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AN INTRODUCTION TO CARTAN GEOMETRIES

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ABSTRACT. This paper is an introduction to Cartan geometries, without proofs, giving a well-motivated definition, and describing the fundamental properties of these geometries. Three quite general structure theorems are given describing certain flat, constant curvature and symmetric geometries. Finally an example is given showing how the Stäckel metrics arise in this context.

1. INTRODUCTION

In this paper we describe the motivation for the definition of Cartan geometries and give some examples and applications. We assume an elementary knowledge of Lie groups and their algebras. A basic reference for all this material is our book [4].

In §2-§4 we give the definition and fundamental properties of Cartan geometries. In §3 we also show how Riemannian geometry is a special case of a Cartan geometry.

In §5 we introduce the notion of "geometric orientability" for use in §6 in describing the complete, torsion-free, geometrically oriented, flat geometries. These results may be interpreted as justification for the statement that Cartan geometries may be regarded as a common generalization of Klein geometries (homogeneous spaces) and Riemannian geometries, as pictured in the following diagram.

$$\begin{array}{ccc} \text{Euclidean Geometry} & \xrightarrow{\text{generalization}} & \text{Klein Geometries} \\ \text{generalization} \downarrow & & \downarrow \text{generalization} \\ \text{Riemannian Geometry} & \xrightarrow{\text{generalization}} & \text{Cartan Geometries} \end{array}$$

In §7 we describe the Cartan space forms and in in §8 we see how symmetric spaces arise and may be classified in this context. In the final §9 we derive the Stäckel metrics as metrics that share their set of unparametrized geodesics with a "strongly distinct" metric.

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2. Principal Bundles

In this section we describe the basic properties of principal bundles of which the homogeneous spaces ("Klein geometries") are the prime examples. We end by defining "model geometries" which will be used as the models for Cartan geometries in §3, in analogy to the way that Euclidean space is a model for Riemannian geometry.

Definition 2.1. *A (right) principal bundle with (Lie) group H consists of*

- *a manifold P*
- *a free proper smooth right H-action $P \times H \to P$*

The main facts about principal bundles are

1. (*Base Space*) There is a unique smooth manifold structure on quotient space $M = P/H$ such that the canonical map $\pi : P \to M$ is smooth. M is called the *base space* of the bundle, and we write $H \to P \to M$ to denote the bundle.
2. (*Local Triviality*) Each point of M has a neighborhood $U \subset M$ such that there is a right H-equivariant diffeomorphism $\psi : \pi^{-1}(U) \approx U \times H$ (a local trivialization of P) making the following diagram commute

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{\psi}{\approx} & U \times H \\ \pi|_U \searrow & & \swarrow \text{proj}_U \\ & U & \end{array}$$

   Given two such diffeomorphisms ψ_1, ψ_2 corresponding to open sets $U_1, U_2 \subset M$, we may set $U_0 = U_1 \cap U_2$ and obtain a right H-equivariant diffeomorphism $\varphi = \psi_2 \circ \psi_1^{-1} : U_0 \times H \to U_0 \times H$ commuting with the canonical projection to U_0. It follows that φ has the form $\varphi(u, h) = (u, k(u)h)$ for some smooth map $k : U_0 \to H$.
3. (*Vector Fields*) An element $v \in \mathfrak{h}$ determines a certain vector field $v^\dagger$ on P in the following way: v determines a left invariant vector field ξ_v^H on H given by $(\xi_v^H)_h = (L_h)_* v$ for $h \in H$, where $L_h : H \to H$ is left multiplication by h. If $\psi : \pi^{-1}(U) \approx U \times H$ is as in 2. above, we define $\xi^\dagger|\pi^{-1}(U) = \psi_*^{-1}(0, \xi_v^H)$. It is easy to check (by varying the choice of local trivializations) that $v^\dagger$ defines a vector field on the whole of P.

Of course the most fundamental examples of principal bundles are the homogeneous spaces $H \subset G \to G/H$, where H is a closed subgroup of G. F. Klein noticed that all classical (Euclidean and non-Euclidean, but not Riemannian) geometry is the study of the geometry of these spaces. For this reason they are called Klein geometries and the pair $(\mathfrak{g}, \mathfrak{h})$ a Klein pair.

