Ulf Lindström

Supersymmetry, a biased review

In: Jarolím Bureš (ed.): Proceedings of the 22nd Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2003. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 71. pp. [41]--66.

Persistent URL: http://dml.cz/dmlcz/701705

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 71 (2003), pp. 41-66

# SUPERSYMMETRY, A BIASED REVIEW

U. LINDSTRÖM

ABSTRACT. This set of lectures contain a brief review of some basic supersymmetry and its representations, with emphasis on superspace and superfields. Starting from the Poincaré group, the supersymmetric extensions allowed by the Coleman-Mandula theorem and its generalisation to superalgebras, the Haag, Lopuszanski and Sohnius theorem, are discussed. Minkowski space is introduced as a quotient space and Superspace is presented as a direct generalization of this. The focus is then shifted from a general presentation to the relation between supersymmetry and complex geometry as manifested in the possible target space geometries for $N = 1$ and $N = 2$ supersymmetric nonlinear sigma models in four dimensions. Gauging of isometries in nonlinear sigma models is discussed for these cases, and the quotient construction is described.

## CONTENTS



---

Invited Lectures at "The 22nd Winter School GEOMETRY AND PHYSICS", Srní, Czech Republic, January 12–19, 2002.
The paper is in final form and no version of it will be submitted elsewhere.

## 1. Introduction

In these lectures I try to give a physicists picture of (some aspects of) supersymmetry and its representations. Since the majority of the audience at the meeting were mathematicians, I presented a lot of background that is normally taken for granted. In taking this course, the choice of what to include and what to leave out becomes even more difficult than is usually the case. Ideally, had I been able to request a similar contribution from a mathematician I myself would have wanted a translation table of the kind "When they say... they mean...", but the present article is no such thing. In the end, in the written version, I have included additional explanations wehere I feel that a few words may clarify the presentation substantially for a newcomer to the subject. One problem is, of course, that I do not know what points may create difficulties. I do not want to make too much of the "clash of cultures", but it is abundantly clear that, e.g., the use of indices creates one communication problem. Rather than modifying the presentation to conform with an index-free notation, however, I have kept to the physicists notation in a hope that a reader may use the text to understand how indices are used to keep track of transformation properties in the physics literature.

I must stress that the title I have given my contribution is correct; I have only included the material I thought I needed to get to the relation between supersymmetry and complex geometry as soon as possible without totally sacrifying the general picture. This is also reflected in my list of references, which is sadly inadequate. However, I believe it includes enough standard texts ([1], [2], [3], [4]) that the reader may find his way to all the basic sources through them. A couple of the general references are further particularly suited for a mathematical audience, namely [5] and [6].

The lectures are divided into two parts, introductory material (Sections 1, 2 and 3) and the relation between supersymmetry and complex geometry (Section 4). For the first part I draw from numerous sources, consiously and subconsiouly. For the second part my main material is the three articles [7], [8] and [9].

## 2. Relativistic symmetries

References for this section are the text books and articles referred to in the introduction along with any good book on quantum field theory such as [10] or [11]. Also the (old) review articles [12] and [13] may provide useful background. If one is more generally interested in graded algebras and their representations samples of the possible references are [14], [15], [16], [17], [18], [19], [20], [21].

### 2.1. The Poincaré algebra.

In theoretical high-energy physics we study the motion of particles, strings and branes in various ambient space times. This means that we are interested in manifolds with a range of dimensions, from zero spacelike and one timelike (the particle) to 25 spacelike and one timelike (the target space[1] of the bosonic string). In all these dimensions we mainly focus on relativistically invariant models, however. Thus the fundamental structure is given by the (tangent space) group $ISO(D-1,1)$, the $D$-dimensional Poincaré group. The generators of its Lie-algebra $\mathfrak{iso}(D-1,1)$ satisfy the following algebra

$$
\begin{aligned}
&[P_a, P_b] = 0 \\
&[M_{ab}, P_c] = \frac{i}{2}\eta_{c[a}P_{b]} \\
&[M_{ab}, M_{cd}] = \frac{i}{2}\eta_{c[a}M_{b]d} - c \leftrightarrow d,
\end{aligned}
\tag{1}
$$

where[2] $P_a$ generate translations, $M_{ab}$ generate Lorentz transformations, and $\eta$ is the Minkowski metric whose relation to the spacetime metric $g$ is

$$
ds^2 = g_{mn}dX^m dX^n = \eta_{ab}e^a_m e^b_n dX^m dX^n, \tag{2}
$$

where the line-element $ds^2$ is expressed in using the coordinates $X^m$ in the space-time. The one forms $e^a = e^a_m dX^m$ are often called "viel-beins" in the physics literature. As a further note on notation, "curved" (space-time) indices are taken from the middle of the alphabet, tangent space indices are from the beginning of the alphabet and the summation convention is used (repeated indices are summed over).

The algebra (1) shows that $P_a$ transforms in the fundamental representation of the Lorentz group (LG), i.e., as a vector, and that $M_{ab}$ itself transforms as an (antisymmetric) second rank tensor.

One is typically interested only in the proper Lorentz group, $SO(D-1,1)^\uparrow$, given by matrices $\Lambda : \Lambda^T\eta\Lambda = \eta$, $det\Lambda = +1$, $\Lambda^0_0 \geq 0$. This semi-simple Lie-group is not simply connected. Its universal covering group is $Spin(D-1,1)$. An element in the fundamental representation of this group is called a spinor and we will denote it by $\Psi_\alpha \in T_S$. All (finite dimensional) representations of the LG may be obtained from tensor products $T_S \otimes T_S \otimes T_S \ldots$, a useful fact that, e.g., later allows us to use pairs of spinor indices to represent vector indices.

The representations of the LG fall into two distinct classes, those with integer spin, the bosons, and those with half integer spin, the fermions. To be more precise, the names refer to the elementary particles that transform in the corresponding representations. Fermions are the constituents of matter and bosons govern the forces in nature. The obey different statistics; many bosons can occupy the same state (cf. Bose-Einstein condensate) while only one fermion can be in a particular state in the Hilbert space (the Pauli exclusion principle).

2.2. **Minkowski space $\mathcal{M}$.** A useful way of representing the Poincaré group is in terms of fields[3] over Minkowski space $\mathcal{M}$, with the generators of the algebra (1) described by differential operators acting on these fields. The Minkowski space itself

[1] Target space and ambient space-time is synonymous in this text.

[2] We use the bracket notation to indicate symmetry or skewness, i.e. (ab) denotes symmetrization and [ab] denotes antisymmetrization, with no combinatoical factors.

[3] I am using the word "field" in the standard physicist way, meaning (usually $C^\infty$) functions.

may be thought of as the quotient of the Poincaré group with the Lorentz group

$$ISO(D-1,1)/SO(D-1,1). \tag{3}$$

Since this is analogous to the way in which Superspace is defined in subsection 3.1, it pays to look at the construction in some detail in this simpler context.

A point in $\mathcal{M}$ is parametrized as

$$h(x) = e^{ix^a P_a}\mathbf{1}, \tag{4}$$

and the group acts by left multiplication

$$h(gx) = h(x') \equiv gh(x) mod SO(D-1,1). \tag{5}$$

For a translation $g = e^{i\xi^a P_a}$, with parameter $\xi$, this yields

$$\begin{aligned} gh &= e^{i\xi^a P_a} e^{ix^a P_a} = e^{i(x^a+\xi^a)P_a + \frac{i}{2}[\xi P, xP] + \ldots} \\ &= e^{i(x^a+\xi^a)P_a} \\ &\equiv h(x'), \end{aligned} \tag{6}$$

where the Baker-Campbell-Hausdorff (BCH) formula becomes trivial since translations commute, as seen in (1). A translation thus induces the following coordinate change:

$$x'^a = x^a + \xi^a, \quad \Rightarrow \delta x^a = \xi^a, \tag{7}$$

where the last relation gives the infinitesimal transformation. The corresponding calculation for a Lorentz transformation $g = e^{i\omega^{ab}M_{ab}}$ with parameter $\omega^{ab}$ is less trivial since the generators $P$ and $M$ do not commute. The BCH formula thus contributes nontrivially. One also has to make use of the quotient structure when calculating

$$\begin{aligned} h(x') &= gh = e^{i\omega^{ab}M_{ab}} e^{ix^c P_c} \\ &= e^{i\omega^{ab}M_{ab}} e^{ix^c P_c} e^{-i\omega^{ab}M_{ab}} \\ &= e^{x^c(e^{i\omega M} P_c e^{-i\omega M})} \\ &= e^{x^c (e^{\omega})_c^a P_a}, \end{aligned} \tag{8}$$

where all operations are performed $modSO(D-1,1)$. The induced action of on the coordinate is thus

$$x'^a = x^b (e^{\omega})_b^a \quad \Rightarrow \delta x^a = x^b \omega_b^a\ . \tag{9}$$

2.3. **Fields over $\mathcal{M}$.** Now that we know howe the Poincaré transformations act on the Minkowski coordinate $x$, we find representations in terms of (scalar) fields $f$ by requiring that they transform as

$$f'(x') = f(x). \tag{10}$$

Under an infinitesimal transformation $x \to x + \delta x$, the fields thus obey

$$\delta f(x) \equiv f'(x) - f(x) = -\delta x^a \partial_a f(x), \tag{11}$$

where $\partial_a \equiv \partial / \partial x^a$, and (11) defines what we are to mean by the infinitesimal variation of a field. Inserting the infinitesimal coordinate transformations in (7) and (9), we find the action of a translation or a Lorentz transformation on a scalar field. We emphasized "scalar" to indicate the alternative possibility that $f$ also transforms in some matrix representation of the Lorentz group. E.g., it may transform as a Lorentz vector, which we indicate by a vector index

(12) $$\delta_\omega f_a = [\omega \cdot M, f_a] = \frac{i}{2}\omega^{bc}\eta_{a[b}f_{c]},$$

(cf. the transformation of $P_a$ in (1)). If it transforms as a spinor instead, we endow $f$ with a spinor index, and a Lorentz transformation reads

(13) $$\delta_\omega f_\alpha = [\omega \cdot M, f_\alpha] = \frac{i}{2}\omega^{bc}(\Gamma_{bc})_\alpha{}^\beta f_\beta,$$

where the Dirac algebra is

(14) $$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \cdot \mathbf{1},$$

and

(15) $$\Gamma_{ab} \equiv \frac{1}{2}\Gamma_{[a}\Gamma_{b]}.$$

Combining (7), (9) and (11) with the possibility of a matrix representation and defining

(16) $$\delta f = i[\omega \cdot M + \xi \cdot P, f],$$

we see that we may represent the generators of $ISO(D-1,1)$ as operators on the fields $f$ as follows:

(17) $$\begin{aligned} P_a &\rightarrow i\partial_a \\ M_{ab} &\rightarrow \frac{i}{2}x_{[a}\partial_{b]} - i\mathbb{M}, \end{aligned}$$

where $\mathbb{M}$ is the appropriate matrix representation.

