin of these generalized (or sometimes called curved) $\mathcal{A}_\infty$-algebras. In particular they are determined by quadratically nilpotent codifferential on conilpotent tensor coalgebra $(\otimes\mathcal{A})^c$ canonically attached to a given type of parabolic geometry. We add moreover many explicit formulas, for example the (generalized) pentagon relation.

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In the last part, based on this example of the generalized $\mathcal{A}_\infty$-algebra and standard constructions in the universal algebra, we give the definition of generalized associative operad over which the (generalized) $\mathcal{A}_\infty$-algebra lives.

1.1. **$\mathcal{A}_\infty$-algebras and Hodge theory.** In this subsection we shall first review a few basic facts concerning Hodge theory and related $\mathcal{A}_\infty$-algebra on a compact Kähler manifold M and than compare it to its counterpart in parabolic geometry. As we shall see, the Hodge theory in parabolic geometry is accompanied by generalized $\mathcal{A}_\infty$-algebra.

Let $\Lambda^\bullet(M)$ be the algebra of real differential forms on M, $\Lambda^{\bullet,\bullet}(M)$ the algebra of complex differential forms with Hodge gradation (on a general manifold only the Hodge filtration) together with differentials of degree one:

$$\partial : \Lambda^{\bullet,\bullet}(M) \to \Lambda^{\bullet+1,\bullet}(M)\,, \qquad \overline{\partial} : \Lambda^{\bullet,\bullet}(M) \to \Lambda^{\bullet,\bullet+1}(M)\,,$$
$$d = \partial + \overline{\partial} : \Lambda^{\bullet}(M) \to \Lambda^{\bullet+1}(M)\,, \qquad d_c = i\partial - i\overline{\partial} : \Lambda^{\bullet}(M) \to \Lambda^{\bullet+1}(M).$$

Their conjugates (w.r. to the Kähler metric and Kähler form Ω) defined by $(\partial -, -) = (-, \partial^\star -)$, $(\overline{\partial} -, -) = (-, \overline{\partial}^\star)$, $(- \wedge \Omega, -) = (-, \Omega^\star \llcorner -)$ are linear operators

$$\partial^\star : \Lambda^{\bullet,\bullet}(M) \to \Lambda^{\bullet-1,\bullet}(M)\,, \qquad \overline{\partial}^\star : \Lambda^{\bullet,\bullet}(M) \to \Lambda^{\bullet,\bullet-1}(M)\,,$$
$$d^\star = \partial^\star + \overline{\partial}^\star : \Lambda^{\bullet}(M) \to \Lambda^{\bullet-1}(M)\,, \qquad d_c^\star = i\partial^\star - i\overline{\partial}^\star : \Lambda^{\bullet}(M) \to \Lambda^{\bullet-1}(M).$$

The Laplace operators (depending on the Kähler metric) are defined by

$$\triangle_\partial = \partial\partial^\star + \partial^\star\partial\,, \qquad \triangle_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^\star + \overline{\partial}^\star\overline{\partial}\,,$$
$$\triangle_d = dd^\star + d^\star d\,, \qquad \triangle_{d_c} = d_c d_c^\star + d_c^\star d_c$$

and simple algebra gives

$$\triangle_\partial = \triangle_{\overline{\partial}} = \frac{1}{2}\triangle_d = \frac{1}{2}\triangle_{d_c}\,.$$

The $sl(2)$-algebra of operators $\{\partial, \overline{\partial}, \Omega^\star\} \in End(\Lambda^{\bullet,\bullet}(M))$

$$[\Omega^\star, \overline{\partial}] = -i\partial^\star\,,\ [\Omega^\star, \overline{\partial}] = i\partial^\star\,,\ [\Omega^\star, d] = id_c^\star\,,$$

determines the Lefschetz decomposition of $\Lambda^\bullet(M)$. Finally, the Hodge decomposition implies that there is an orthogonal decomposition of $\omega \in \Lambda^{\bullet,\bullet}(M)$ on the harmonic part and the images of $\partial, \partial^\star$:

$$\omega = \omega|_{Ker(\triangle)} \oplus \{\partial\triangle_\partial^{-1}\partial^\star(\omega) + \partial^\star\triangle_\partial^{-1}\partial^\star(\omega)\} \tag{1}$$

and similarly for $\overline{\partial}, d, d_c$.

Let us recall a few standard facts concerning the origin of $\mathcal{A}_\infty$-algebras in Hodge theory. We shall start with review of strong homotopy associative algebras for Kähler manifolds, see [6].

- The real $\mathcal{A}_\infty$-algebra of a Kähler manifold M is based on the subcomplex of the de Rham complex ($\mathbb{Z}/2$-graded differential algebra)

$$(\Lambda^\bullet(M)|_{Ker(d_c^\star)}, d) \subset (\Lambda^\bullet(M), d) \tag{2}$$

with strong homotopy retract $Q = d_c\Delta_d^{-1}\Omega^\star$. By elementary algebra

$$\begin{aligned}&1-[d,Q] = 1-[d,d_c\Delta_d^{-1}\Omega^\star] = (dd_c = -d_cd) = \\ &1+d_c[d,\Delta_d^{-1}\Omega^\star] = (d\Delta_d^{-1} = \Delta_d^{-1}d) = 1+d_c\Delta_d^{-1}[d,\Omega^\star] = \\ &([\Omega^\star,d] = d_c^\star) = 1-d_c\Delta_d^{-1}d_c^\star = \Pi_{Ker(d^\star)}\,,\end{aligned}$$

where the last equality follows from Hodge theory (i.e. $1 - d_c\Delta_d^{-1}d_c^\star$ is the projector $\Pi_{Ker(d^\star)} : \Lambda^\bullet(M) \to \Lambda^\bullet(M)|_{Ker(d_c^\star)}$). The space $\Lambda^\bullet(M)|_{Ker(d_c^\star)}$ carries a structure of $\mathcal{A}_\infty$-algebra, such that we have $\mu_1 = d$, $\mu_2(-,-) = (1 - d_c\Delta_d^{-1}d_c^\star)(-\wedge-)$ etc. Note that the product of two $d^\star$-closed forms is again $d^\star$-closed form.

- The complex $\mathcal{A}_\infty$-algebra of a Kähler manifold M is based on the subcomplex of the Dolbeault complex ($\mathbb{Z}/2$-graded differential algebra)

(3) $$(\Lambda^{\bullet,\bullet}(M)|_{Ker(\partial^\star)},\partial) \subset (\Lambda^{\bullet,\bullet}(M),\partial)$$

  with strong homotopy retract $Q = i\partial\Delta_{\bar\partial}^{-1}\Omega^\star$. A simple algebra gives

(4) $$1-[d,Q] = 1-\partial\Delta_{\bar\partial}^{-1}\partial^\star = \Pi_{Ker(\partial^\star)}\,,$$

  which is again by Hodge theory the projector

$$\Pi_{Ker(\partial^\star)} : \Lambda^{\bullet,\bullet}(M) \to \Lambda^{\bullet,\bullet}(M)|_{Ker(\partial^\star)}.$$

  The space $\Lambda^{\bullet,\bullet}(M)|_{Ker(\partial^\star)}$ has canonical structure of $\mathcal{A}_\infty$-algebra, such that $\mu_1 = \partial$, $\mu_2(-,-) = (1-\partial\Delta_{\bar\partial}^{-1}\partial^\star)(-\wedge-)$ etc. In particular the product of two $\partial^\star$-closed forms is again $\partial^\star$-closed.

- ($\mathcal{A}_\infty$-algebra of Calabi-Yau manifolds)
  In the case the base manifold M is a Calabi-Yau manifold there is, for a suitable Q, d, an $\mathcal{A}_\infty$-algebra structure on the graded algebra $\Lambda^\bullet(TM)\otimes\Lambda^\bullet(\overline{T}^\star M)$ with multiplication given by the wedge product. Its full definition together with applications in the context of Mirror symmetry can be found in [3].

We would like to emphasize that Hodge theory just reviewed is based on two differential operators, e.g. $\partial, \partial^\star$. This will not be the case in our example called **parabolic geometries**, which will occupy almost the rest of this article.

- Let $(\mathcal{G},\mathfrak{g},P,M,\omega)$ be a parabolic geometry on M given by a principal fiber bundle $\mathcal{G}\to M$ with typical fiber P and a Cartan connection ω, $\omega : T_u\mathcal{G} \xrightarrow{\sim} \mathfrak{g}$ ($\forall u\in\mathcal{G}$). For every finite dimensional irreducible $\mathfrak{g}$-module $\mathbb{W}$, the twisted exterior differential operator $d^{\mathfrak{g}}$ is a linear map (acting on smooth sections)

(5) $$d^{\mathfrak{g}} : \Gamma(M,\Lambda^i\mathfrak{m}^\star\otimes\mathbb{W}) \longrightarrow \Gamma(M,\Lambda^{i+1}\mathfrak{m}^\star\otimes\mathbb{W})\,,$$

  where $\mathfrak{m} := \mathfrak{g}/\mathfrak{p}$ (resp. $(\mathfrak{g}/\mathfrak{p})^\star \simeq \mathfrak{m}^\star$ via Killing-Cartan form) is isomorphic to a typical copy of the tangent (resp. cotangent) space in a given point of M (the isomorphism comes from the Cartan connection ω). The Eilenberg-Chevalley algebra codifferential $\delta_{\mathfrak{m}^\star}$ acts on the chain complex

(6) $$\delta_{\mathfrak{m}^\star} : C_\bullet(\mathfrak{m}^\star,\mathbb{W}) \to C_{\bullet-1}(\mathfrak{m}^\star,\mathbb{W})$$

  and it is P-equivariant. It follows that $\delta_{\mathfrak{m}^\star}$ descends to an algebraic operator $\delta_{T^\star M}$ on M. Harmonic theory for the couple $(d^{\mathfrak{g}},\delta_{T^\star M})$, given by the differential operator of first order and the algebraic operator is based on the differential operator of first order $\Box^{\mathfrak{g}} = d^{\mathfrak{g}}\delta_{T^\star M} + \delta_{T^\star M} d^{\mathfrak{g}}$, such that its inverse $\Box^{\mathfrak{g}^{-1}}$ is differential operator of finite order.