Example 2.2. Euclidean geometry is the study of the homogeneous space G/H, where

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in Gl_{n+1}(\mathbb{R}) \;\middle|\; v \in \mathbb{R}^n,\ A \in O_n(\mathbb{R}) \right\} = \text{Euc}_n(\mathbb{R})$$

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in Gl_{n+1}(\mathbb{R}) \;\middle|\; A \in O_n(\mathbb{R}) \right\} = O_n(\mathbb{R})$$

The group G is the group of Euclidean motions of $\mathbb{R}^n$, and H is the subgroup fixing the origin. We have $H \subset G$ and $G/H = \mathbb{R}^n$, and the projection $\pi : G \to \mathbb{R}^n$ maps

$\binom{1\,0}{v\,A} \mapsto v$. The Lie algebras of these groups are

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix} \in M_{n+1}(\mathbb{R}) \;\middle|\; v \in \mathbb{R}^n,\ A \in \mathfrak{o}_n(\mathbb{R}) \right\} = \mathfrak{euc}_n(\mathbb{R})$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in M_{n+1}(\mathbb{R}) \;\middle|\; A \in \mathfrak{o}_n(\mathbb{R}) \right\} = \mathfrak{o}_n(\mathbb{R})$$

where $\mathfrak{o}_n(\mathbb{R})$ denotes the space of skew symmetric $n \times n$ matrices. The adjoint action of H on $\mathfrak{g}$ is conjugation: $\mathrm{Ad}(h)w = h^{-1}wh$. Note that the canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ is invariant under the adjoint action and that this action of H on the $\mathbb{R}^n$ factor is just the standard left multiplication action.

Example 2.3. (Real) projective geometry is the study of the homogeneous space G/H, where

$$G = PSl_{n+1}(\mathbb{R}) = Sl_{n+1}(\mathbb{R})/Z \text{ where } Z = \begin{cases} \{I\} & \text{if } n+1 \text{ is odd} \\ \{\pm I\} & \text{if } n+1 \text{ is even} \end{cases}$$

$$H = \left\{ h \in G \;\middle|\; h = \begin{pmatrix} \Delta & a \\ 0 & y \end{pmatrix}, \Delta \in \mathbb{R}^\times \right\}$$

The group G is the group of projective motions of $\mathbb{RP}^n$, and H is the subgroup fixing the point $[1:0:\cdots:0]$. We have $H \subset G$ and $G/H = \mathbb{RP}^n$. The projection $\pi : G \to \mathbb{RP}^n$ maps $\binom{\Delta\, b}{v\, A} \mapsto [\Delta : v_1 : \cdots : v_n]$. The Lie algebras of these groups are

$$\mathfrak{g} = \mathfrak{psl}_{n+1}(\mathbb{R}) = \mathfrak{sl}_{n+1}(\mathbb{R}) \text{ the } (n+1) \times (n+1) \text{ matrices of trace } 0$$

$$\mathfrak{h} = \left\{ h \in H \;\middle|\; \begin{pmatrix} a & b \\ 0 & A \end{pmatrix}, a \in \mathbb{R} \right\}$$

The adjoint action of H on $\mathfrak{g}$ is conjugation: $\mathrm{Ad}(h)w = h^{-1}wh$.

We are going to be using Klein geometries as "flat model geometries" for the Cartan geometries to be introduced in §3. But for this purpose we will not need *all* the data that comes with a Klein geometry. So we make the definition:

Definition 2.4. A *model geometry* consists of

- A Klein pair $(\mathfrak{g}, \mathfrak{h})$ which is effective (i.e., $\mathfrak{h}$ contains no non-zero ideal of $\mathfrak{g}$.)
- A Lie group H with Lie algebra $\mathfrak{h}$.
- A representation, denoted $\mathrm{Ad} : H \to Gl_{\mathrm{Lie}}(\mathfrak{g})$, extending the adjoint representation $\mathrm{Ad}_{\mathfrak{h}} : H \to Gl_{\mathrm{Lie}}(\mathfrak{h})$.

It is clear how a Klein geometry (G, H) will give rise to a model geometry.