2.4. **Internal symmetries.** In addition to the transformation properties under the LG described in the previous subsection, the fields may also transform in some representation of an internal symmetry group $\mathcal{G}$. We indicate this by an additional index $i$ on the fields. Thus, e.g., $f^i_\alpha(x)$ is a spinor field which transforms in some matrix representation of $\mathcal{G}$,

(18) $$\delta_\lambda f^i_\alpha = \lambda^I (B_I)^i_j f^j_\alpha,$$

where $\lambda$ is a transformation parameter which is taken to depend on $x \in \mathcal{M}$ for gauge symmetry. Popular internal (gauge) symmetry groups are $\mathcal{G} = U(1)$ (electro magnetism), $\mathcal{G} = SU(2)$ (weak interactions) and $\mathcal{G} = SU(3)$ (strong interactions).

It is of course tempting, in the name of unification, to try to find a larger group which encompasses both the Poincaré group and the internal symmetry group in a non-tivial way. All such attepts came to an halt in the late 1960's due to the famous "No-Go" theorem of Coleman and Mandula (CM) [22], where the requirements of a relativistic quantum theory are used to limit the possibilities. In brief (leaving out some technicalities) it states that if

(1) the S-matrix is based on a local relativistic field theory in space time,
(2) there are only a finite number of different particles associated with one particle states at a given mass,
(3) there is an energy gap between the vacuum and the one-particle states,

then:

> The most general Lie-algebra of symmetries of the S-matrix has generators $P_a$, $M_{ab}$ and $B_I$, where the $B_I$'s are Lorentz scalars and belong to a compact Lie-group $\mathcal{G}$.

The setting for this theorem is really $D = 4$, so the conclusion is that the group structure has to be $SO(3,1) \otimes \mathcal{G}$.

2.5. **Supersymmetry.** In the 1970's, with the advent of supersymmetry, it was realized that there is a loop-hole in the CM theorem, and it was extended by Haag, Lopuszanski and Sohnius to allow for $Z_2$ graded Lie-algebras [23]. The result may be most simply stated by giving the most general super-algebra allowed (in $D = 4$). In addition to the Poincaré algebra (1), we also have

(19)
$$\begin{aligned}
&[M_{ab}, B_I] = 0\\
&[P_a, B_I] = 0\\
&[B_I, B_J] = ic_{IJ}^{\ K} B_K\\
&\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}(\Gamma^a C)_{\alpha\beta}P_a + C_{\alpha\beta}Z^{ij}(\Gamma_5 C)_{\alpha\beta}Y^{ij}\\
&[M_{ab}, Q^i_\alpha] = \frac{1}{2}(\Gamma_{ab})_\alpha^{\ \beta} Q^i_\beta\\
&[B_I, Q^i_\alpha] = (B_I)_i^{\ j} Q^i_\alpha\\
&[P_a, Q^i_\alpha] = 0\\
&[\mathcal{O}, Z] = [\mathcal{O}, Y] = 0,
\end{aligned}$$

where the three first realations say that the generators of the group [4] $\mathcal{G} = O(\mathcal{N})$ are Poincaré scalars, in accordance with the CM theorem. The radically new structure is carried by the $\mathcal{N}$ odd generators $Q$. As seen from (19) they are translationally invariant Lorentz spinors that carry a non-trivial representation of the internal group $\mathcal{G}$. They come together with the anticommutator $\{\ ,\ \}$, under which they close to a translation plus terms that depend on the central charges $Z$ and $Y$. (The last relation in (19) is meant to indicate that $Z$ and $Y$ commute with all generators.) The central charges are antisymmetric in their $\mathcal{G}$ indices, and C, finally, is the charge conjugation matrix.

The spinors in (19) are Majorana spinors, i.e., they obey the "reality condition" $Q = C\bar{Q}^T$. In general, in $D$ dimensions the spinors have $2^{[\frac{D}{2}]}$ complex components (where square bracket denotes 'integer part of'). Depending on the dimension $D$, one may impose "reality" and/or chirality conditions on the spinors according to the following table (adapted from [24]) [5]

---

[4]Note that the internal group has to be $O(\mathcal{N})$ or a subgroup thereof. $c_{IJ}^{\ K}$ are its structure constants.

[5]The table refers to space-time signature (-++...). In (-++...+-), which is sometimes considered, there are other possibilities. Note also that we are discussing conditions over and above the Dirac equation which is always assumed for the spinor fields.

| D | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| Spinor type | M | MW | M | W | D | W | D | W | M | MW |
| Real spinor dim | 32 | 16 | 16 | 16 | 16 | 8 | 8 | 4 | 2 | 1 |
| Real/Complex | R | R | R | C | C | C | C | C | R | R |
| N | 1 | 2 | 2 | 2 | 2 | 4 | 4 | 8 | 16 | (16,16) |
| | | 1 | 1 | 1 | 1 | 2 | 2 | 4 | 8 | (8,8) |
| | | | | | | 1 | 1 | 2 | 4 | (4,4) |
| | | | | | | | | 1 | 2 | (2,2) |
| | | | | | | | | | 1 | (1,1) |

Table 1

In this table, $M$ denotes Majorana, $D$ denotes Dirac and $W$ denotes Weyl conditions. The Majorana condition was given above, Dirac just means a Dirac spinor, i.e., no additional constaraints, and the Weyl condition in even $D$ dimensions is

$$\psi = P_{-}\psi = \frac{1}{2}(1 - \Gamma_{D+1})\psi, \tag{20}$$

where $\psi$ is called a Weyl spinor and $\Gamma_{D+1}$ is the totally antisymmetrized product of the $D$ Dirac-matrices (suitably normalized to make $P_{-}$ a projection operator). In four dimensions one may impose either the Majorana or the Weyl condition, as we will see below. In two dimensions, finally, by $(p, q)$ we have indicated the possibility of having separate right and left moving supersymmetries. [6]

The reader may ask what restricts the entries in Table 1 to $D \leq 11$ and/or $N \leq 16$. The reason is as follows. We know what equations various spins should obey, and we also know the spin content of the irreducible representations of spersymmetry (see subsection 3.3). For $N = 8$ in $D = 4$ the spin content is $(2, 3/2, 1, 1/2, 0)$, whereas higher $N$ will necessarily contain spin $\geq 2$. But "higher spin" field theories (with a finite number of higher spin fields) are in general unphysical as interacting theories. Since $N = 8$ in $D = 4$ is $N = 1$ in $D = 11$, this sets the limit.

2.6. $D = 4$ **Supersymmetry.** In $D = 4$ we illustrate explicitly the equivalence between Majorana and Weyl spinors as well as how to build the tensors from spinor representations.

In $D = 4$ the Weyl projection operators are given by $P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5)$. We utilize the isomorphism $Spin(3,1) \approx SL(2,\mathbb{C})$ to introduce a convenient notation for the Weyl spinors. Let $\Psi_\alpha$ now be a spinor that transforms with the $SL(2,\mathbb{C})$ matrix $N_\alpha^\beta$, and denote by $\bar{\Psi}^{\dot{\alpha}}$ a spinor that transforms in the conjugate representation according to $\bar{N}_{\dot{\alpha}}^{\dot{\beta}}$. Introducing also the two sets of $SL(2,\mathbb{C})$ matrices $\sigma_a = (1, \underline{\sigma})$ and $\tilde{\sigma}_a = (1, -\underline{\sigma})$ with $\underline{\sigma}$ being the Pauli matrices, we use a representation of the Dirac algebra where

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix}. \tag{21}$$

The relations between $\sigma$ and $\tilde{\sigma}$ may be stated as

$$(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma_a)_{\beta\dot{\beta}}, \tag{22}$$

[6] One may in fact introduce this possibility also, e.g., in $D = 6$ and 10.

where

$$(\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (\epsilon_{\dot\alpha\dot\beta}) = -(\epsilon^{\alpha\beta}) = -(\epsilon_{\dot\alpha\dot\beta}). \tag{23}$$

In this representation

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \tag{24}$$

and a Majorana spinor may be written

$$\Psi = \begin{pmatrix} \Psi_\alpha \\ \bar\Psi^{\dot\alpha} \end{pmatrix}. \tag{25}$$

It is a straightforward matter to convince one-self that (25) indeed satisfies $C\bar\Psi^T = \Psi$, and that

$$P_+\Psi = \begin{pmatrix} \Psi_\alpha \\ 0 \end{pmatrix} \tag{26}$$

$$P_-\Psi = \begin{pmatrix} 0 \\ \bar\Psi^{\dot\alpha} \end{pmatrix}, \tag{27}$$

thus explicitly demonstrating the equivalence between the Majorana and Weyl representations in four dimensions. In fact, in four dimensions the latter representation is often preferred. In that context it is also convenient to represent vector indices as pairs of spinor indices according to

$$V_a \to (\sigma^a)_{\alpha\dot\alpha} V_a \equiv V_{\alpha\dot\alpha}. \tag{28}$$

This notation becomes particularly useful when discussing representations of susy in superspace.

## 3. Superspace

The main references for this section are the books [1], [2], [3] and [4], but also the review article [24].

3.1. **Induced representation.** Superspace is defined via the natural generalization of the Minkowski-space construction described in Section 2.2 above. Denoting the graded ($N = 1$) Poincaré group by SISO(D-1,1) and specifying to $D = 4$, the relevant part of the superalgebra reads

$$\begin{aligned} \{Q_\alpha, \bar Q_{\dot\alpha}\} &= P_{\alpha\dot\alpha} \\ [M_{\alpha\beta}, Q_\gamma] &= \frac{i}{2}\epsilon_{\gamma(\alpha}Q_{\beta)} \\ [M_{\alpha\beta}, P_{\gamma\dot\gamma}] &= \frac{i}{2}\epsilon_{\gamma(\alpha}P_{\beta)\dot\gamma} \\ [M_{\alpha\beta}, M_{\gamma\delta}] &= \frac{i}{2}(\epsilon_{\gamma(\alpha}M_{\beta)\delta} + \gamma \leftrightarrow \delta), \end{aligned} \tag{29}$$

where the antisymmetric generator of Lorentz transformations, $M_{ab}$, is represented by its irreducible (symmetric) spinor parts according to

$$M_{ab} \approx M_{\alpha\dot\alpha\beta\dot\beta} = i\epsilon_{\alpha\beta}\bar M_{\dot\alpha\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}. \tag{30}$$