The subsequence of twisted de Rham sequence ($\mathbb{Z}/2$-graded differential algebra)

$$(\Lambda^\bullet(M) \otimes W|_{H_\bullet(\mathfrak{m}^\star, \mathbb{W})}, \mathcal{D}) \subset (\Lambda^\bullet(M) \otimes W, d^{\mathfrak{g}}), \tag{7}$$

where W is the bundle on M induced from a finite dimensional $\mathfrak{g}$-mod $\mathbb{W}$, is called BGG sequence. The rôle of a retraction homotopy from the twisted de Rham complex to BGG sequence is played by $Q = \Box^{\mathfrak{g}^{-1}}\delta_{T^\star M}$, such that the BGG sequence is in the image of $\Pi^{\mathfrak{g}} = 1 - [d^{\mathfrak{g}}, Q]$. Note that $\Pi^{\mathfrak{g}}$ is not in general a projector, but rather fulfills $\Pi^{\mathfrak{g}2} = \Pi^{\mathfrak{g}} + QR^{\mathfrak{g}}Q$ for the curvature $R^{\mathfrak{g}}$ (see the next section for more properties of these operators).

2. $\mathcal{A}_\infty$-ALGEBRA STRUCTURE ON PLACES OF BGG SEQUENCES

2.1. Summary of basic operations and their properties. In this article, we shall restrict ourselves to the case of the **regular normal** parabolic geometry (see the next notation for the explanation of adjectives regular and normal).

Notation 2.1. *We shall first recall the set of operators entering the definition of regular normal parabolic geometry (see for example, [1]):*

- $d^{\mathfrak{g}} : J^1(\Lambda^k(T^\star M) \otimes W) \to \Lambda^{k+1}(T^\star M) \otimes W$ *is the twisted exterior covariant derivative (twisted de Rham differential) associated to a $\mathfrak{g}$-module $\mathbb{W}$ and acting on the first jet bundle of the tensor product bundle.*
- *proj* $\circ -$ *denotes the composition with the projection from $Ker(\delta)$ to the Lie algebra homology $Ker(\delta)/Im(\delta)$, and $-\circ$ repr denotes the choice of a representative from the quotient space $Ker(\delta)/Im(\delta)$ in $Ker(\delta)$. Except a few cases, we shall suppress explicit notation of these two operations. The set of differential operators*

$$\Pi_k : \Gamma(\Lambda^k T^\star M \otimes W) \to \Gamma(\Lambda^k T^\star M \otimes W), \, k \in \mathbb{N} \tag{8}$$

then have their images in $Ker(\delta)$, vanish on $Im(\delta)$ and induce identity map on the Eilenberg-Chevalley homology of δ.
- $R^{\mathfrak{g}}$ *is the curvature of $d^{\mathfrak{g}}$, i.e. $(d^{\mathfrak{g}})^2 = R^{\mathfrak{g}}$. For the regular parabolic geometry it is the zero-order differential operator (i.e. it is an algebraic operator). The action of $R^{\mathfrak{g}}$ on $s \in \Gamma(\Lambda^k T^\star M \otimes W)$ is given by wedge product with 2-form part of $K_M \in \Gamma(M, \Lambda^2 T^\star M \otimes \mathfrak{g}_M)$ followed by the action of $\mathfrak{g}_M$ part of K_M on W-values of the section s, i.e.*

$$R^{\mathfrak{g}} : s \in \Gamma(\Lambda^k T^\star M \otimes W) \to Tr(X \to K_M \wedge X \cdot s) \in \Gamma(\Lambda^{k+2} T^\star M \otimes W), \tag{9}$$

where $X \in \mathfrak{g}_M$ and $\mathfrak{g}_M$ is the vector bundle on M induced from the $\mathfrak{p}$-module $\mathfrak{g}$.
- *The Kostant's quabla differential operator $\Box^{\mathfrak{g}}$ and the chain homotopy deformation retract (differential) operator $Q^{\mathfrak{g}}$. Their explicit form will be discussed in the next paragraph.*

Let us briefly summarize a few basic facts necessary for the construction of a generalized $\mathcal{A}_\infty$-algebra structure on the set of pieces of BGG resolutions. The source for all these results (although without much details) is [1]. Note that in many formulas we suppress the superscript $\mathfrak{g}$, i.e. we shall write Q instead of $Q^\mathfrak{g}$, $\Box$ instead of $\Box^\mathfrak{g}$ etc.

- The homomorphism $d\Pi$ is a map $d\Pi : Ker\delta \to Ker\delta$:

$$\delta d\Pi = (\Box - d\delta)\Pi = (\delta\Pi = 0) = \Box\Pi = (\Box\Pi = \delta RQ = \delta R\Box^{-1}\delta) = \delta R\Box^{-1}\delta\,,$$

and so instead of BGG operator $\mathcal{D} := \Pi d\Pi$ one can use the shorthand notation $\mathcal{D} = d\Pi$. We shall often add a subscript to $\mathcal{D}$, which simply means that we restrict $\mathcal{D}$ to a particular graded component.

- (Leibniz rule). Let us use a shorthand notation for the sheaves of sections, i.e. $\mathcal{O}(H_k(W))$ instead of $\Gamma(M, V(H_k(W)))$ (which means the space of sections of the vector bundle $V(H_k(W))$ induced from the $\mathfrak{p}$-module $H_k(W)$).

For $s_1 \in \mathcal{O}(H_k(W_1)), s_2 \in \mathcal{O}(H_l(W_2))$

$$\begin{aligned}\mathcal{D}_{k+l}(s_1 \cup s_2) = \\ &(\mathcal{D}_k s_1 \cup s_2) + (-)^k(s_1 \cup \mathcal{D}_l s_2) + [\Pi_{k+l+1}((QR\Pi_k s_1 \wedge \Pi_l s_2) + \\ &(-)^k\Pi_k s_1 \wedge QR\Pi_l s_2 - RQ(\Pi_k s_1 \wedge \Pi_l s_2))]\,,\end{aligned}$$

because

$$\begin{aligned}d\Pi^2(\Pi s_1 \wedge \Pi s_2) = \\ &(\Pi^2 = \Pi + QRQ) = d\Pi(\Pi s_1 \wedge \Pi s_2) + dQRQ(\Pi s_1 \wedge \Pi s_2) = \\ &(d\Pi = \Pi d + QR - RQ) = \Pi d(\Pi s_1 \wedge \Pi s_2) + QR(\Pi s_1 \wedge \Pi s_2) \\ &- RQ(\Pi s_1 \wedge \Pi s_2) + dQRQ(\Pi s_1 \wedge \Pi s_2) = (Id = \Pi + dQ + Qd) \\ &= \Pi d(\Pi s_1 \wedge \Pi s_2) - \Pi RQ(\Pi s_1 \wedge \Pi s_2) + Im(\delta) = (d(\Pi s_1 \wedge \Pi s_2) \\ &= (d\Pi s_1 \wedge \Pi s_2) + (-)^k(\Pi s_1 \wedge d\Pi s_2)) = \Pi(d\Pi s_1 \wedge \Pi s_2) + \\ &(-)^k\Pi(\Pi s_1 \wedge d\Pi s_2) - \Pi RQ(\Pi s_1 \wedge \Pi s_2) = (Id = \Pi + dQ + Qd) \\ &= \{\Pi(\Pi d\Pi s_1 \wedge \Pi s_2) + \Pi(dQd\Pi s_1 \wedge \Pi s_2) + \Pi(Qd^2\Pi s_1 \wedge \Pi s_2)\} + \\ &(-)^k\{1 \leftrightarrow 2\} - \Pi RQ(\Pi s_1 \wedge \Pi s_2) = (d\Pi :\ Ker\delta \to Ker\delta\,, \\ &Q(Ker\delta) = 0) \\ &= (\mathcal{D}_k s_1 \cup s_2) + (-)^k(s_1 \cup \mathcal{D}_l s_2) + [\Pi_{k+l+1}((QR\Pi_k s_1 \wedge \Pi_l s_2) \\ &+ (-)^k\Pi_k s_1 \wedge QR\Pi_l s_2 - RQ(\Pi_k s_1 \wedge \Pi_l s_2))]\,.\end{aligned}$$

2.2. **First four structure operations of the generalized $\mathcal{A}_\infty$-algebra structure on BGG sequence.** The purpose of this section is to explain the origin of the structure operations of the generalized $\mathcal{A}_\infty$-algebra on the BGG sequence.