From the algebra we exponentiate to get the group elements. This requires introduction of Grassmann valued (anti commuting) spinor parameters. [7] A general group element is thus written

$$g = e^{i(\xi\cdot P+\epsilon\cdot Q+\omega\cdot M)}, \tag{31}$$

where we use the short hand $\epsilon\cdot Q$ for $\epsilon^\alpha Q_\alpha + \bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha}$, and $\omega\cdot M$ for $\omega^{\alpha\beta}M_{\alpha\beta}+\bar\omega^{\dot\alpha\dot\beta}\bar M_{\dot\alpha\dot\beta}$. In analogy with our discussion of Minkowski space $\mathcal{M}$, we parametrize a point in the neighbourhood of the identity in $SISO(3,1)/SO(3,1)$ by $x$ and $\theta$ according to

$$h(x,\theta)\equiv h(Z^A) = e^{i(x\cdot P+\theta\cdot Q)}, \tag{32}$$

(where $Z^A = x^a, \theta^\alpha, \bar\theta^{\dot\alpha}$), and find the action on $x$ and $\theta$ through $h(Z') = g\cdot h$ mod SO(3,1). We then represent the generators as differential operators on superfields, i.e., on funtions $\phi(z)$. We first state the result and then supply the necessary explanations. The operators are (cf. (17))

$$\begin{aligned} P_{\alpha\dot\alpha} &= i\frac{\partial}{\partial x^{\alpha\dot\alpha}} \equiv i\partial_{\alpha\dot\alpha} \\ Q_\alpha &= i\frac{\partial}{\partial\theta^\alpha}+\frac{1}{2}\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \equiv i\partial_\alpha + \frac{1}{2}\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \\ \bar Q_{\dot\alpha} &= i\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+\frac{1}{2}\theta^{\alpha}\partial_{\alpha\dot\alpha} \equiv i\bar\partial_{\dot\alpha} + \frac{1}{2}\theta^{\alpha}\partial_{\alpha\dot\alpha} \\ M_{\alpha\beta} &= \frac{i}{2}x_{(\alpha}^{\ \dot\gamma}\partial_{\beta)\dot\gamma} + i\theta_{(\alpha}\partial_{\beta)} - i\mathbb{M}_{\alpha\beta}. \end{aligned} \tag{33}$$

Here the $\theta$'s, like all the spinors, are anticommuting, which means that derivatives with respect to $\theta$ is defined as Berezin integrals/derivatives [25]. We take the derivatives to act from the left according to $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$, and the corresponding integration is $\int d\theta^\alpha\theta^\beta = \delta_\alpha^\beta$. There is a wealth of results on the geometry of superspace, but we shall only need a few items.

The covariant derivatives are differential operators $D$ that anti-commute with the supersymmetry generators $\{D,Q\}=0$. They are

$$\begin{aligned} D_\alpha &= \partial_\alpha + \frac{i}{2}\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \\ \bar D_\alpha &= \bar\partial_{\dot\alpha} + \frac{i}{2}\theta^{\alpha}\partial_{\alpha\dot\alpha}, \end{aligned} \tag{34}$$

and their existence might have been anticipated from the fact that left and right multiplication commutes and that the $Q$'s were defined using left multiplication. From the point of view of superspace geometry their (anti) commutation relation

$$\{D_\alpha, \bar D_{\dot\alpha}\} = i\partial_{\alpha\dot\alpha}, \tag{35}$$

signals that even "flat" superspace has torsion. The usefulness of the $D$'s lies in the fact that they anticommute with the $Q$'s, since this allow us to impose invariant conditions on the super fields.

[7] This exponentiation with parameters that are nilpotent is not mathematically well defined. For this reason, mathematicians prefer to extend the functions on the previously discussed quotient by allowing them to depend on Grassmann parameters instead. Operationally, I believe the net result amounts to the same thing. See, e.g., [6] for a stringent definintion of Superspace from this point of view.

3.2. **Superfields.** The differential operators (33) allow us to represent supersymmetry on fields oveer superspace $\mathcal{M}^{(p,q)}$ as we represent the Poincaré group on fields over $\mathcal{M}$ (here $(p,q)$ denotes $p$ bosonic and $q$ spinorial coordinates). For example, using the explicit form of the $Q$'s given in (33) we evaluate the anticommutator of two $Q$'s acting on a scalar superfield $\phi(z)$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}\phi(z) = i\partial_{\alpha\bar{\alpha}}\phi(z). \tag{36}$$

From the point of view of functions on Minkowski space, a superfield is a collection of ordinary fields over $\mathcal{M}$. This is seen if we make a formal Taylor expansion in the Grassmann coordinates $\theta$ of, e.g., a real (so called vector) superfield

$$\begin{aligned}\phi(z) \;=\; & C(x) + \theta^\alpha\chi_\alpha(x) + \bar{\theta}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}(x) - \theta^2 M(x) - \bar{\theta}^2\bar{M}(x) + \theta^\alpha\bar{\theta}^{\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) \\ & -\bar{\theta}^2\theta^\alpha\lambda_\alpha(x) - \theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_\alpha(x) + \theta^2\bar{\theta}^2 B(x).\end{aligned} \tag{37}$$

Although the Taylor series quickly terminates in this case ($N = 1$ in four dimensions), it is more economical to define the component fields using the covariant derivatives (34). With | denoting "the $\theta$ independent part of", the above component fields are

$$\begin{aligned} &\phi(z)| = C(x) \qquad D_\alpha\phi(z)| = \chi_\alpha(x) \qquad \bar{D}_{\dot{\alpha}}(z)| = \bar{\chi}_{\dot{\alpha}}(x) \\ &[D_\alpha, \bar{D}_{\dot{\alpha}}]\phi(z)| = A_{\alpha\dot{\alpha}}(x) \qquad D^2\phi(z)| = M(x) \qquad \bar{D}^2\phi(z)| = \bar{M}(x) \\ &-\bar{D}^2 D_\alpha\phi(z)| = \lambda(x) \qquad D^2\bar{D}_{\dot{\alpha}}\phi(z)| = \bar{\lambda}(x) \qquad D^2\bar{D}^2\phi(z)| = B(x), \end{aligned} \tag{38}$$

where $D^2 \equiv D^\alpha D_\alpha$. The "supermultiplet" of $\mathcal{M}$ fields represented by $\phi(z)$ and transforming into each other under supersymmetry transformations is thus a collection of scalar, spinor and vector fields [8]

$$(C, \chi, \bar{\chi}, A_a, M, \bar{M}, \lambda, \bar{\lambda}, B). \tag{39}$$

In analogy to (16), a supersymmetry transformation of a superfield is

$$\delta\phi(z) = i[\epsilon^\alpha Q_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \phi]. \tag{40}$$

To illustrate the result on the component fields, we first introduce the concept of a <u>chiral</u> superfield, which is simply a (complex) superfield $\Phi$ which satisfies

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{41}$$

As mentioned earlier, this is a covariant condition, i.e., $\bar{D}_{\dot{\alpha}}\delta\Phi = \delta(\bar{D}_{\dot{\alpha}}\Phi)$, as is seen from (40) and $\{Q, \bar{D}\} = 0$. We define the components of $\Phi$ as follows

$$\Phi| = \mathcal{A}(x), \qquad D_\alpha\Phi| = \lambda_\alpha(x), \qquad D^2\Phi| = \mathcal{F}(x). \tag{42}$$

With, correspondingly,

$$\delta\Phi| = \delta\mathcal{A}(x), \qquad D_\alpha\delta\Phi| = \delta\lambda_\alpha(x), \qquad D^2\delta\Phi| = \delta\mathcal{F}(x), \tag{43}$$

we find the component transformations

$$\delta\mathcal{A} = -\epsilon^\alpha\lambda_\alpha, \qquad \delta\lambda_\alpha = \epsilon_\alpha\mathcal{F} - i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\mathcal{A}, \qquad \delta\mathcal{F} = -i\bar{\epsilon}^{\dot{\alpha}}\partial^\alpha_{\dot{\alpha}}\lambda_\alpha. \tag{44}$$

In fact, starting from these transformations, one may show that the algebra clooses on all the fields.

[8]In a physical model, not all of these fields will be dynamical. They have different mass-dimensions and some of the fields will be auxiliary.

**3.3. Representations.** In the previous subsection we saw how supersymmetry may be represented on superfields. In particular, a chiral superfield was seen to be a smaller representation than an arbitrary superfield (it has fewer component fields). The question thus arises of what are the irreducible representations of supersymmetry.

Staying with the superfields in four dimensions, we first give the projection operators that project onto the irreducible superfields. This is in analogy with the way a Lorentz vector $V_a$ is split into irreducible pieces according to

$$V_a = [(\Pi^L + \Pi^T)V]_a, \tag{45}$$

where

$$(\Pi^L)_a^{\ b} \equiv \partial^{-2}\partial_a\partial^b \qquad (\Pi^T)_a^{\ b} \equiv \partial^{-2}\delta_a^{[b}\partial^{c]}\partial_c, \tag{46}$$

(such that $\Pi^L + \Pi^T = 1$). Explicitly

$$(\Pi^L V)_a = \partial^{-2}\partial_a(\partial^b V_b) \equiv \partial^{-2}\partial_a S, \qquad (\Pi^T V)_a = \partial^{-2}\delta_a^b\partial^c\partial_{[c}V_{b]} \equiv \partial^{-2}\partial^c F_{ca}, \tag{47}$$

where $S$ and $F$ are irreducible representations of the Poincaré group. The corresponding operators on superfields are (for $N = 1$ in four dimensions)

$$\begin{aligned} \Pi_0 &= \partial^{-2}D^2\bar{D}^2 \\ \Pi_1 &= -\partial^{-2}D^\alpha\bar{D}^2D_\alpha \\ \Pi_2 &= \partial^{-2}\bar{D}^2D^2, \end{aligned} \tag{48}$$

such that $\Pi_0 + \Pi_1 + \Pi_2 = 1$. This is not quite sufficient, $\Pi_1$ has to be further specified as $\Pi_{1\pm}$, where

$$\Pi_{1\pm}\psi = -\partial^{-2}D^\alpha\bar{D}^2D_\alpha\frac{1}{2}(\psi + \bar{\psi}). \tag{49}$$

Then an arbitrary (complex) scalar superfield $\Phi$ contains a chiral superfield, two vector superfields and an antichiral superfield (in that order) according to

$$\Phi = \Pi_0\Phi + \Pi_{1\pm}\frac{1}{2}(\Phi + \bar{\Phi}) + \Pi_2\Phi, \tag{50}$$

thus displaying the irreducible parts of the superfield.

Another question regarding representations of supersymmetry has to do with the particle content and representations as states in a Hilbert space. We will use Wigners "little group" method to find those.

The $N$-extended supersymmetry algebra in four dimensions involves the anticommutator (see (19))

$$\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}(\Gamma^a C)_{\alpha\beta}P_a. \tag{51}$$

For particles with mass $m \neq 0$ we choose $P_a = (-m, 0, 0, 0)$. Rescaling the charges, $Q \rightarrow \tilde{Q}$, by a factor $(m)^{-\frac{1}{2}}$ we have the Clifford algebra

$$\{\tilde{Q}^i_\alpha, \tilde{Q}^j_\beta\} = \delta_{i\alpha,j\beta}, \qquad i\alpha = 1, \ldots, 4N. \tag{52}$$

Rewriting this in Weyl notation,

$$\begin{aligned} \{\tilde{Q}^i_\alpha, \tilde{Q}^j_\beta\} &= \{\bar{\tilde{Q}}^i_{\dot\alpha}, \bar{\tilde{Q}}^j_{\dot\beta}\} = 0 \\ \{\tilde{Q}^i_\alpha, \bar{\tilde{Q}}^j_{\dot\beta}\} &= \delta_{ij}\delta_{\alpha\dot\beta}, \end{aligned} \tag{53}$$

we recognize a set of $2N$ pairs of annihilation and creation operators, $\tilde{Q}^i_\alpha \equiv a^i_\alpha$ and $\bar{\tilde{Q}}^i_{\dot\alpha} \equiv a^{\dagger i}_{\dot\alpha}$. Introducing the Clifford vaccum $|0\rangle$ such that $a^i_\alpha|0\rangle = 0$, a general state is

$$|n_{11}, n_{12}, \ldots, n_{1N}, n_{21}, \ldots, n_{2N}\rangle = \prod_{\alpha=1,2\ i=1,2,\ldots N} (a^\dagger i_{\dot\alpha})^{n_{\alpha i}}|0\rangle, \tag{54}$$

where $n_{\alpha i}$ denotes the occupation number of the state created by $a^\dagger i_{\dot\alpha}$. There are clearly $2^{2N}$ such states. For example, when $N = 1$, the possibilities are

$$|0\rangle, \qquad a^\dagger{}_{\dot\alpha}|0\rangle, \qquad a^\dagger{}_{\dot\alpha}a^\dagger{}_{\dot\beta}|0\rangle = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}a^\dagger \cdot a^\dagger|0\rangle, \tag{55}$$

representing a Lorentz scalar, a spinor (two states) and a scalar respectively.

The massless representations are similarily derived. Starting from the massless four momentum $P_a = (-P, 0, 0, P)$, we have the anticommutation relation

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2\delta_{ij}P_{\alpha\dot\beta}, \tag{56}$$

where

$$P_{\alpha\dot\beta} = 2P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{57}$$

In a way analogous to the massive case, this leads to a set of annihilation and creation operators, $a^i$ and $a^{\dagger i}$ which step down and up half a unit in helicity and satisfy

$$\{a^i, a^{\dagger i}\} = \delta^{ij}. \tag{58}$$

Again we introduce a Clifford vacuum which is annihilated by $a^i$, and create the states with $a^{\dagger i}$. Since we only have one set of operators for each $i$, instead of two, there are $2^N$ states in the massless representation.

With $j_{MAX} \equiv J$ depending on the helicity of the Clifford vacuum, we find the following table of massless states and their multiplicity for various number of supersymmetries $N$.

| N= Helicity | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| J | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| J-$\frac{1}{2}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| J-1 | | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| J-$\frac{3}{2}$ | | | 1 | 4 | 10 | 20 | 35 | 56 |
| J-2 | | | | 1 | 5 | 15 | 35 | 70 |
| J-$\frac{5}{2}$ | | | | | 1 | 6 | 21 | 56 |
| J-3 | | | | | | 1 | 7 | 28 |
| J-$\frac{7}{2}$ | | | | | | | 1 | 8 |
| J-4 | | | | | | | | 1 |

Table 2

The multiplicity is simply the binomial coefficient

$$\binom{N}{k} \tag{59}$$

for the totally antisymmetric product of $k$ creation operators. One observes from Table 2 that for $N = 8$ we need a $J$ of at least 2, for $N = 4$ $J$ has to be at least 1, for the smallest (CPT conjugate) multiplet with helicities $J, \cdots - J$. For other $N$, e.g, for $N = 1$, we cannot choose a $J$ to satisfy CPT conjugation, and for a physical theory we have to add the charge conjugated states.

## 4. Supersymmetry and complex geometry

The complex geometry in this section is from the books [26], [27], and the background material relating to supersymmetry is found in [28], [29], [30]. The main part of the section, however, is from the articles [7], [8] and [9].

4.1. **Notation.** In this subsection we collect some definitions and notation needed later.
For any $d = 2n$ dimensional real manifold $\mathcal{M}$ with coordinates $x^i$, we may locally introduce complex coordinates as

(60) $$z^i = \left\{ \begin{array}{l} z^A = x^i + ix^{i+n} \\ \bar{z}^{\bar{A}} = x^i - ix^{i+n} \end{array} \right\} \qquad i = 1, \ldots, n.$$

A mixed second rank tensor $J^i_j$ such that $J^i_m J^m_j = -1$ is called an almost complex structure on $\mathcal{M}$. A metric $g_{ij}$ which preserves $J^i_j$

(61) $$J^i_j g_{im} J^m_n = g_{jn}, \qquad \Rightarrow J^i_j g_{in} \equiv J_{jn} = -J_{nj},$$

is called an almost hermitean metric. To make everything globally well defined and ensure that there exist canonical complex coordinate patches related by holomorphic transition functions, integrability conditions are needed. They may be phrased as the vanishing of the Nijenhuis tensor

(62) $$N_{ij}{}^k = J^n_{[i}\partial_{|n|}J^k_{j]} + J^k_n \partial_{[j} J^n_{i]} = 0.$$

The integrability conditions remove "almost" from the definitions above. In the canonical coordinates the complex structure takes the form

(63) $$J^i_j = \begin{pmatrix} i\delta^B_A & 0 \\ 0 & -i\delta^{\bar{B}}_{\bar{A}} \end{pmatrix},$$

and the components $g_{AB}$ and $g_{\bar{A}\bar{B}}$ of the hermitean metric vanish.

If we further require the fundamental 2-form

(64) $$\omega \equiv J^i_j g_{ik} dx^j \wedge dx^k = 2ig_{A\bar{A}} dz^A \wedge d\bar{z}^{\bar{A}},$$

to be closed, the manifold is Kähler. In such a manifold, the Levi-Civita covariant derivative of the complex structure vanishes

(65) $$\nabla_i J^k_j = 0,$$

and the metric has a Kähler potential $K$

(66) $$g_{A\bar{A}} = \frac{\partial^2 K}{\partial z^A \partial \bar{z}^{\bar{A}}}.$$

The converse is also true, if $\nabla J = 0$ then the Nijenhuis tensor vanishes and $g = \partial\bar{\partial}K$.

When the manifold carries three covariantly constant complex structures $J^{(X)}$, $X = 1, 2, 3$, and these complex structures satisfy the $SU(2)$ algebra

$$J_i^{(X)j} J_j^{(Y)k} = -\delta^{XY}\delta_i^k + \epsilon^{XYZ} J_i^{(Z)k}, \tag{67}$$

the geometry is hyperkähler.

The various spaces described in this section are also characterized by their holonomy. The holonomy group $\mathcal{H}_p$ at a point $p \in \mathcal{M}$ is the subgroup of the tangent space group obtained by paralell transporting vectors around closed loops in $\mathcal{M}$. The restriction to contractible loops is the restricted holonomy $\mathcal{H}_{p'}$. When $\mathcal{M}$ is simply connected $\mathcal{H}_p \approx \mathcal{H}_{p'}$ and is always a subgroup of $GL(d, \mathbb{R})$. When $\Gamma$ is the Levi-Civita connection, the holonomy group is further a subgroup of $O(d)$, so a subgroup of $O(2n)$ for a complex manifold. For a Kähler manifold it is smaller: E.g., if $\mathcal{M}$ is Ricci-flat the holonomy is $\subset SU(n)$.

4.2. **Nonlinear sigma models.** A link between supersymmetry and complex geometry was first established in the context of supersymmetric non-linear sigma models, (NLSM's), [29], [30], [28]. A sigma model is a map from a manifold $\mathcal{M}$, oftent taken to be space-time, and a Target space $\mathcal{T}$

$$\Phi^A : \mathcal{M} \to \mathcal{T}, \tag{68}$$

mapping the coordinates

$$x^a \in \mathcal{M} \to \mathcal{T} \ni \Phi^A(x). \tag{69}$$

This map is obtained by extremizing the action

$$S = \int dx \; G_{AB}(\Phi) \; \partial_a \Phi^A \partial_b \Phi^B \eta^{ab}, \tag{70}$$

which gives the equation

$$\eta^{ab} \partial_a \Phi^B \nabla_B \partial_b \Phi^A = 0, \tag{71}$$

with

$$\nabla_A V^B = \partial_A V^B + \Gamma_{AC}^{\;A} V^C, \tag{72}$$

the target space covariant derivative. The Levi-Civita connection $\Gamma$ is formed from the target space metric $G_{AB}$,

$$\Gamma_{AC}^{\;A} \equiv \frac{1}{2} G^{CD}(G_{D(A,B)} - G_{AB,C}). \tag{73}$$

To get an indication of how the relation between NLSM's and complex geometry we look at an example:

The complex projective space

$$\mathbb{CP}^{(n)} = U(n+1)/U(n) \times U(1), \tag{74}$$

is Kähler. Now $U(n+1)/U(n) \times U(1)$ may be thought of as the surface

$$u^A \in \mathbb{C}^{n+1}; \quad \sum_{I=1}^{n+1} \bar{u}^{\bar{A}} u^A = 1, \tag{75}$$

in $\mathbb{C}^{n+1}$. The space $\mathbb{CP}^{(n)}$ is thus given by the equivalence class [9]

$$(u^1, \dots, u^{n+1}) \approx e^{i\phi}(u^1, \dots, u^{n+1}), \tag{76}$$

with $u^A$ as in (75). How may we describe this in terms of a NLSM? We want the model to incorporate the structure of the manifold and to also provide us with a metric on that manifold. If we promote the coordinates in (75) to functions from some space $\mathcal{M}$ and take the sigma model action to be

$$S = \int dx \partial_a \bar{u}^{\bar{A}}(x) \partial^a u^A(x); \qquad \sum_{I=1}^{n+1} \bar{u}^{\bar{A}}(x) u^A(x) = 1, \tag{77}$$

we have a start. But we still have to encode the independence of phases (76) at each point. In physics terms this is the question of how to promote the rigid $U(1)$ symmetry to a local one, to gauge a sigma model. It entails introducing a gauge field $A_a(x)$ via minimal coupling

$$S \to S_G = \int dx (\partial_a + iA_a(x)) \bar{u}^{\bar{A}}(x) (\partial^a - iA_a(x)) u^A(x). \tag{78}$$

Next we eliminate $A$ by extremizing $S_G$. This does not break the gauge invariance, it means that we choose a particular $A$ expressed in the other fields. In terms of that particular $A$

$$S_G \to \int dx \left( \partial_a \bar{u}^{\bar{A}} \partial^a u^A + \frac{1}{4} (\bar{u}^{\bar{A}} \overleftrightarrow{\partial}_a u^A)(\bar{u}^{\bar{B}} \overleftrightarrow{\partial}^a u^B) \right). \tag{79}$$

Finally, we rewrite this in coordinates that solve the constraint in (75)

$$\begin{aligned} u^a &= \frac{1}{\sqrt{1 + z \cdot \bar{z}}} z^A, \qquad A = 1, \dots, n \\ u^{n+1} &= \frac{1}{\sqrt{1 + z \cdot \bar{z}}}, \end{aligned} \tag{80}$$

where $z \cdot \bar{z} \equiv z^A \bar{z}^{\bar{A}}, A = 1, \dots, n$. This gives

$$S_G = \int dx \frac{1}{1 + z \cdot \bar{z}} \left( \delta^{AB} - \frac{z^A \bar{z}^{\bar{B}}}{1 + z \cdot \bar{z}} \right) \partial_a \bar{z}^{\bar{A}} \partial^a z^B. \tag{81}$$

We recognize the Fubini-Study metric on $\mathbb{CP}^{(n)}$. The corresponding Kähler potential is $K = ln(1 + z \cdot \bar{z})$.