Let us recall that we shall consider regular (i.e. the geometric weights of the curvature are negative) parabolic geometry with exterior covariant derivative $d^\mathfrak{g}$. This means that $d^\mathfrak{g}K_M = 0$ (differential Bianchi identity) and $\delta_{T^*M}K_M = 0$, i.e. K_M is uniquely determined by its (δ_{T^*M}-homology) class $\mathcal{K} := [K_M]_{\delta_{T^*M}}$. Because $K_M = \Pi_2\mathcal{K}$, we have $\mathcal{D}_2\mathcal{K} = 0$ for $\mathcal{D}_2$ (invariant differential operator) on BGG torsion-free sequence of the $\mathfrak{g}$-module $\mathfrak{g}$. Moreover, for $s \in \mathcal{O}(H_k(W))$, we get

$$\mathcal{D}_{k+1}\mathcal{D}_k s = \mathcal{K} \cup s = \Pi_{k+2}(\Pi_2\mathcal{K} \wedge \Pi_k s) = \Pi_{k+2}(K_M \wedge \Pi_k s) = \Pi_{k+2}R^\mathfrak{g}\Pi_k s\,,$$

where we used homomorphism (given by the $\mathfrak{g}$-action on W) $\mathfrak{g} \otimes W \to W$ and we again suppressed the symbol "*proj* $\circ$" for the projection on δ_{T^*M}-homology.

Definition 2.2. *Suppose $R \subset R[\mathfrak{g}]$ is a subring of the ring of finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$, freely generated by the set of irreducible representations $W_1, \dots, W_n$. For example, one can take $R \simeq \mathbb{C}[W_1, \dots, W_{rank(\mathfrak{g})}]$ with $W_1, \dots, W_{rank(\mathfrak{g})}$ the set of fundamental representations of $\mathfrak{g}$. Using the notation $W := W_1 \oplus \dots W_n$, we define the structure of Lie algebra on the vector space given by the semi-direct product $\mathfrak{g} \ltimes W$:*

$$(a_1, w_1) \circ (a_2, w_2) = ([a_1, a_2], 0) \ , \ a_1, a_2 \in \mathfrak{g} \, , \, w_1, w_2 \in W$$

(with the commutative ring structure on W). The universal enveloping algebra $\mathcal{U}(\mathfrak{g} \ltimes W)$ is $\mathfrak{g}$-module via

$$(\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})) \to End(\mathcal{U}(\mathfrak{g} \ltimes W)) \, .$$

An equivalent definition relies on the notion of the semiholonomic enveloping algebra. It is defined by

$$\begin{aligned} (9) \qquad & T_{\mathbb{C}}(\mathfrak{g} \oplus W)/\{I = a \otimes \phi - \phi \otimes a - a \cdot \phi\} = \\ & = \{\mathbb{C} \oplus \mathbb{C} \otimes (\mathfrak{g} \oplus W) \oplus \mathbb{C} \otimes^2 (\mathfrak{g} \oplus W) \oplus \dots\}/I \, , \, \phi \in T_{\mathbb{C}}(\mathfrak{g} \oplus W), a \in \mathfrak{g} \, , \end{aligned}$$

such that the action of $\mathfrak{g}$ is diagonal on $\mathfrak{g} \oplus W$ (i.e. $ad_a \otimes Id + Id \otimes a$) and extended diagonally to the whole tensor algebra.

For any differential form s with values in the associated vector bundle of this Lie algebra, we have

$$\mathcal{D}_{k+1}\mathcal{D}_k s = \mathcal{K} \cup s - s \cup \mathcal{K} \ , \ \mathcal{D}_2\mathcal{K} = 0 \tag{10}$$

and for $s_1 \in \mathcal{O}(H_k(W_1))$, $s_2 \in \mathcal{O}(H_l(W_2))$, equation (10) reads as

$$\begin{aligned} \mathcal{D}_{k+l}(s_1 \cup s_2) = & (\mathcal{D}_k s_1 \cup s_2) + (-)^k (s_1 \cup \mathcal{D}_l s_2) \\ & - \langle \mathcal{K}, s_1, s_2 \rangle + \langle s_1, \mathcal{K}, s_2 \rangle - \langle s_1, s_2, \mathcal{K} \rangle \end{aligned}$$

with

$$\begin{aligned} \langle \mathcal{K}, s_1, s_2 \rangle &= \Pi_{k+l+1}(\Pi_2\mathcal{K} \wedge Q(\Pi_k s_1 \wedge \Pi_l s_2)) - \Pi_{k+l+1}(Q(\Pi_2\mathcal{K} \wedge \Pi_k s_1) \wedge \Pi_l s_2) \, , \\ \langle s_1, \mathcal{K}, s_2 \rangle &= (-)^k \Pi_{k+l+1}(\Pi_k s_1 \wedge Q(\Pi_2\mathcal{K} \wedge \Pi_l s_2)) - \Pi_{k+l+1}(Q(\Pi_k s_1 \wedge \Pi_2\mathcal{K}) \wedge \Pi_l s_2) \, , \\ \langle s_1, s_2, \mathcal{K} \rangle &= (-)^k \Pi_{k+l+1}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_2\mathcal{K})) - \Pi_{k+l+1}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_2\mathcal{K}) \, . \end{aligned}$$

More generally, for $s_1 \in \mathcal{O}(H_k(W_1)), s_2 \in \mathcal{O}(H_l(W_2)), s_3 \in \mathcal{O}(H_m(W_3))$ one can define a trilinear differential pairing

$$\langle -, -, - \rangle \, : \, \mathcal{O}(H_k(W_1)) \otimes \mathcal{O}(H_l(W_2)) \otimes \mathcal{O}(H_m(W_3)) \to \mathcal{O}(H_{k+l+m-1}(W_4))$$

by

$$\begin{aligned} \langle s_1, s_2, s_3 \rangle = & (-)^k \Pi_{k+l+m-1}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) \\ & - \Pi_{k+l+m-1}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) \end{aligned} \tag{11}$$

using for all terms a fixed homomorphism $W_1 \otimes W_2 \otimes W_3 \otimes \to W_4$ of $\mathfrak{g}$-modules.

The deviation from non-associativity of the cup product $\cup$ together with (in the non-flat case appearing) quartylinear products of the form $\langle \mathcal{K}, s_1, s_2, s_3\rangle$ is measured by

$$\begin{aligned} \mathcal{D}_{k+l+m-1}\langle s_1,s_2,s_3\rangle = (s_1\cup s_2)\cup s_3 - s_1\cup(s_2\cup s_3) - \langle \mathcal{D}_k s_1, s_2, s_3\rangle \\ -(-)^k\langle s_1,\mathcal{D}_l s_2, s_3\rangle - (-)^{k+l}\langle s_1,s_2,\mathcal{D}_m s_3\rangle \\ +\langle \mathcal{K},s_1,s_2,s_3\rangle - \langle s_1,\mathcal{K},s_2,s_3\rangle \\ +\langle s_1,s_2,\mathcal{K},s_3\rangle - \langle s_1,s_2,s_3,\mathcal{K}\rangle \end{aligned} \tag{12}$$

for $s_1\in\mathcal{O}(H_k(W_1)), s_2\in\mathcal{O}(H_l(W_2)), s_3\in\mathcal{O}(H_m(W_2))$. Let us verify equation (12). It follows from the definition that

$$\begin{aligned} (s_1\cup s_2)\cup s_3 &= \Pi_{k+l+m}(\Pi_{k+l}(\Pi_k s_1\wedge\Pi_l s_2)\wedge\Pi_m s_3) \\ &+ \Pi_{k+l+m}(Q(\Pi_2\mathcal{K}\wedge Q(\Pi_k s_1\wedge\Pi_l s_2))\wedge\Pi_m s_3) \\ &- \Pi_{k+l+m}(Q(Q(\Pi_k s_1\wedge\Pi_l s_2)\wedge\Pi_2\mathcal{K})\wedge\Pi_m s_3)\,, \\ s_1\cup(s_2\cup s_3) &= \Pi_{k+l+m}(\Pi_k s_1\wedge\Pi_{l+m}(\Pi_l s_2\wedge\Pi_m s_3)) \\ &+ \Pi_{k+l+m}(\Pi_k s_1\wedge Q(\Pi_2\mathcal{K}\wedge Q(\Pi_l s_2\wedge\Pi_m s_3))) \\ &- \Pi_{k+l+m}(\Pi_k s_1\wedge Q(Q(\Pi_l s_2\wedge\Pi_m s_3)\wedge\Pi_2\mathcal{K})) \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\langle \mathcal{D}_k s_1,s_2,s_3\rangle = \\ &\qquad \Pi_{k+l+m}((-)^{k+1}\Pi_{k+1}d\Pi_k s_1\wedge Q(\Pi_l s_2\wedge\Pi_m s_3)) - \\ &\qquad \Pi_{k+l+m}(Q(\Pi_{k+1}d\Pi_k s_1\wedge\Pi_l s_2)\wedge\Pi_m s_3)\,, \\ &\langle s_1,\mathcal{D}_l s_2,s_3\rangle = \\ &\qquad \Pi_{k+l+m}((-)^{k}\Pi_k s_1\wedge Q(\Pi_{l+1}d\Pi_l s_2\wedge\Pi_m s_3)) - \\ &\qquad \Pi_{k+l+m}(Q(\Pi_k s_1\wedge\Pi_{l+1}d\Pi_l s_2)\wedge\Pi_m s_3)\,, \\ &\langle s_1,s_2,\mathcal{D}_m s_3\rangle = \\ &\qquad \Pi_{k+l+m}((-)^{k}\Pi_k s_1\wedge Q(\Pi_l s_2\wedge\Pi_{m+1}d\Pi_m s_3)) - \\ &\qquad \Pi_{k+l+m}(Q(\Pi_k s_1\wedge\Pi_l s_2)\wedge\Pi_{m+1}d\Pi_m s_3)\,. \end{aligned}$$

Moreover, the identity $\Pi = Id - dQ - Qd$ gives

$$\begin{aligned} dQ(\Pi_k s_1\wedge\Pi_l s_2) = (\Pi_k s_1\wedge\Pi_l s_2) - \Pi_{k+l}(\Pi_k s_1\wedge\Pi_l s_2) \\ - Q(d\Pi_k s_1\wedge\Pi_l s_2) - (-)^l Q(\Pi_k s_1\wedge d\Pi_l s_2)\,. \end{aligned}$$

Because $\Pi d\Pi = d\Pi$ on $Ker(\delta)$, we have

$$\begin{aligned} d\Pi\Pi &= (d\Pi = \Pi d + QR - RQ) = \Pi d\Pi + QR\Pi = (QR\Pi\subset Im(\delta)) = \\ &= \Pi d\Pi = \Pi^2 d + \Pi QR - \Pi RQ = \Pi d + QRQd - \Pi RQ = \Pi d - \Pi RQ\,. \end{aligned}$$

Both latter displays enter the computation of LHS of equation (12):

$$
\begin{aligned}
\mathcal{D}_{k+l+m-1}\langle s_1, s_2, s_3\rangle = \\
&\Pi_{k+l+m} d((-)^k \Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) \\
&- \Pi_{k+l+m} d(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) - \\
&- \Pi_{k+l+m} RQ((-)^k \Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) + \\
&\Pi_{k+l+m} RQ(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) \\
&= (-)^k \Pi_{k+l+m}(\Pi_{k+1} d\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) + \\
&(-)^k \Pi_{k+l+m}(Q(\Pi_2 \mathcal{K} \wedge \Pi_k s_1) \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) - \\
&- (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_2 \mathcal{K}) \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) \\
&- \Pi_{k+l+m}(\Pi_k s_1 \wedge \Pi_{l+m}(\Pi_l s_2 \wedge \Pi_m s_3)) - \\
&- \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_{l+1} d\Pi_l s_2 \wedge \Pi_m s_3)) \\
&- \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(Q(\Pi_2 \mathcal{K} \wedge \Pi_l s_2) \wedge \Pi_m s_3)) + \\
&+ \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(Q(\Pi_l s_2 \wedge \Pi_2 \mathcal{K}) \wedge \Pi_m s_3)) - \\
&(-)^l \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_{m+1} d\Pi_m s_3)) - \\
&- (-)^l \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge Q(\Pi_2 \mathcal{K} \wedge \Pi_m s_3))) + \\
&(-)^l \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge Q(\Pi_m s_3 \wedge \Pi_2 \mathcal{K}))) + \\
&+ \Pi_{k+l+m}(\Pi_{k+l}(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) + \\
&\Pi_{k+l+m}(Q(\Pi_{k+1} d\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) + \\
&+ \Pi_{k+l+m}(Q(Q(\Pi_2 \mathcal{K} \wedge \Pi_k s_1) \wedge \Pi_l s_2) \wedge \Pi_m s_3) - \\
&\Pi_{k+l+m}(Q(Q(\Pi_k s_1 \wedge \Pi_2 \mathcal{K}) \wedge \Pi_l s_2) \wedge \Pi_m s_3) + \\
&+ (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_{l+1} d\Pi_l s_2) \wedge \Pi_m s_3) \\
&+ (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge Q(\Pi_2 \mathcal{K} \wedge \Pi_l s_2)) \wedge \Pi_m s_3) - \\
&- (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_2 \mathcal{K})) \wedge \Pi_m s_3) + \\
&(-)^{k+l} \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_{m+1} d\Pi_m s_3) + \\
&+ (-)^{k+l} \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge Q(\Pi_2 \mathcal{K} \wedge \Pi_m s_3)) - \\
&(-)^{k+l} \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge Q(\Pi_m s_3 \wedge \Pi_2 \mathcal{K})) - \\
&- (-)^k \Pi_{k+l+m}(\Pi_2 \mathcal{K} \wedge Q(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3))) + \\
&\Pi_{k+l+m}(\Pi_2 \mathcal{K} \wedge Q(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3)) .
\end{aligned}
$$

After the identification of quarty-linear terms

$$
\begin{aligned}
&\langle \mathcal{K}, s_1, s_2, s_3\rangle = \\
&\quad - (-)^k \Pi_{k+l+m}(\Pi_2\mathcal{K} \wedge Q(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)))+ \\
&\quad + (-)^k \Pi_{k+l+m}(Q(\Pi_2\mathcal{K} \wedge \Pi_k s_1) \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3))+ \\
&\quad \Pi_{k+l+m}(Q(Q(\Pi_2\mathcal{K} \wedge \Pi_k s_1) \wedge \Pi_l s_2) \wedge \Pi_m s_3)+ \\
&\quad + \Pi_{k+l+m}(\Pi_2\mathcal{K} \wedge Q(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3)) \\
&\quad - \Pi_{k+l+m}(Q(\Pi_2\mathcal{K} \wedge Q(\Pi_k s_1 \wedge \Pi_l s_2)) \wedge \Pi_m s_3)\,,
\end{aligned}
$$

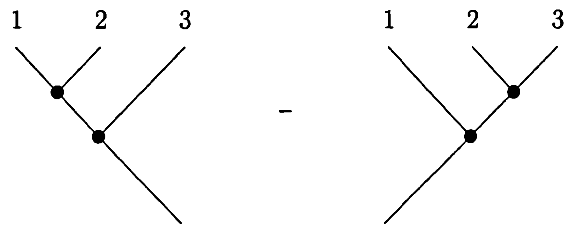
$$
\begin{aligned}
&\langle s_1, \mathcal{K}, s_2, s_3\rangle = \\
&\quad \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(Q(\Pi_2\mathcal{K} \wedge \Pi_l s_2) \wedge \Pi_m s_3))+ \\
&\quad + (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_2\mathcal{K}) \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3))- \\
&\quad (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge Q(\Pi_2\mathcal{K} \wedge \Pi_l s_2) \wedge \Pi_m s_3))+ \\
&\quad - \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_2\mathcal{K} \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3))) \\
&\quad + \Pi_{k+l+m}(Q(Q(\Pi_k s_1 \wedge \Pi_2\mathcal{K}) \wedge \Pi_l s_2) \wedge \Pi_m s_3)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\langle s_1, s_2, \mathcal{K}, s_3\rangle = \\
&\quad \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(Q(\Pi_l s_2 \wedge \Pi_2\mathcal{K}) \wedge \Pi_m s_3))- \\
&\quad - (-)^l \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge Q(\Pi_2\mathcal{K} \wedge \Pi_m s_3)))- \\
&\quad (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_2\mathcal{K})) \wedge \Pi_m s_3)+ \\
&\quad + (-)^{k+l} \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge Q(\Pi_2\mathcal{K} \wedge \Pi_m s_3)) \\
&\quad + \Pi_{k+l+m}(Q(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_2\mathcal{K}) \wedge \Pi_m s_3)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\langle s_1, s_2, s_3, \mathcal{K}\rangle = \\
&\quad \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(Q(\Pi_l s_2 \wedge \Pi_m s_3) \wedge \Pi_2\mathcal{K}))- \\
&\quad - (-)^l \Pi_{k+l+m}(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge Q(\Pi_m s_3 \wedge \Pi_2\mathcal{K})))- \\
&\quad (-)^k \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge Q(\Pi_l s_2 \wedge \Pi_m s_3)) \wedge \Pi_2\mathcal{K})+ \\
&\quad + (-)^{k+l} \Pi_{k+l+m}(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge Q(\Pi_m s_3 \wedge \Pi_2\mathcal{K})) \\
&\quad + \Pi_{k+l+m}(Q(Q(\Pi_k s_1 \wedge \Pi_l s_2) \wedge \Pi_m s_3) \wedge \Pi_2\mathcal{K})\,,
\end{aligned}
$$

we directly recognize all terms of the non-flat homotopy associativity equation (12). □

Equation 12 can be conveniently represented by the same picture as in the standard case (i.e. with trivial curvature endomorphism):



2.3. The structure of codifferential on generalized $\mathcal{A}_\infty$-algebra. Equations (10), (11) and (12) are the first four defining operations of generalized $\mathcal{A}_\infty$-algebra. In this subsection we shall describe in detail the whole generalized $\mathcal{A}_\infty$-algebra. Let us first recall its axioms, [4].

Definition 2.3. *A generalized $\mathcal{A}_\infty$-algebra is a $\mathbb{Z}$-graded vector space $\mathcal{A}$ (over the field $\mathbb{C}$) together with a collection of multilinear maps*

$$\mu_k : \otimes^k \mathcal{A} \to \mathcal{A}\,,\ k \in \mathbb{N}_0 \tag{14}$$

of degree $deg(\mu_k) = 2 - k$, satisfying the compatibility condition

$$\sum_{\substack{j+k=m+1\\ j\geq 1, k\geq 0}} \sum_{l=0}^{j-1} (-)^{k+l+kl+k\sum_{i=1}^{l}|s_i|}\mu_j(s_1, \dots, s_l, \mu_k(s_{l+1}, \dots, s_{k+l}), s_{k+l+1}, \dots, s_m) = 0 \tag{15}$$

for all $s_i \in \mathcal{A}$ of degree $|s_i|$.

Note that in the previous subsection we used the notation $\langle -, -, \dots, - \rangle$ for multilinear map μ.