This example is our first encounter with a quotient construction, which will play an important role later.

[9]There are only phase independence left from the projective requirement due to the constraint in (75).

4.3. **The bosonic quotient construction.** In the example in the previous subsection, we start from the Kähler manifold $\mathbb{C}^{n+1}$ with potential $K = \bar{u}^A u^A$ and construct another, $\mathbb{CP}^{(n)}$ as the space of gauge orbits by "gauging the isometries" and choosing a particular gauge potential that extremizes the action.[10] These ideas generalize.

Suppose we have a NLSM (70) with target space $\mathcal{T}$:

$$S = \int dx\, G_{AB}(\Phi)\ \partial_a\Phi^A\partial_b\Phi^B\eta^{ab}, \tag{82}$$

and an isometry, i.e., a vector field $k(\Phi)$ such that

$$\delta\Phi^A = \lambda^q k_q^B(\Phi) = [\lambda^q k_q^B(\Phi)\partial/\partial\Phi^B, \Phi^A] = \mathcal{L}_{\lambda\cdot k}\Phi^A, \qquad \mathcal{L}_{\lambda\cdot k}G_{AB} = 0, \tag{83}$$

where $\mathcal{L}_{\lambda\cdot k}$ denotes Lie derivative along $k$. Since the variation of the action (82) is

$$\delta S = \int dx\, \left(\partial_a\Phi^A\partial_b\Phi^B\eta^{ab}\mathcal{L}_{\lambda\cdot k}G_{AB}\right), \tag{84}$$

it follows that an isometry is an invariance of the action $S$. In a general situation, the isometry will be non-abelian

$$[k_q, k_p] = c_{qp}^{\ \ r}k_r, \tag{85}$$

corresponding to a non-abelian isometry group $\mathcal{G}$.

We gauge the isometry (83) by substituting

$$\partial_a\Phi^A \to \nabla_a\Phi^A \equiv \partial_a\Phi^A - A_a^q k_q^A = \partial_a\Phi^A - [A_a^q k_q^B\partial/\partial\Phi^B, \Phi^A] \tag{86}$$

(cf. (78)). This results in

$$S \to S_G = \int dx\, G_{AB}(\Phi)\ \nabla_a\Phi^A\nabla_b\Phi^B\eta^{ab}. \tag{87}$$

Proceeding as in the example, we eliminate $A$ by extremising $S_G$, which gives

$$A_a^q = \mathbb{H}^{-1pq}G_{AB}k_p^A\partial_a\Phi^B, \tag{88}$$

where

$$\mathbb{H}_{qp} \equiv k_q^A G_{AB}k_p^B, \tag{89}$$

and

$$S_G = \int dx\, \left(G_{AB} - \mathbb{H}^{-1pq}k_{pA}k_{qB}\right)\partial_a\Phi^A\partial^a\Phi^B \equiv \int dx\tilde{G}_{AB}\partial_a\Phi^A\partial^a\Phi^B. \tag{90}$$

It is easy to see that the new target space metric $\tilde{G}_{AB}$ projects onto the original manifold modulo the $k$-orbits, i.e., it is a metric on the quotient space $\mathcal{T}/\mathcal{G}$.

The quotient construction described above gives a new target space geometry (the coset) starting from one which has isometries. It remains to see under which conditions the two geometries are of the same typer (as in the $\mathbb{C}^{n+1} \to \mathbb{CP}^{(n)}$ example). In particular we shall be interested in when the supersymmetry of the original NLSM is carried over to the quotient. To that end we first need to study supersymmetic NLSM's.

[10] We are glossing over the presense of the constraint in (75) to bring out the essentials.

4.4. $N = 1$ **Supersymmetric nonlinear sigma models.** We want to find a supersymmetric extension of (70), and first need to introduce actions in superspace. Schematically, an action is written

$$S = \int dx d\theta \mathcal{L}(\Phi, D_\alpha \Phi, \dots), \tag{91}$$

where the measure $dxd\theta$ will depend on the kind of superspace under consideration. Explicitly, for $\mathcal{M}^{(p,q)}$ the full superspace measure is $d^p x d^q \theta$. (We will mostly consider $(p,q) = (4,4)$.) As introduced in subsection 3.1, the integral over $\theta$ is equivalent to a derivative, $\int d\theta = \partial/\partial\theta = D_\theta|$, a fact that we will often use. E.g., the invariance of this action under a susy transformation may be shown as follows (recall that $|$ denotes "the $\theta$-independent part of")

$$\begin{aligned} \delta S &= \int dx d\theta \delta\mathcal{L} = \int dx d\theta \epsilon \cdot Q\mathcal{L} = \int dx D^q \epsilon \cdot Q\mathcal{L}| \\ &= \int dx \epsilon \cdot Q D^q \mathcal{L}| = \int dx \epsilon \cdot D D^q \mathcal{L}| \doteq 0, \end{aligned} \tag{92}$$

where the last relation $\doteq 0$ means "$= 0$ up to total derivatives", and follows from the supersymmetry algebra for a product of more than $q$ $D$'s.

Specializing to four dimensions, the superfields $\Phi$ that contain scalar component fields but not vectors are the chiral superfields $\bar{D}_{\dot\alpha}\Phi = D_\alpha \bar{\Phi} = 0$. A general action in superspace for a set $\Phi^A$ of such fields is

$$S = \int d^4x d^2\theta d^2\bar{\theta} K(\Phi^A, \bar{\Phi}^{\bar{A}}), \tag{93}$$

where we use Weyl spinor notation. Keeping the definition (42) in subsection 3.2 in mind, we expand the action, again exploring the relation between $\theta$-derivatives and integrals

$$\begin{aligned} &\int d^4x d^2\theta d^2\bar{\theta} K(\Phi^A, \bar{\Phi}^{\bar{A}}) \\ &= \int d^4x D^2 \bar{D}^2 K| = \int d^4x \left(-2K_{A\bar{B}} \partial A^A \partial \bar{A}^{\bar{A}} + \dots\right) \end{aligned} \tag{94}$$

where indices on $K$ denote derivatives with respect to $\Phi$'s and where only the purely bosonic content is displayed in the last equaility. This shows that as a bosonic sigma model, we are dealing with a Kähler target space with Kähler potential $K$ and metric $K_{A\bar{B}}$ (ignoring the $-2$, which is due to our conventions). In keeping with this observation, we rename the lowest component in (42), $A^A \to z^A$ (and $\lambda \to \psi$), and interpret the remaining terms in the expansion geometrically. We find

$$\begin{aligned} \int d^4x D^2 \bar{D}^2 K| = \int d^4x \Big\{ &-2G_{A\bar{B}} \left(\partial z^A \partial \bar{z}^{\bar{B}} - i\psi^A \partial \bar{\psi}^{\bar{B}} - i\partial\psi^A \bar{\psi}^{\bar{B}} - \mathcal{F}^A \bar{\mathcal{F}}^{\bar{B}}\right) \\ &+ \Gamma_{\bar{A}\bar{B}C} (\mathcal{F}^C \bar{\psi}^{\bar{B}} \cdot \bar{\psi}^{\bar{A}} - 2i\psi^C \partial \bar{z}^{\bar{B}} \bar{\psi}^{\bar{A}}) \\ &+ \Gamma_{AB\bar{C}} (\bar{\mathcal{F}}^{\bar{C}} \psi^B \cdot \psi^A + 2i\psi^A \partial z^B \bar{\psi}^{\bar{C}}) \\ &+ (R_{C\bar{A}D\bar{B}} + G^{E\bar{E}} \Gamma_{CD\bar{E}} \Gamma_{\bar{B}\bar{D}E}) \psi^D \cdot \psi^C \bar{\psi}^{\bar{B}} \cdot \bar{\psi}^{\bar{A}} \Big\}, \end{aligned} \tag{95}$$

where we have introduced the curvature $R_{C\bar{A}D\bar{B}}$ and connection $\Gamma_{\bar{A}\bar{B}\bar{C}}, \Gamma_{ABC}$ for the metric $G_{A\bar{B}}$. The spinorial contractions are indicated by '$\cdot$' or are the obvious ones. Eliminating the auxiliary field $\mathcal{F}$ finally gives the fully geometric form of the component action

$$(96)\quad \int d^4x \left\{ -2G_{A\bar{B}} \left( \partial z^A \partial \bar{z}^{\bar{B}} - i\psi^A \mathcal{D} \bar{\psi}^{\bar{B}} - i\mathcal{D}\psi^A \bar{\psi}^{\bar{B}} \right) + R_{C\bar{A}D\bar{B}} \psi^D \cdot \psi^C \bar{\psi}^{\bar{B}} \cdot \bar{\psi}^{\bar{A}} \right\},$$

where the $\Gamma \partial z \psi \bar{\psi}$ terms are incorporated in the derivative terms $\psi \mathcal{D} \bar{\psi}$.

To conclude this section, a supersymmetric NLSM in four dimensions necessarily has a Kähler target space $\mathcal{T}$. The canonical complex coordinates are the lowest components of the chiral superfields. Integrability etc. is thus manifest (locally).