In particular the $\mathbb{Z}$-grading on $\mathcal{A}$, $\mathcal{A} \simeq \{\oplus_k \mathcal{A}_k\}_{k\in\mathbb{Z}}$, is compatible with multilinear operations μ_k in such a way that

$$\mu_k : \mathcal{A}_{(i_1-1)+1} \otimes \mathcal{A}_{(i_2-1)+1} \cdots \otimes \mathcal{A}_{(i_k-1)+1} \longrightarrow \mathcal{A}_{((i_1-1)+\dots+(i_k-1)+1)+1}\,.$$

Let us consider the tensor algebra $\otimes\mathcal{A}$ of $\mathcal{A}$ and the corresponding projections $p_k : \otimes\mathcal{A} \to \otimes\mathcal{A}^k$ ($k \in \mathbb{N}_0$), such that $\mu_k : \otimes\mathcal{A} \to \mathcal{A}$ factors through p_k. Let us extend

$$(\mu := \sum_{k\in\mathbb{N}_0} \mu_k) : \otimes\mathcal{A} \to \mathcal{A} \tag{16}$$

to a coderivation μ^c of the tensor coalgebra of $\mathcal{A}$

$$\mu^c : (\otimes\mathcal{A})^c \to (\otimes\mathcal{A})^c \tag{17}$$

inductively by

- $p_0\mu^c = 0$ ($p_0\mu^c : \otimes\mathcal{A} \to \mathbb{C}$);
- $p_1\mu^c = \mu$ ($p_1\mu^c : \otimes\mathcal{A} \to \mathcal{A}$);
- $\Delta\mu^c = (1 \otimes \mu^c + \mu^c \otimes 1)\Delta$,

where $\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the conilpotent coproduct on $\otimes\mathcal{A}$, defined by

$$\triangle(a_1 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_k) \tag{18}$$

for $a_1 \otimes a_1 \otimes \cdots \otimes a_k \in \otimes^k \mathcal{A}$.

Lemma 2.4. *We have the equality* $(\mu^c)^2 = \mu^c\mu^c = (\mu\mu^c)^c$, *i.e.* $(\mu^c)^2 = 0$ *iff* $\mu\mu^c = 0$.

Proof. The proof follows from the definition of the coproduct. In particular,

- $p_0(\mu\mu^c)^c = p_0(\mu\mu^c) = (p_0\mu)\mu^c = 0 = (p_0\mu)\mu^c = \mu^c\mu^c$;
- $p_1(\mu\mu^c)^c = \mu\mu^c = (p_1\mu^c)\mu^c = p_1(\mu^c\mu^c) = p_1(\mu^c)^2$;
- by definition, $\mu\mu^c$ is extended to the coalgebra $\otimes\mathcal{A}$ by

$$\triangle(\mu\mu^c)^c = (\mu^c \otimes 1 + 1 \otimes \mu^c)\triangle\mu^c, \tag{19}$$

which is in correspondence with $\triangle\mu^c\mu^c = (\mu^c \otimes 1 + 1 \otimes \mu^c)\triangle\mu^c$.

Combining all three observations together yields the desired equality. □

It is well known that the existence of $\mathcal{A}_\infty$-structure on the vector space $\mathcal{A}$ is equivalent to the existence of conilpotent coderivation μ^c on the tensor coalgebra $(\otimes\mathcal{A})^c$, [4].

Our aim in the remaining part of this section is the construction of this codifferential in the case of parabolic geometry. In this case the $\mathcal{A}_\infty$-algebra $\mathcal{A}$ has the following structure. Let $U \subset M$ be an open set on the manifold M with parabolic structure. Then $\mathcal{A}$ is the sheaf of (smooth) sections associated to the presheaf

$$\mathcal{A}|_U = \Gamma(U, \oplus_k H_k(\mathcal{U}(\mathfrak{g} \ltimes W))), \tag{20}$$

and the first four multilinear maps are

$$\begin{aligned} \mu_0 : 1 &\longrightarrow \mathcal{A} \quad , \quad \mu_0(1) = \mathcal{K} \in \mathcal{O}(H_2(\mathfrak{g})), \\ \mu_1 : \mathcal{A} &\longrightarrow \mathcal{A} \quad , \quad \mu_1 = \mathcal{D}, \\ \mu_2 : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathcal{A} \quad , \quad \mu_2 = \cup_{\mathfrak{g}}, \\ \mu_3 : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathcal{A} \quad , \quad \mu_3 = \langle -, -, \rangle_{\mathfrak{g}}. \end{aligned} \tag{21}$$

Each of the members in the collection of maps $\{\mu_k\}_{k\in\mathbb{N}}$ acts, by definition of $\mathcal{A}$, on the sections of vector bundles induced from Lie algebra homology modules. Let us now define inductively the collection of multilinear maps $\{\lambda_k\}_{k\in\mathbb{N}_0}$ acting on sections of vector bundles induced from Lie algebra chain bundles:

$$\begin{aligned} \lambda_k(\alpha_1, \ldots, \alpha_k) = \\ = \sum_{\substack{i+j=k \\ i\geq 1, j\geq 1}} (-)^{(j-1)(i+|\alpha_1|+\cdots+|\alpha_i|)} Q\lambda_i(\alpha_1, \ldots, \alpha_i) \wedge Q\lambda_j(\alpha_{i+1}, \ldots, \alpha_k), \end{aligned} \tag{22}$$

where we define $Q\lambda_1 := -p_1$. The relation between $\{\mu_k\}_{k\in\mathbb{N}_0}$ and $\{\lambda_k\}_{k\in\mathbb{N}_0}$ is (for $k \geq 2$)

$$\mu_k(s_1, \ldots, s_k) = \Pi\lambda_k(\Pi s_1, \ldots, \Pi s_k) \ , \ s_i \in \mathcal{A} \,\forall i = 1, \cdots, k \tag{23}$$

and explicitly $\mu_0 = \mathcal{K}$, $\mu_1(s_1) = \mathcal{D}s_1$.

Lemma 2.5. *The curvature $R^{\mathfrak{g}}$ acts on $\mathcal{A}$ via*

$$R^{\mathfrak{g}} = -\lambda_2(K_M \otimes 1 + 1 \otimes K_M)\,, \tag{24}$$

Proof. First of all note that the wedge operator $\wedge$ (applied on two elements of $\mathcal{A}$ means to lift them by Π to chain complex, then to use the wedge on forms followed by representation projection on irreducible $\mathfrak{g}$-module and finally apply Π once again) has degree one, because it maps

$$\begin{aligned} \wedge &: \mathcal{A}\otimes\mathcal{A}\to\mathcal{A} \\ |\wedge| &: |a_1|+|a_2|\to|a_1|+|a_2|+1 \end{aligned} \tag{25}$$

(or, in degrees shifted by one, $\{|a_1|+1\}+\{|a_2|+1\}\to\{(|a_1|+|a_2|+1)+1\}$). The rest follows from the fact, that as an element of $K_M\in\mathcal{A}$ has shifted degree 1. □

Lemma 2.6. *The recursive definition of λ,*

$$\lambda_m = \sum_{j+k=m,j\geq 1,k\geq 1} Q\lambda_j \wedge Q\lambda_k\,,$$

is equivalent to

$$\lambda = \lambda_2((Q\lambda + Q\lambda_1)\otimes(Q\lambda+Q\lambda_1))\Delta\,. \tag{26}$$

Proof. The (recursive) definition follows from the associativity of λ_2 (λ_2 is just the associative wedge product). The composition of (reduced) coproduct with $(Q\lambda + Q\lambda_1)\otimes(Q\lambda+Q\lambda_1)$ reproduces $\sum_{j+k=m,j\geq1,k\geq1} Q\lambda_j\otimes Q\lambda_k$ in the recursive definition, and then the application of λ_2 gives the result. □

Lemma 2.7. *The following* (*recursion*) relations are satisfied:

- $$\lambda\lambda^c = \lambda_2((Q\lambda - p_1)\otimes Q\lambda\lambda^c + Q\lambda\lambda^c\otimes(Q\lambda-p_1))\Delta \tag{27}$$

- $$\begin{aligned} d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}},Q]\lambda)^c = \lambda_2((Q\lambda - p_1)\otimes Q(d^{\mathfrak{g}}\lambda+\lambda(d^{\mathfrak{g}})^c-\\ \lambda([d^{\mathfrak{g}},Q]\lambda)^c) + Q(d^{\mathfrak{g}}\lambda+\lambda(d^{\mathfrak{g}})^c-\\ \lambda([d^{\mathfrak{g}},Q]\lambda)^c)\otimes(Q\lambda-p_1))\Delta \end{aligned} \tag{28}$$

- $$\begin{aligned}&\lambda_2((Q\lambda - p_1)\otimes Q\lambda K_M^c + \\ &Q\lambda K_M^c \otimes (Q\lambda - p_1))\Delta - \\ &-\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\ &\quad K_M \otimes (Q\lambda - p_1)))^c = \lambda_2(Q\lambda_2((Q\lambda - p_1)\otimes Q\lambda K_M^c + \\ &\quad Q\lambda K_M^c \otimes (Q\lambda - p_1))\Delta \otimes (Q\lambda - p_1) + \\ &\quad (Q\lambda - p_1)\otimes Q\lambda_2((Q\lambda - p_1)\otimes Q\lambda K_M^c + \\ &\quad Q\lambda K_M^c \otimes (Q\lambda - p_1))\Delta)\Delta - \\ &\quad \lambda_2(Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\ &\quad K_M \otimes (Q\lambda - p_1)))^c \otimes (Q\lambda - p_1) + \\ &\quad (Q\lambda - p_1)\otimes Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\ &\quad K_M \otimes (Q\lambda - p_1)))^c)\Delta\,.\end{aligned}$$