4.5. **Isometries in Kähler spaces.** In complex coordinates, the isometry (83) becomes

$$(97)\quad \begin{aligned} \delta z^A &= \lambda^q k_q^A \\ \delta \bar{z}^{\bar{A}} &= \lambda^q \bar{k}_q^{\bar{A}}. \end{aligned}$$

For a holomorphic isometry $k = k(\Phi)$ and $\bar{k} = \bar{k}(\bar{\Phi})$, and the requirement that they leave $S$ in (93) invariant means that they have to leave $K$ invariant up to a Kähler gauge transformation

$$(98)\quad \delta K(\Phi, \bar{\Phi}) = \lambda^q (K_A k_q^A(\Phi) + K_{\bar{A}} \bar{k}_q^{\bar{A}}(\bar{\Phi})) = \lambda^q (\eta_q(\Phi) + \bar{\eta}_q(\bar{\Phi})).$$

The right hand side of (98) will give zero in the superspace integral due to the (anti)chirality of the fields. In fact, the condition (98) for the holomorphic $k$'s is sufficient to show that $\mathcal{L}_k \partial \bar{\partial} K = 0$, i.e., that they generate an isometry and satisfy Killing's equation

$$(99)\quad \nabla_A k_{\bar{A}} + \nabla_{\bar{A}} \bar{k}_A = 0.$$

The relation (98) only determines $\eta$ up to an imaginary constant. This is reflected in an ambiguity in the (real) Killing potential $X_q(\Phi, \bar{\Phi})$ defined by

$$(100)\quad \begin{aligned} k_q^A K_A &= iX_q + \eta_q \\ \bar{k}_q^{\bar{A}} K_{\bar{A}} &= -iX_q + \bar{\eta}_q. \end{aligned}$$

Clearly $X_q$ is correspondingly defined only up to a real constant. From (100) it follows, using the properties of $k$, that

$$(101)\quad \begin{aligned} k_{q\bar{B}} &\equiv k_q^A K_{A\bar{B}} = iX_{q\bar{B}} \\ \bar{k}_{qB} &= -iX_{qB}, \end{aligned}$$

hence the name "Killing potential". It further follows that

$$(102)\quad \bar{k}_q^{\bar{B}} X_{p\bar{B}} + k_p^B X_{qB} = 0,$$

and, hence, that

$$(103)\quad \delta X_p = i\lambda^q (\bar{k}_{[q}^{\bar{A}} X_{p]\bar{A}} + k_{[q}^A X_{p]A}).$$

This expression for the transformation of the Killing potential will be needed later.

For holomorphic Killing vectors, the algebra (85) becomes

$$(104)\quad K_{[p}^A k_{q]}^B{}_{,A} = c_{pq}{}^r k_r^B,$$

and its complecx conjugate. In conjunction with the transformation of the Kähler potential $K$

$$\delta K = \lambda^q (K_A k_q^A + K_{barA} \bar{k}_q^{\bar{A}}) = \lambda^q(\eta_q + \bar{\eta}_q), \tag{105}$$

and analyticity, we may use (104) to derive the transformations of $\eta$:

$$\begin{aligned} k_{[p}^A \eta_{q]}^A &= c_{pq}^{\ r} \eta_r + i o_{pq} \\ \bar{k}_{[p}^{\bar{A}} \bar{\eta}_{q]}^A &= c_{pq}^{\ r} \eta_r - i o_{pq}. \end{aligned} \tag{106}$$

It is important for gaugin the isometries that the $\eta$'s transform equivariantly. When the constants $o_{pq}$ are not removable, they thus constitute an obstruction to gauging the isometries. From the Jacobi identities one finds that they have to satisfy $o_{p[q} c_{rs]}^{\ t} = 0$. This is the case if $o_{pq} = c_{pq}^{\ r} \xi_r$ for some real constant $\xi$, and the shift $\eta \to \eta + i\xi$ then removes the obstructions, except for invariant abelian subgroups. Indeed, for semi-simple groups, even non-compact ones, we may choose $\xi_q = c_{qp}^{\ r} o_{rs} g^{ps}$, with $g$ the Killing metric.

As an illustration of the previous discussion, let us look at the an example where the isometry group is Abelian and the obstructions not removable. Take the Kähler potential to be $K = \Phi\bar{\Phi}$ corresponding to the flat metric $G = 1$. Then the translations generated by

$$\begin{aligned} k_1 &= \partial/\partial\Phi + \partial/\partial\bar{\Phi} \equiv \partial + \bar{\partial} \\ k_2 &= i(\partial - \bar{\partial}), \end{aligned} \tag{107}$$

are isometries. From the variation $\delta K$, we find

$$\eta_1 = \Phi, \qquad \eta_2 = -i\Phi. \tag{108}$$

Calculating the effect of a transformation as in (106), we have

$$k_{[1}^A \eta_{2]A} = -2i \quad \Rightarrow o_{12} = -2i. \tag{109}$$

Since the isometry is abelian, this obstruction is not removable, and the implication is that we can only gauge a linear combination of $k_1$ and $k_2$.

4.6. **Gauging isometries in $N = 1$ susy sigma models.** Let us first discuss some generalities before taking on the isometries. We study chiral fields that transform under some representation of a Yang-Mills group $\mathcal{G}$

$$\Phi^{A\prime} = \left(e^{i\Lambda}\right)_B^A \Phi^B, \qquad \bar{\Phi}^{\bar{A}\prime} = \bar{\Phi}^{\bar{B}} \left(e^{-i\bar{\Lambda}}\right)_{\bar{B}}^{\bar{A}}, \tag{110}$$

where $\Lambda_B^A \equiv \Lambda^q (T_q)_B^A$ with $\Lambda^q(x, \theta, \bar{\theta})$ a chiral superfield and $T_q$ the generators of the lie-algebra $\mathfrak{g}$ of $\mathcal{G}$

$$[T_q, T_p] = i c_{pq}^{\ r} T_r. \tag{111}$$

Since $\Lambda \neq \bar{\Lambda}$, the group $\mathcal{G}$ acts on $\Phi$ and $\bar{\Phi}$ through its complexification $\mathcal{G}^C$ ($\Rightarrow T_q \to (T_q, iT_q)$).

The gauge potential is an adjoint vector superfield $V = V^q T_q$. It transforms as

$$e^{V'} = \left(e^{i\bar{\Lambda}}\right) e^V \left(e^{-i\Lambda}\right), \tag{112}$$

which means that we may define a superfield $\tilde{\Phi}^A$ from $\bar{\Phi}^A$ which transforms in $\mathcal{G}_\Lambda$ rather than in $\mathcal{G}_{\bar{\Lambda}}$:

$$\tilde{\Phi} \equiv \bar{\Phi}^B \left(e^V\right)_B^{\ A}, \qquad \Rightarrow \tilde{\Phi}' = \left(\tilde{\Phi} e^{-i\Lambda}\right). \tag{113}$$

This is precisely what is needed to construct invariant actions. To this end we also introduce the gauge covariant superspace derivatives

$$\nabla_{\mathcal{A}} = (\nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}) \equiv (e^{-V} D_\alpha e^V, \bar{D}_{\dot{\alpha}}, -\{\nabla_\alpha, \nabla_{\dot{\alpha}}\}). \tag{114}$$

With these tools we write the gagued NLSM action in superspace as (cf. (94))

$$\int d^4x d^2\theta d^2\bar{\theta} K(\Phi^A, \bar{\Phi}^{\bar{A}}) \rightarrow \int d^4x d^2\theta d^2\bar{\theta} K(\Phi^A, \tilde{\Phi}^{\bar{A}}), \tag{115}$$

and instead of (42) we use an the component expansion

$$z^A = \Phi^A|, \qquad \psi_\alpha = \nabla_\alpha \Phi^A|, \qquad \mathcal{F}^A = \nabla^2 \Phi^A|, \tag{116}$$

for the chiral fields along with

$$A^q_{\alpha\dot{\alpha}} = i(\nabla_{\alpha\dot{\alpha}}| - \partial_{\alpha\dot{\alpha}})^q, \qquad \lambda^q_\alpha = i\bar{D}^2 \nabla_\alpha| \equiv W_\alpha|, \qquad D^{q'} = -\frac{i}{2}\{\nabla^\alpha, W_\alpha\}|, \tag{117}$$

for the physical components of the vector multiplet.

Invariance of the ungauged action (94) under an isomorphism was shown above to be equivalent to $\delta K = \lambda(\eta + \bar{\eta})$. Now that we are promoting the constant $\lambda$ to a superfield $\Lambda$ (and $\bar{\Lambda}$), this is no-longer necessarily true. E.g.,

$$\int d^2\theta d^2\bar{\theta} \lambda^q \bar{\eta}_q = 0, \tag{118}$$

but

$$\int d^2\theta d^2\bar{\theta} \Lambda^q \bar{\eta}_q \neq 0, \tag{119}$$

in general. To amend this, we introduce a new chiral superfield $\zeta$ with transformation properties (in the ungauged case)

$$\delta\zeta = \eta_q(\Phi^A)\lambda^q, \qquad \delta\bar{\zeta} = \bar{\eta}_q(\bar{\Phi}^{\bar{A}})\lambda^q. \tag{120}$$

If the isometry under consideration is generated by $k$, we now define new holomorphic Killing vectors $k'$ in the enlarged target space with coordinates $\Phi$, $\bar{\Phi}$, $\zeta$, $\bar{\zeta}$:

$$k'_q \equiv k^A_q \partial_A + \eta_q \partial_\zeta, \qquad \bar{k}'_q \equiv \bar{k}^{\bar{A}}_q \bar{\partial}_{\bar{A}} + \bar{\eta}_q \bar{\partial}_{\bar{\zeta}}. \tag{121}$$

The symmetries generated by $k'$ leave the new Kähler potential $K' \equiv K(\Phi, \bar{\Phi}) - \zeta - \bar{\zeta}$ *invariant.* The new action is independent of $\zeta$ but it has important consequences for the gauged. It is this Kähler potential we will use when gaguging the isometries.

The transformations (110) describe transformations linearly realized on $\Phi$ and does not cover general isometries in arbitrary coordinates. To cover the general case we must gauge the isometries acting as in (83), or (83) for the holomorphic $k$'s:

$$\delta\Phi^A = \Lambda^q k^A_q, \qquad \delta\bar{\Phi}^{\bar{A}} = \bar{\Lambda}^q \bar{k}^{\bar{A}}_q. \tag{122}$$

The appropriate generalization of (112) is

$$\tilde{\Phi}^A \equiv e^{\mathcal{L}_{iV\cdot k}} \bar{\Phi}, \tag{123}$$

i.e., the action of an exponentiated Lie-derivative along the direction $iV^q\bar{k}_q$, representing a finite gauge transformation with parameter $V^q$. Accordingly, for the case at hand with Killing vectors $k'$ as in (121)

$$
\begin{aligned}
(124)\qquad \tilde{\bar{\zeta}} &= e^{\mathcal{L}_{iV\cdot \bar{k}'}}\bar{\zeta} \\
&= \left(1+\left(\frac{e^{\mathcal{L}'}-1}{\mathcal{L}'}\right)\mathcal{L}'\right)\bar{\zeta} = \bar{\zeta} + i\left(\frac{e^{\mathcal{L}'}-1}{\mathcal{L}'}\right)\bar{\eta}_q V^q \\
&= \bar{\zeta} + i\left(\frac{e^{\mathcal{L}}-1}{\mathcal{L}}\right)\bar{\eta}_q V^q,
\end{aligned}
$$

where $\mathcal{L}' \equiv \mathcal{L}_{iV\cdot\bar{k}'}$, and the prime is removed in the last equality because $V$ and $\eta$ are independent of $\zeta$. As noted earlier, the $\bar{\zeta}$ term, is irrelevant in the action, and thus the gauged action is

$$
\begin{aligned}
(125)\qquad S &= \int d^4x d^2\theta d^2\bar{\theta}\left(K(\Phi,\tilde{\bar{\Phi}}) - \zeta - \tilde{\bar{\zeta}}\right) \\
&= \int d^4x d^2\theta d^2\bar{\theta}\left(K(\Phi,\tilde{\bar{\Phi}}) - i\left(\frac{e^{\mathcal{L}}-1}{\mathcal{L}}\right)\bar{\eta}_q V^q\right).
\end{aligned}
$$

Finally, we use the relation (100) to eliminate $\eta$ in favour of the Killing potential $X$ (which also entails removing the tilde from $\Phi$ in $K$)

$$
(126)\qquad \int d^4x d^2\theta d^2\bar{\theta}\left(K(\Phi,\bar{\Phi}) + \left(\frac{e^{\mathcal{L}}-1}{\mathcal{L}}\right)X_q V^q\right).
$$

The last term is hermitean, although not manifestly so. Through the ambiguity in the definition of $X$ there is the possibility to include a so called Fayet-Iliopoulos term for each $U(1)$ factor in $\mathcal{G}$, i.e., a term of the type $c_q V^q$. (See the discussion of obstructions in subsection 4.5 above.) The action (126) solves the problem of gauging isometries of $N=1$ supersymmetric NLSM's.