Proof. Using Lemma 2.6 with $Q\lambda_1 = -p_1$, we get

- $$\begin{aligned}&\lambda\lambda^c = \\ &\quad \lambda_2((Q\lambda - p_1)\otimes (Q\lambda - p_1))\Delta\lambda^c = \\ &\quad \lambda_2((Q\lambda - p_1)\otimes (Q\lambda - p_1))(1\otimes \lambda^c + \lambda^c \otimes 1)\Delta = \\ &\quad (p_1\lambda^c = \lambda) = \lambda_2((Q\lambda - p_1)\otimes Q\lambda\lambda^c + \\ &\quad Q\lambda\lambda^c \otimes (Q\lambda - p_1))\Delta - \lambda_2((Q\lambda - p_1)\otimes \lambda + \\ &\quad \lambda\otimes (Q\lambda - p_1))\Delta = \lambda_2((Q\lambda - p_1)\otimes Q\lambda\lambda^c + \\ &\quad Q\lambda\lambda^c \otimes (Q\lambda - p_1))\Delta\,,\end{aligned}$$

  because the second term

$$\begin{aligned}&\lambda_2((Q\lambda - p_1)\otimes \lambda + \lambda\otimes (Q\lambda - p_1))\Delta = \\ &\qquad \lambda_2((Q\lambda - p_1)\otimes \lambda_2((Q\lambda - p_1)\otimes (Q\lambda - p_1))\Delta + \\ &\qquad \lambda_2((Q\lambda - p_1)\otimes (Q\lambda - p_1))\Delta \otimes (Q\lambda - p_1))\Delta\end{aligned}$$

  cancels out due to the associativity of λ_2, i.e.

$$\lambda_2(-\otimes \lambda_2(-\otimes -)) = -\lambda_2(\lambda_2(-\otimes -)\otimes -)$$

  or $(-\wedge(-\wedge -)) = ((-\wedge -)\wedge -)$. In the last equality with the wedge product the minus sign on the right hand side disappeared in comparison with the last but one equation containing λ_2 due to passing the first λ_2 through the second λ_2 (in the shifted (suspended) degrees has λ_2 degree one).

- The Leibniz rule yields $d^{\mathfrak{g}}\lambda_2(-\otimes -) = -\lambda_2(d^{\mathfrak{g}} - \otimes -) - \lambda_2(-\otimes d^{\mathfrak{g}} -)$ ($d^{\mathfrak{g}}$, $|d^{\mathfrak{g}}| = 1$, passes through elements with suspended grading by one, i.e. λ_2 has suspended degree 1).

$$
\begin{aligned}
d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c = \\
& d^{\mathfrak{g}}\lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))\triangle + \\
& \lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))\triangle(d^{\mathfrak{g}})^c - \\
& \lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))\triangle([d^{\mathfrak{g}}, Q]\lambda)^c = \\
& \lambda_2(-d^{\mathfrak{g}}(Q\lambda - p_1) \otimes (Q\lambda - p_1) + \\
& (Q\lambda - p_1) \otimes -d^{\mathfrak{g}}(Q\lambda - p_1))\triangle + \\
& \lambda_2((Q\lambda - p_1)(d^{\mathfrak{g}})^c \otimes (Q\lambda - p_1) + \\
& (Q\lambda - p_1) \otimes (Q\lambda - p_1)(d^{\mathfrak{g}})^c)\triangle - \\
& - \lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1)([d^{\mathfrak{g}}, Q]\lambda)^c + \\
& (Q\lambda - p_1)([d^{\mathfrak{g}}, Q]\lambda)^c \otimes (Q\lambda - p_1)) = \\
& \lambda_2((Q\lambda - p_1) \otimes \{-d^{\mathfrak{g}}(Q\lambda - p_1) + (Q\lambda - p_1)(d^{\mathfrak{g}})^c - \\
& (Q\lambda - p_1)([d^{\mathfrak{g}}, Q]\lambda)^c\} + \\
& \{-d^{\mathfrak{g}}(Q\lambda - p_1) + (Q\lambda - p_1)(d^{\mathfrak{g}})^c - \\
& (Q\lambda - p_1)([d^{\mathfrak{g}}, Q]\lambda)^c\} \otimes (Q\lambda - p_1))\triangle .
\end{aligned}
$$

The term in any of the last two curly bracket can be reorganized as

$$
\begin{aligned}
& - d^{\mathfrak{g}}(Q\lambda - p_1) + (Q\lambda - p_1)(d^{\mathfrak{g}})^c - (Q\lambda - p_1)([d^{\mathfrak{g}}, Q]\lambda)^c = (d^{\mathfrak{g}}Q + Qd^{\mathfrak{g}} = [d^{\mathfrak{g}}, Q]) = \\
& - [d^{\mathfrak{g}}, Q]\lambda + Qd^{\mathfrak{g}}\lambda + d^{\mathfrak{g}} + Q\lambda(d^{\mathfrak{g}})^c - d^{\mathfrak{g}} - Q\lambda([d^{\mathfrak{g}}, Q]\lambda)^c + [d^{\mathfrak{g}}, Q]\lambda \\
& = Q(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c)
\end{aligned}
$$

which is the formula (27).

- Let us treat both terms of the LHS (28) separately. Using recursive definition of λ, we get

$$
\begin{aligned}
& \lambda_2((Q\lambda - p_1) \otimes Q\lambda K_M^c + \\
& Q\lambda K_M^c \otimes (Q\lambda - p_1))\triangle = \\
& \qquad \lambda_2(Q\lambda_2((Q\lambda - p_1)K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes (Q\lambda - p_1)K_M^c)\triangle \otimes (Q\lambda - p_1) + \\
& \qquad + (Q\lambda - p_1) \otimes Q\lambda_2((Q\lambda - p_1)K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes (Q\lambda - p_1)K_M^c)\triangle)\triangle \\
& \qquad = \lambda_2(Q\lambda_2(Q\lambda K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes Q\lambda K_M^c)\triangle \otimes (Q\lambda - p_1) + \\
& \qquad + (Q\lambda - p_1) \otimes Q\lambda_2(Q\lambda K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes Q\lambda K_M^c)\triangle)\triangle + \\
& \qquad \lambda_2(Q\lambda_2(-K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes -K_M^c)\triangle \otimes (Q\lambda - p_1) + \\
& \qquad + (Q\lambda - p_1) \otimes Q\lambda_2(-K_M^c \otimes (Q\lambda - p_1) + \\
& \qquad (Q\lambda - p_1) \otimes -K_M^c)\triangle)\triangle
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\quad K_M \otimes (Q\lambda - p_1)))^c = \\
&\qquad \lambda_2((Q\lambda - p_1)\otimes(Q\lambda - p_1))\Delta(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))^c \\
&\qquad = \lambda_2((Q\lambda - p_1)(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))^c\otimes(Q\lambda - p_1) + \\
&\qquad (Q\lambda - p_1)\otimes(Q\lambda - p_1)(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))^c)\Delta \\
&\qquad = \lambda_2(Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))^c\otimes(Q\lambda - p_1) + \\
&\qquad (Q\lambda - p_1)\otimes Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))^c)\Delta + \\
&\qquad \lambda_2(-Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1))\otimes(Q\lambda - p_1) + \\
&\qquad (Q\lambda - p_1)\otimes -Q\lambda_2((Q\lambda - p_1)\otimes K_M + \\
&\qquad K_M\otimes(Q\lambda - p_1)))\Delta\,.
\end{aligned}
\tag{30}
$$

In the difference of LHS of the last two equations exactly the second terms of RHS cancel out, i.e.

$$
\begin{aligned}
&(29) - (30) = \\
&\quad \lambda_2(Q\lambda_2(Q\lambda K_M^c\otimes(Q\lambda - p_1) + (Q\lambda - p_1)\otimes Q\lambda K_M^c)\Delta\otimes(Q\lambda - p_1) + \\
&\quad (Q\lambda - p_1)\otimes Q\lambda_2(Q\lambda K_M^c\otimes(Q\lambda - p_1) + (Q\lambda - p_1)\otimes Q\lambda K_M^c)\Delta)\Delta - \\
&\quad \lambda_2(Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + K_M\otimes(Q\lambda - p_1)))^c\otimes(Q\lambda - p_1) + \\
&\quad (Q\lambda - p_1)\otimes Q\lambda(Q\lambda_2((Q\lambda - p_1)\otimes K_M + K_M\otimes(Q\lambda - p_1)))^c)\Delta
\end{aligned}
$$

which proves the third equation. □

The following Theorem lies in the heart of this article.

Theorem 2.8. *The linear map μ^c is quadratically nilpotent coderivation (i.e. (co)differential) on tensor conilpotent coalgebra $\otimes\mathcal{A}$, i.e.*

$$
\mu^c\mu^c = 0\,. \tag{31}
$$

Proof. As follows from Lemma2.4, it is sufficient to prove $\mu\mu^c = 0$.

$$
\begin{aligned}
&\mu\mu^c = \\
&\quad \Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi[\Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi]^c = (p_0[\Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi]^c = 0, \\
&\quad p_1\mu^c = \mu) = \Pi d^{\mathfrak{g}}\Pi[\Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi] + \Pi\lambda[\Pi^2(K_M + d^{\mathfrak{g}} + \lambda)\Pi]^c = \\
&\quad \Pi d^{\mathfrak{g}}\Pi^2 K_M\Pi + \Pi d^{\mathfrak{g}}\Pi^2 d^{\mathfrak{g}}\Pi + \Pi d^{\mathfrak{g}}\Pi^2\lambda\Pi + \Pi\lambda[\Pi^2(K_M + d^{\mathfrak{g}} + \lambda)\Pi]^c\,.
\end{aligned}
\tag{32}
$$

Because

$$
\begin{aligned}
&proj\Pi d^{\mathfrak{g}}\Pi^2 = \\
&\qquad (proj\Pi d^{\mathfrak{g}}\Pi = proj\Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)) = proj\Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)\Pi = \\
&\qquad (\delta\Pi = 0) = proj\Pi d^{\mathfrak{g}}\Pi = proj\Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)\,,
\end{aligned} \tag{33}
$$

the first term of (32) is trivial:

$$
\begin{aligned}
\Pi d^{\mathfrak{g}}\Pi^2 K_M\Pi &= \Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)K_M\Pi = (\delta K_M = 0 \Longrightarrow K_M\Pi \in Ker\delta) = \\
&= \Pi d^{\mathfrak{g}}\Pi K_M\Pi = \Pi d^{\mathfrak{g}}K_M\Pi = \Pi d^{\mathfrak{g}}K_M\Pi = (d^{\mathfrak{g}}K_M = 0) = 0.
\end{aligned}
$$

The second term of (32) reduces to

$$
\Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)d^{\mathfrak{g}}\Pi = (d^{\mathfrak{g}}\Pi \in Ker\delta) = \Pi(d^{\mathfrak{g}})^2\Pi = \Pi R^{\mathfrak{g}}p_1\Pi\,,
$$

the third term of (32) gives $\Pi d^{\mathfrak{g}}\Pi^2\lambda\Pi = \Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)\lambda\Pi$, the fourth term of (32) can be reduced using

- $[\Pi^2 K_M]^c\Pi = K_M^c\Pi$,
- $[\Pi^2 d^{\mathfrak{g}}]^c\Pi = (\Pi d^{\mathfrak{g}}\Pi repr = (d^{\mathfrak{g}} - QR^{\mathfrak{g}})\Pi repr) = [\Pi(d^{\mathfrak{g}} - QR^{\mathfrak{g}})]^c\Pi = (\Pi Q = 0) = [(d^{\mathfrak{g}} - QR^{\mathfrak{g}})]^c\Pi$,
- $[\Pi^2\lambda]^c\Pi = (\Pi^2 = \Pi + QR^{\mathfrak{g}}Q) = [\Pi\lambda + QR^{\mathfrak{g}}Q\lambda]^c\Pi$

and so we acquire

$$
\begin{aligned}
(32) = {} & \Pi R^{\mathfrak{g}}p_1\Pi + \Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)\lambda\Pi \\
& + \Pi\lambda[K_M + d^{\mathfrak{g}} - QR^{\mathfrak{g}}p_1 + \Pi\lambda + QR^{\mathfrak{g}}Q\lambda]^c\Pi\,.
\end{aligned} \tag{34}
$$

The previous terms can be collected and further simplified in the following way:

- $$\Pi d^{\mathfrak{g}}\lambda\Pi + \Pi\lambda(d^{\mathfrak{g}})^c\Pi + \Pi\lambda(\Pi\lambda)^c\Pi = \Pi(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c + \lambda(\Pi\lambda)^c)\Pi\,,$$

- $$\begin{aligned} -\Pi\lambda(QR^{\mathfrak{g}}p_1)^c\Pi + \Pi\lambda(QR^{\mathfrak{g}}Q\lambda)^c\Pi &= -\Pi\lambda(QR^{\mathfrak{g}}(p_1 - Q\lambda))^c\Pi = (\text{Lemma}(2.5)) = \\ &\Pi\lambda(Q\lambda_2(K_M \otimes (p_1 - Q\lambda) + (p_1 - Q\lambda) \otimes K_M)^c)\Pi\,, \end{aligned}$$

- $$\begin{aligned} \Pi\lambda K_M^c\Pi &= \Pi\lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1)) \cdot \Delta K_M^c\Pi = \\ &\Pi\lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))(K_M^c \otimes 1 + 1 \otimes K_M^c) \cdot \Delta\Pi\,, \end{aligned}$$

- $$\begin{aligned} &\Pi R^{\mathfrak{g}}p_1\Pi - \Pi R^{\mathfrak{g}}Q\lambda\Pi = \\ &\qquad \Pi R^{\mathfrak{g}}(p_1 - Q\lambda)\Pi = \Pi\lambda_2(K_M \otimes 1 + 1 \otimes K_M)(Q\lambda - p_1)\Pi = \\ &\qquad (\text{Lemma}(2.5)) = \Pi\lambda_2(K_M \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes K_M)\Pi\,, \end{aligned}$$

and so we get

$$
\begin{aligned}
(34) = {} & \Pi(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c + \lambda(\Pi\lambda)^c)\Pi + \Pi\lambda(Q\lambda_2(K_M \otimes (p_1 - Q\lambda) \\
& + (p_1 - Q\lambda) \otimes K_M)^c)\Pi \\
& + \Pi\lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))(K_M^c \otimes 1 + 1 \otimes K_M^c)\Delta\Pi \\
& + \Pi\lambda_2(K_M \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes K_M)\Pi\,.
\end{aligned} \tag{35}
$$

Concerning the first term of (35), we use

$$(\Pi\lambda)^c = ((Id - d^{\mathfrak{g}}Q - Qd^{\mathfrak{g}})\lambda)^c = \lambda^c - ([d^{\mathfrak{g}}, Q]\lambda)^c$$

(note the appearance of the $\mathbb{Z}_2$-graded commutator and $|Q| = 1 = |d^{\mathfrak{g}}|$). The third and the fourth terms of (35) can be combined,

$$\begin{aligned}&\Pi\lambda_2((Q\lambda - p_1) \otimes (Q\lambda - p_1))(K_M^c \otimes 1 + 1 \otimes K_M^c)\Delta\Pi + \\ &\qquad \Pi\lambda_2(K_M \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes K_M)\Pi = \\ &\qquad\qquad \Pi\lambda_2(Q\lambda K_M^c \otimes (Q\lambda - p_1) + \\ &\qquad\qquad (Q\lambda - p_1) \otimes Q\lambda K_M^c)\Delta\Pi\end{aligned}$$

and so finally

$$\begin{aligned}&(35) = \\ &\quad \Pi(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - ([d^{\mathfrak{g}}, Q]\lambda)^c)\Pi + \Pi(\lambda\lambda^c)\Pi + \Pi\lambda_2(Q\lambda K_M^c \otimes (Q\lambda - p_1) + \\ &\quad (Q\lambda - p_1) \otimes Q\lambda K_M^c)\Delta\Pi + \Pi\lambda(Q\lambda_2(K_M \otimes (p_1 - Q\lambda) + (p_1 - Q\lambda) \otimes K_M)^c)\Pi\,.\end{aligned}$$

Now the application of recursion relation in the Lemma 2.7 to all terms (36) makes them trivial due to the associativity of λ_2. This completes the proof. □

2.4. **Generalized pentagon (associahedron) condition.** In the previous discussion we have explored the generalized associator on the set of pieces of BGG sequences, i.e. the expression $(s_1 \cup s_2) \cup s_3 - s_1 \cup (s_2 \cup s_3)$, in the framework of generalized $\mathcal{A}_\infty$-algebra.

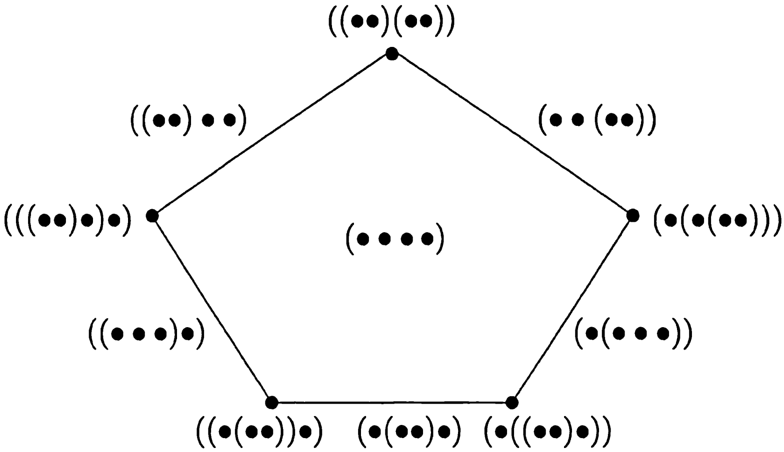
In this subsection, we shall write down explicit formulas for the generalized associahedron as the next multilinear operation of $\mathcal{A}_\infty$-algebra. Equation (15) implies, for $m = 4$ and $s_1 \in \mathcal{O}(H_k(W_1)), s_2 \in \mathcal{O}(H_l(W_2)), s_3 \in \mathcal{O}(H_m(W_3)), s_3 \in \mathcal{O}(H_n(W_4))$, the following coherence condition:

$$\begin{aligned}&0 = \\ &\mu_1(\mu_4(s_1, s_2, s_3, s_4)) - \mu_2(\mu_3(s_1, s_2, s_3), s_4) \\ &\quad - (-)^k\mu_2(s_1, \mu_3(s_2, s_3, s_4)) + \mu_3(\mu_2(s_1, s_2), s_3, s_4) - \mu_3(s_1, \mu_2(s_2, s_3), s_4) \\ (36)\quad &\quad + \mu_3(s_1, s_2, \mu_2(s_3, s_4)) - \{\mu_4(\mu_1(s_1), s_2, s_3, s_4) + (-)^k\mu_4(s_1, \mu_1(s_2), s_3, s_4) \\ &\quad + (-)^{k+l}\mu_4(s_1, s_2, \mu_1(s_3), s_4) + (-)^{k+l+m}\mu_4(s_1, s_2, s_3, \mu_1(s_4))\} \\ &\quad + \mu_5(\mu_0(1), s_1, s_2, s_3, s_4) - \mu_5(s_1, \mu_0(1), s_2, s_3, s_4) + \mu_5(s_1, s_2, \mu_0(1), s_3, s_4) \\ &\quad - \mu_5(s_1, s_2, s_3, \mu_0(1), s_4) + \mu_5(s_1, s_2, s_3, s_4, \mu_0(1))\,,\end{aligned}$$