We close this subsection with a simple example of a Kähler quotient construction.

Let us again look at the $\mathbb{CP}^{(n)}$ model discussed in subsection 4.2, but now from the point of view of superspace. We start from the flat space Kähler potential in $\mathbb{C}^{(n+1)}$ which is

$$
(127)\qquad K(\Phi,\bar{\Phi}) = \sum_{A=1}^{n+1}\Phi^A\bar{\Phi}^{\bar{A}} = e^{(\phi^0+\bar{\phi}^0)}\left(1+\sum_{a=1}^{n}\phi^a\bar{\phi}^{\bar{a}}\right),
$$

where the last equality involves an obvious field redefinition and displays one of the isometries of the model. The corresponding Killing vector is

$$
(128)\qquad \partial\phi^A = \lambda^q k_q^A\partial_A = i\lambda^q\delta_q^0\partial_0, \qquad \partial\bar{\phi}^{\bar{A}} = \lambda^q\bar{k}_q^{\bar{A}}\bar{\partial}_{\bar{A}} = -i\lambda^q\delta_q^0\bar{\partial}_0.
$$

We gauge this isometry by letting $\phi^0+\bar{\phi}^0 \to \phi^0+\bar{\phi}^0+V$, which amounts to introducing $\tilde{\bar{\Phi}}^0$ in this case. We may also use the freedom discussed after (126) to include a Fayet-Iliopoulos term. Hence

$$
(129)\qquad K(\Phi,\bar{\Phi}) \to K(\Phi,\tilde{\bar{\Phi}}) - cV = e^{(\phi^0+\bar{\phi}^0)}\left(1+\sum_{a=1}^{n}\phi^a\bar{\phi}^{\bar{a}}\right).
$$

We find the new quotient Kähler potential by extremizing the corresponding action with respect to $V$:

$$\delta V \Rightarrow V = -ln(1+\sum \phi^a\bar{\phi}^{\bar{a}}) - \phi^0 - \bar{\phi}^0 + const., \tag{130}$$

which gives the quotient Kähler potential

$$K'(\phi,\bar{\phi}) = c\ ln(1+\sum \phi^a\bar{\phi}^{\bar{a}}), \tag{131}$$

and again we recognize the $\mathbb{CP}^{(n)}$ Kähler potential for the metric (81).

4.7. **$N = 2$ supersymmetric nonlinear sigma models.** To formulate the $N = 2$ supersymmetric NLSM's we ideally want a $N = 2$ superspace where both supersymmetries are manifest. This means that we need to introduce a second set of $\theta$'s and extend the integration measure accordingly, so that an action will be written as

$$S = \int d^4x d^4\theta d^4\bar{\theta}\mathcal{L}. \tag{132}$$

However, such an action cannot accomodate a Lagrangian $\mathcal{L}(\Phi_H,\bar{\Phi}_H)$, where $\Phi_H$ is the smallest $N = 2$ representation, a so called hypermultiplet corresponding a pair of $N = 1$ chiral superfields. To be more precise, a dimensional analysis of the measure shows that such an action will not have the right bosonic content for a NLSM. There are ways around this. Enlarging the superspace by additional bosonic coordinates one may find invariant subspaces and corresponding subintegrals that give correct results. We do not discuss these *projective* supererspaces [31]–[38] and *harmonic* superspaces [39] here, though. Instead our discussion of $N = 2$ NLSM's will be entirely in terms of $N = 1$ superfields in $N = 1$ superspace. Our starting point will thus be the action (94)

$$\int d^4x d^2\theta d^2\bar{\theta} K(\Phi^A,\bar{\Phi}^{\bar{A}}). \tag{133}$$

One supersymmetry is thus manifest due to the $N = 1$ formalism. For the second supersymmetry we make the ansatz

$$\delta\Phi^A = \bar{D}^2(\bar{\epsilon}\bar{\Omega}^A), \qquad \delta\bar{\Phi}^{\bar{A}} = D^2(\epsilon\Omega^{\bar{A}}). \tag{134}$$

The reason for the covariant derivatives is for the second supersymmetry to commute with the first, and they come squared to preserve the (anti-) chirality of the fields. Here $\Omega = \Omega(\Phi,\bar{\Phi})$, so they represent the general situation. The requirements of closure of the supersymmetry algebra and invariance of the action will constrain the $\Omega$'s and reveal an interesting target space geometry.

The superfield transformation parameter satisfies

$$\bar{D}_{\dot{\alpha}}\epsilon = D^2\epsilon = \partial_{\alpha\dot{\alpha}}\epsilon = 0. \tag{135}$$

Closure of the non-manifest supersymmetry means that

$$[\delta_{\epsilon^1},\delta_{\epsilon^2}]\Phi^A = i(\bar{D}^{\dot{\alpha}}\bar{\epsilon}^2 D^{\alpha}\epsilon^1 - (1\leftrightarrow 2))\partial_{\alpha\dot{\alpha}}\Phi^A, \tag{136}$$

and implies

$$\bar{\Omega}^A_{\bar{B}}\Omega^{\bar{B}}_B = -\delta^A_B, \qquad \bar{\Omega}^C_{[\bar{D}}\bar{\Omega}^A_{\bar{B}]} = 0, \tag{137}$$

along with their hermitean conjugate.(An additional condition turns out to be a field equation of the model.) Additional subscript again represent derivatives with respect to the fields $\Phi$ and $\bar{\Phi}$. Invariance of the action (133) implies the further constraints

$$K_{A\bar{B}}\bar{\Omega}^{A}_{\bar{C}} = -K_{A\bar{C}}\bar{\Omega}^{A}_{\bar{B}}$$
$$K_{A\bar{B}}\bar{\Omega}^{A}_{\bar{C}\bar{D}} + K_{A\bar{C}\bar{D}}\bar{\Omega}^{A}_{\bar{B}} = 0 \tag{138}$$
$$K_{A\bar{B}}\bar{\Omega}^{A}_{\bar{C}D} + K_{A\bar{B}D}\bar{\Omega}^{A}_{\bar{C}} = 0.$$

Putting all this together, we conclude that there are two additional integrable complex structures (from (137)) which are covariantly constant with respect to the hermitean Kähler metric $G = \partial\bar{\partial}K$ (from (138)). There are thus three covariantly constant complex structures. The (lowest components of the) chiral superfields are the canonical coordinates for one of them. The complex structures are

$$J^{(3)i}_{j} = \begin{pmatrix} i\delta_{AB} & 0 \\ 0 & -i\delta_{\bar{A}\bar{B}} \end{pmatrix}, \; J^{(1)i}_{j} = \begin{pmatrix} 0 & \Omega^{\bar{A}}_{B} \\ \bar{\Omega}^{A}_{\bar{B}} & 0 \end{pmatrix}, \; J^{(2)i}_{j} = \begin{pmatrix} 0 & i\Omega^{\bar{A}}_{B} \\ -i\bar{\Omega}^{A}_{\bar{B}} & 0 \end{pmatrix}. \tag{139}$$

We conclude that the symmetries and invariances of a $N = 2$ susy NLSM requires the target space $\mathcal{T}$ to be hyperkähler.

4.8. **Isometries in hyperkähler spaces.** The isometries we will consider in the $N = 2$ case are tri holomorphic, i.e., whereas a holomorphic Killing vector preserves the fundamental two-form (64) $\omega = 2iK_{A\bar{A}}dz^{A} \wedge d\bar{z}^{\bar{A}}$ corresponding to $J^{(3)}$, a triholomorphic Killing vector preserves in addition the two-forms related to $J^{(1)}$ and $J^{(2)}$, which means that

$$\rho_{\bar{B}[\bar{C}}\nabla_{\bar{D}]}\bar{k}^{\bar{B}} \equiv K_{A\bar{B}}\bar{\Omega}^{A}_{[\bar{C}}\nabla_{\bar{D}]}\bar{k}^{\bar{B}} = 0. \tag{140}$$

This defines $\rho$ (and $\bar{\rho}$ through the hermitean conjugate relation). Such a Killing vector has a Killing potential with respect to each $J$, or, in arbitrary coordinates, $k^{i}J^{(X)}_{ij} = -X^{(X)}{}_{,j}$. We combine $X^{(1)}$ and $X^{(2)}$ to a holomorphic potential $P$ and an antiholomorphic potential $\bar{P}$ with respect to $J^{(2)} \pm iJ^{(2)}$, respectively,

$$k^{A}\rho_{AB} = -P_{,B}\,, \qquad \bar{k}^{\bar{A}}\bar{\rho}_{\bar{A}\bar{B}} = -\bar{P}_{,\bar{B}}\,. \tag{141}$$

These ingredients are all needed to describe the gauging of isometries of $N = 2$ NLSM's.