and (or) in terms of invariant differential BGG operators, (36) amounts to

$$\begin{aligned}&\mathcal{D}_{k+l+m+n-1}\langle s_1, s_2, s_3, s_4\rangle = \langle s_1, s_2, s_3\rangle \cup s_4 + (-)^k s_1 \cup \langle s_2, s_3, s_4\rangle - \\ &\qquad \langle s_1 \cup s_2, s_3, s_4\rangle + \langle s_1, s_2 \cup s_3, s_4\rangle - \langle s_1, s_2, s_3 \cup s_4\rangle + \\ (37)\quad &\qquad \langle \mathcal{D}_k s_1, s_2, s_3, s_4\rangle + (-)^k\langle s_1, \mathcal{D}_l s_2, s_3, s_4\rangle + \\ &\qquad (-)^{k+l}\langle s_1, s_2, \mathcal{D}_m s_3, s_4\rangle + (-)^{k+l+m}\langle s_1, s_2, s_3, \mathcal{D}_n s_4\rangle - \\ &\qquad \langle \mathcal{K}, s_1, s_2, s_3, s_4\rangle + \langle s_1, \mathcal{K}, s_2, s_3, s_4\rangle - \langle s_1, s_2, \mathcal{K}, s_3, s_4\rangle \\ &\qquad + \langle s_1, s_2, s_3, \mathcal{K}, s_4\rangle - \langle s_1, s_2, s_3, s_4, \mathcal{K}\rangle\,.\end{aligned}$$

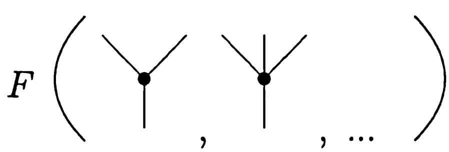
In the conventional graphical form are equations (36), (37) represented by picture (where we use parenthesis instead of trees):



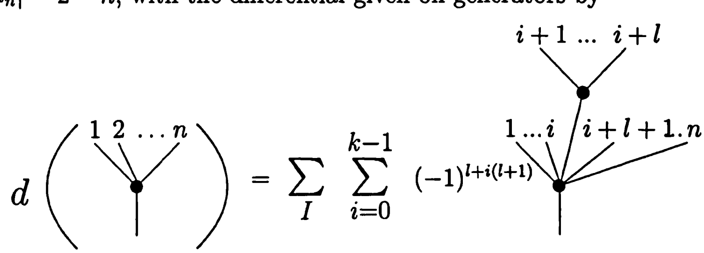
3. Generalized graded $\mathcal{A}_\infty$-operad

The structure of the $\mathcal{A}_\infty$-algebra on the set of pieces of BGG sequences, studied in the previous sections, is an example of generalized strongly homotopy algebra, i.e. a standard $\mathcal{A}_\infty$-algebra equipped with a strong unit $\mu_0 : 1 \to \mathcal{A}$. In this section we are going to define the operad over which this algebra lives.

In the standard situation, [4], the differential graded operad $\mathcal{A}_\infty$ for strongly homotopy $\mathcal{A}_\infty$-algebras is the free operad living in the category of differential graded vector spaces:

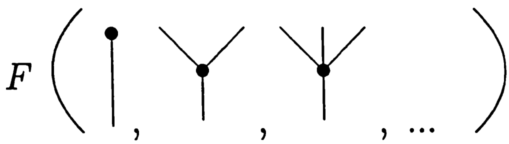


where the first term represents μ_2, the second μ_3 etc., the grading of n-ary operation is $deg(\mu_n) = |\mu_n| = 2 - n$, with the differential given on generators by

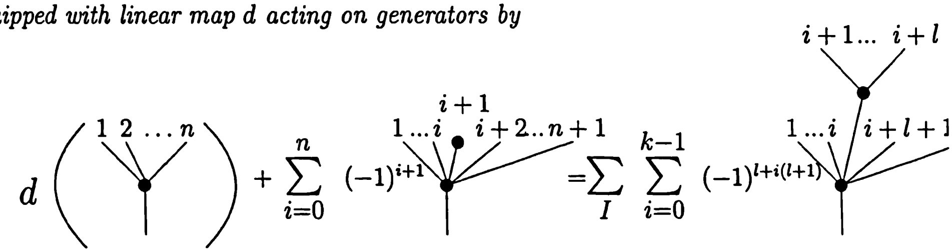


and extended by derivation property. In the previous summation, we have used the notation $I := \{k + l = n + 1 \mid k = 2, \dots n - 1,\ l = 2, 3, \dots n - 1\}$, and in what follows we shall use for these relations the notation $\{R_n\}_{n \in \mathbb{N}_0}$.
Let us consider the case of generalized $\mathcal{A}_\infty$-algebra, which (probably for the first time) appeared in [2], and corresponding generalized $\mathcal{A}_\infty$-operad:

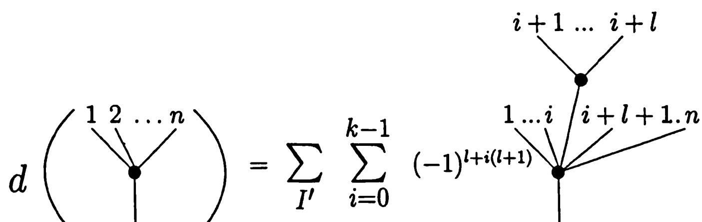
Definition 3.1. *The generalized $\mathcal{A}_\infty$-operad is the free graded (non-differential) operad*



equipped with linear map d acting on generators by



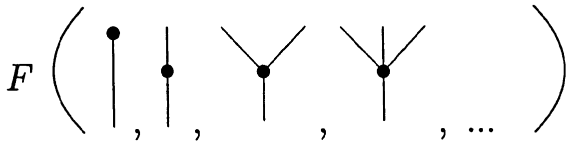
where $I := \{k + l = n + 1 \mid k = 2, \dots n - 1,\ l = 2, 3, \dots n - 1\}$, or in a more compact form



where $I' := \{k + l = n + 1 \mid k = 2, \dots n - 2, n - 1, n + 1,\ l = 0, 2, 3, \dots n - 1\}$ and the linear map d is extended by derivation property.

Note that this operad lives in the monoidal category of graded vector spaces with a distinguished (and not necessarily nilpotent) endomorphism.

Remark 3.2. *The previous definition, based on the preferred choice of the linear map d, can be rewritten in the following way. The generalized $\mathcal{A}_\infty$-operad is the quotient space of the free graded operad*

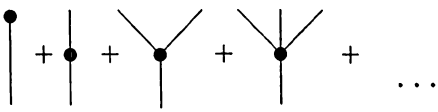


by the ideal generated by $\{R_n\}_{n\in\mathbb{N}_0}$.

Note that this definition is more standard (classical) in the sense that the generalized $\mathcal{A}_\infty$*-operad lives in the monoidal category of graded vector spaces.*

Note that we have tacitly omitted in the previous definition the adjective differential. The reason for that is $d^2 \neq 0$, i.e. the linear map d is not a differential in $Hom(\mathcal{A}^{\otimes k}, \mathcal{A})$. Consequently, we do not get (on the tree level) the (graph) complex but instead only the (graph) sequence.

Remark 3.3. *The results of the previous section concerning the codifferential property of the map* $\mu^c : (\otimes\mathcal{A})^c \longrightarrow (\otimes\mathcal{A})^c$:

$\mu^c =$ 

make it possible to compute the homology of μ^c *(recall that* $(\mu^c)^2 = 0$*), i.e.* $H^\star((\otimes\mathcal{A})^c, \mu^c)$.

Regrettably we did not find in the literature any explicitly computed example of this homology of the generalized $\mathcal{A}_\infty$-algebra.

Example 3.4. *In the case of* $n = 3$*, we have*

$$\begin{aligned}[d, \mu_3](-,-,-) + \sum_{i=1\,(i\to\mathcal{K})}^{4} (-)^i \mu_4(-,\dots,i,\dots,-) \\ = \mu_2(\mu_2(-,-),-) - \mu_2(-,\mu_2(-,-))\end{aligned} \tag{38}$$

with $\mu_2 = \langle -,- \rangle_{\mathfrak{g}} = \cup_{\mathfrak{g}}, \mu_3 = \langle -,-,- \rangle_{\mathfrak{g}}, \mu_4 = \langle -,-,-,- \rangle_{\mathfrak{g}}$ *and BGG operators* $d = \mathcal{D}$. *Taking into account the gradings of evaluation elements* $s_1, s_2, s_3 \in \mathcal{A}$, *we reproduce immediately equation* (12).

Remark 3.5. (Corrections) *In the sources cited in the article there are some missprints.*

In the reference [5] *there is mistake on p.* 2 *in the last equation concerning the sign* ϵ. *The summation over* $|a_i|$ *should end by* $|a_s|$.

In the reference [6] *there is mistake on p.* 4, *equation* (3). *In the first summation there should be* $k + l = n$ *instead of* $k + l = n + 1$.

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