4.9. **Gauging isometries in $N = 2$ susy sigma models.** When the $N = 2$ model (133) has triholomorphic isometries they may be gauged in a manner that closely follows the description in subsection 4.6. The new features to do with $N = 2$ supersymmetry is that the scalar superfields now come in pairs of chiral $N = 1$ field that together constitute a $N = 2$ hyper multiplet. Also the vector superfield $V^{q}$ gets an $N = 2$ partner, a chiral superfield $S^{q}$.

$$\Phi^{A} \to (\Phi^{A}_{+}, \Phi^{A}_{-}), \qquad V^{q} \to (V^{q}, S^{q}). \tag{142}$$

The second supersymmetry (134) is affected by the gauging in that $\Omega(\Phi, \bar{\Phi}) \to \Omega(\Phi, \tilde{\bar{\Phi}})$. In addition, for the $N = 2$ vector multiplet it reads

$$\delta e^{V} = \epsilon\bar{S}e^{V} + e^{V}S\bar{\epsilon}, \delta S = -iW^{\alpha}D_{\alpha}\epsilon \tag{143}$$

(see (117)). The gauge transformations with parameter $\Lambda$ are as in (122) with the additional

$$\delta S = i[\Lambda, S]. \tag{144}$$

The gauged action, invariant under the local isometries, is the generalization of (126)

$$\int d^4xd^2\theta d^2\bar\theta \left[K(\Phi,\bar\Phi) + \left(\frac{e^{\mathcal{L}}-1}{\mathcal{L}}\right) X_q V^q + g_{pq} S^p \left(e^{-V}\bar S e^V\right)^q\right] + \left\{\int d^4xd^2\theta (iS^q P_q) + h.c.\right\}. \tag{145}$$

The possibility to add Fayet-Iliopoulos terms discussed for $N = 1$ generalizes to $N = 2$

$$S_{FI} = \int d^4xd^2\theta d^2\bar\theta c_q V^q + \left\{\int d^4xd^2\theta i\hat c_q S^q + h.c.\right\}, \tag{146}$$

again with a sum over abelian factors.

The action (145) is the starting point for the $N = 2$ quotient, the hyperkähler quotient construction [9], and we end with an example of this [7].

Starting from the action

$$\int d^4xd^2\theta d^2\bar\theta \left(\bar\Phi_+^{\bar A}\Phi_+^A e^V + \Phi_-^A\bar\Phi_-^{\bar A}e^{-V} - cV\right) + \left\{\int d^4xd^2\theta (\Phi_-^A\Phi_+^A - bS) + h.c.\right\}, \tag{147}$$

which is a gauged flat ($\mathbb{C}^{2(n+1)}$) $N = 2$ action with Fayet-Iliopoulos terms ($c$ and $b$), invariant under the gauged abelian isometries

$$\Phi'_\pm = e^{\pm i\Lambda}\Phi_\pm, \qquad \bar\Phi'_\pm = e^{\mp\bar\Lambda}\bar\Phi_\pm, \qquad V' = V + i(\bar\Lambda - \Lambda). \tag{148}$$

We extremise this action with respect to the $N = 2$ vector multiplet, i.e., with respect to $V$ and $S$

$$\begin{aligned} \delta V &\Rightarrow \bar\Phi_+^{\bar A}\Phi_+^A e^V - \Phi_-^A\bar\Phi_-^{\bar A}e^{-V} = c \\ \delta S &\Rightarrow \Phi_-^A\Phi_+^A = b, \end{aligned} \tag{149}$$

where the last relation is known as the "moment map". With the gauge choice

$$\Phi_+^{n+1} = \Phi_-^{n+1} \equiv \phi, \tag{150}$$

and the redefinitions

$$\Phi_\pm^a \equiv U_\pm^a\phi, \qquad a = 1,\ldots,n, \tag{151}$$

these moment map constraints are solved. Further defining

$$M_\pm \equiv \bar\Phi_\pm^A\Phi_\pm^A, \tag{152}$$

we solve the $V$ equations in (149) and rewrite the action (147) as

$$\int d^4xd^2\theta d^2\bar\theta \left[\sqrt{c^2 + 4M_+M_-} - c\left(ln(c + \sqrt{c^2+4M_+M_-}\,) - lnM_+\right)\right]. \tag{153}$$

The Lagrangian density inside the square brackets is the new Kähler potential on the $N = 2$ quotient, which also has a hyperkähler target space $\mathcal{T}$. The quotient Kähler potential is a generalization of that of the $\mathbb{CP}^n$ models and yields the Calabi metrics.

**Acknowledgment**: I gratefully acknowledge all my collaborators in Supersymmetry. In particular Nigel Hitchin, Chris Hull, Anders Karlhede and Martin Roček. The work is supported in parts by EU contract HPNR-CT-2000-0122 and by VR grant 650-1998368.

## References

[1] Gates, S. J., Grisaru, M. T., Roček, M. and Siegel, W., *Superspace, or One Thousand and One Lessons in Supersymmetry*, Front. Phys. **58**, 1 (1983). [arXiv:hep-th/0108200].

[2] Bagger, J. and Wess, J., *Supersymmetry and Supergravity*, JHU-TIPAC-9009, Princeton Univ. Press.

[3] Freund, P. G., *Introduction to Supersymmetry*, Cambridge, UK: University Press (1986), 152 pp. (Cambridge Monographs on Mathematical Physics).

[4] Buchbinder, I. L. and Kuzenko, S. M., *Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace*, Bristol, UK: IOP (1998), 656 pp.

[5] Freed, D. S., *Five Lectures on Supersymmetry*, Providence, USA: AMS (1999), 119 pp.

[6] Deligne, P. *et al.*, *Quantum Fields and Strings: A Course for Mathematicians*, Vol. 1, 2, Providence, USA: AMS (1999), 1–1501.

[7] Lindström, U. and Roček, M., *Scalar Tensor Duality and $N = 1$, $N = 2$ Nonlinear Sigma Models*, Nucl. Phys. B **222**, 285 (1983).

[8] Hull, C. M., Karlhede, A., Lindström, U. and Roček, M., *Nonlinear Sigma Models and Their Gauging In and Out of Superspace*, Nucl. Phys. B **266**, 1 (1986).

[9] Hitchin, N. J., Karlhede, A., Lindström, U. and Roček, M., *Hyperkähler Metrics and Supersymmetry*, Commun. Math. Phys. **108**, 535 (1987).

[10] Peskin, M. E. and Schroeder, D. V., *An Introduction to Quantum Field Theory*, Reading, USA: Addison-Wesley (1995), 842 pp.

[11] Weinberg, S., *The Quantum Theory of Fields. Vol. 1: Foundations*, Cambridge, UK: University Press (1995), 609 pp.

[12] Fayet, P. and Ferrara, S., *Supersymmetry*, Phys. Rept. **32**, 249 (1977).

[13] Salam, A. and Strathdee, J., *Supersymmetry and Superfields*, Fortsch. Phys. **26**, 57 (1978).

[14] Corwin, L., Ne'eman, Y. and Sternberg, S., *Graded Lie Algebras in Mathematics and Physics (Bose-Fermi Symmetry)*, Rev. Mod. Phys. **47**, 573 (1975).

[15] Pais, A. and Rittenberg, V., *Semisimple Graded Lie Algebras*, J. Math. Phys. **16**, 2062 (1975) [Erratum-ibid. **17**, 598 (1975)].

[16] Rittenberg, V. and Scheunert, M., *Elementary Construction of Graded Lie Groups*, J. Math. Phys. **19**, 709 (1978).

[17] Scheunert, M., Nahm W. and Rittenberg, V., *Irreducible Representations of the $Osp(2,1)$ and $Spl(2,1)$ Graded Lie Algebras*, J. Math. Phys. **18**, 155 (1977).

[18] Scheunert, M., Nahm W. and Rittenberg, V., *Graded Lie Algebras: Generalization of Hermitian Representations*, J. Math. Phys. **18**, 146 (1977).

[19] Scheunert, M., Nahm W. and Rittenberg, V., *Classification of All Simple Graded Lie Algebras Whose Lie Algebra is Reductive. 2. (Construction of the Exceptional Algebras)*, J. Math. Phys. **17**, 1640 (1976).

[20] Scheunert, M., Nahm W. and Rittenberg, V., *Classification of All Simple Graded Lie Algebras Whose Lie Algebra is Reductive.* 1, J. Math. Phys. **17**, 1626 (1976).

[21] Nahm, W., Rittenberg V. and Scheunert, M., *The Classification of Graded Lie Algebras*, Phys. Lett. B **61**, 383 (1976).

[22] Coleman, S. R. and Mandula, J., *All Possible Symmetries Of The S Matrix*, Phys. Rev. **159**, 1251 (1967).

[23] Haag, R., Lopuszanski, J. T. and Sohnius, M., *All Possible Generators of Supersymmetries of the S Matrix*, Nucl. Phys. B **88**, 257 (1975).

[24] Roček, M., *Introduction to supersymmetry*, In Boulder 1992, Proceedings, Recent directions in particle theory 101–139.

[25] Berezin, F. A., *The Method of Second Quantization*, Academic Press, New York, 1966.
[26] Yano, K., *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon, Oxford, 1965.
[27] Yano, K. and Kon, M., *Structures of Manifolds*, Series in Pure Mathematics, Vol.3, World Scientific, Singapore, 1984.
[28] Alvarez-Gaume, L. and Freedman, D. Z., *A Simple Introduction to Complex Manifolds*, ITP-SB-80-31.
[29] Zumino, B., *Supersymmetry and Kähler Manifolds*, Phys. Lett. B **87**, 203 (1979).
[30] Alvarez-Gaume, L. and Freedman, D. Z., in: *Unification of the Fundamental Particle Interactions*, Ferrara, S., Ellis, J. and van Nieuwenhuizen, P. (Eds.), New York, Plenum, 1980.
[31] Gates, S. J., Hull, C. M. and Roček, M., *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nucl. Phys. B **248**, 157 (1984).
[32] Karlhede, A., Lindström, U. and Roček, M., *Selfinteracting tensor multiplets in $N = 2$ Superspace*, Phys. Lett. B **147**, 297 (1984).
[33] Grundberg, J. and Lindström, U., *Actions for linear multiplets in six-dimensions*, Class. Quant. Grav. **2**, L33 (1985).
[34] Lindström, U. and Roček, M., *New hyperkähler metrics and new supermultiplets*, Commun. Math. Phys. **115**, 21 (1988).
[35] Lindström, U. and Roček, M., "*$N = 2$ Superyang-Mills theory in projective superspace*, Commun. Math. Phys. **128**, 191 (1990).
[36] Lindström, U., Ivanov, I. T. and Roček, M., *New $N = 4$ superfields and sigma models*, Phys. Lett. B **328**, 49 (1994) [arXiv:hep-th/9401091].
[37] Lindström, U., Kim, B. B., and Roček, M., *The nonlinear multiplet revisited*, Phys. Lett. B **342**, 99 (1995) [arXiv:hep-th/9406062].
[38] Gonzalez-Rey, F., Roček, M., Wiles, S., Lindström U. and von Unge, R., *Feynman rules in $N = 2$ projective superspace. I: Massless hypermultiplets*, Nucl. Phys. B **516**, 426 (1998) [arXiv:hep-th/9710250].
[39] See Galperin, A. S., Ivanov E. A., Ogievetsky V. I. and Sokatchev E. S., *Harmonic Superspace*, Cambridge, UK: Univ. Press (2001), 306 pp. and references therein.

DEPARTMENT OF PHYSICS
STOCKHOLM UNIVERSITY
SCFAB
S-106 91 STOCKHOLM
SWEDEN
*E-mail address*: ul@physto.se