Definition 2.5. A model is *first order* or *higher order* according as the homomorphism $\mathrm{Ad} : H \to Gl(\mathfrak{g}/\mathfrak{h})$ is injective or not. It is *primitive* if $\mathfrak{h}$ is a maximal proper subalgebra of $\mathfrak{g}$.

For example, while both the Euclidean and projective models are primitive, the Euclidean model is first order while the projective model is higher order.

Definition 2.6. A Klein pair $(\mathfrak{g}, \mathfrak{h})$ is called *reductive* if there is an $\mathrm{Ad}(H)$ submodule $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.[1]

The Euclidean model is reductive, while the projective model is not.

A generalization of the notion of Klein geometry is:

Definition 2.7. A *locally Klein geometry* is a triple (Γ, G, H) where
- G is a Lie group
- $H \subset G$ is a closed subgroup
- $\Gamma \subset G$ is a subgroup acting by left multiplication on G/H as a group of covering transformations with $\Gamma \backslash G/H$ connected

The *space* of the locally Klein geometry is $\Gamma \backslash G/H$.

Definition 2.8. We say the locally Klein geometry is *geometrically oriented*[2] if G is connected.

3. The Maurer-Cartan Form

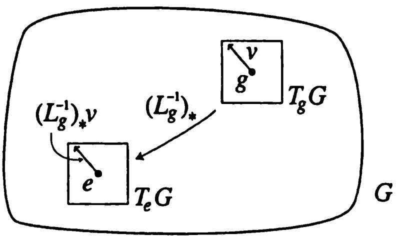
In this section we give properties of Lie groups that allow us to give a rough[3] characterization of homogeneous spaces. This sets the stage for the definition of Cartan geometries in §3. These geometries will be defined by dropping a two of the conditions characterizing homogeneous spaces.

Definition 3.1. The *(left) Maurer-Cartan form* on G is a $\mathfrak{g}$-valued 1-form $\omega_G : TG \to \mathfrak{g}$ defined by

$$\omega_G(v) = (L_g^{-1})_* v \quad \text{where} \quad v \in T_g G$$

where $L_g : G \to G$ denotes left multiplication by g (i.e., $\gamma \mapsto g\gamma$.)

The geometry of this situation is pictured in the following diagram.



The main facts about the Maurer-Cartan form are:
1. *Relation to* ξ_v^G. If $v \in \mathfrak{g} = T_e G$ then $\omega(\xi_v^G) = v$ everywhere.
2. *Invariance.* $L_g^* \omega = \omega$ and $R_g^* \omega = \mathrm{Ad}(g^{-1})\omega$ (where $R_g \gamma = \gamma g$).
3. *Regularity.* $\omega_G : TG \to \mathfrak{g}$ is an isomorphism for all $g \in G$.

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[1] Of course such a decomposition, even if it exists, will not always be unique. The reader may consider the example $\mathfrak{g} = \mathfrak{so}_n \oplus \mathfrak{so}_n$, with $\mathfrak{h} \subset \mathfrak{g}$ the diagonal copy of $\mathfrak{so}_n$. Then we have many choices for $\mathfrak{p}$, for example $\mathfrak{p}_1 = \mathfrak{so}_n \oplus 0$ and $\mathfrak{p}_2 = 0 \oplus \mathfrak{so}_n$.

[2] This notion will be generalized in definition 5.2.

[3] The roughness will be removed in §6.

4. *Completeness.* For any smooth function $f : \mathbb{R} \to \mathfrak{g}$, the vector field X_f on $\mathbb{R} \times G$ given by $(X_f)_{(t,g)} = (\partial_t, \omega_G^{-1} f(t)|_g)$ is complete[4].
5. *Structural equation, or "flatness".* $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$, (Note that the 2-form $[\omega_G, \omega_G]$ is defined by $\frac{1}{2}[\omega_G, \omega_G](u, v) = [\omega_G(u), \omega_G(v)]$.)
6. *"Darboux derivative".* If $f : M \to G$ is any smooth map, then the $\mathfrak{g}$-valued 1-form on M given by $\omega_f \stackrel{def}{=} f^*\omega_G$ also satisfies a version of the structural equation, namely $d\omega_f + \frac{1}{2}[\omega_f, \omega_f] = 0$.
7. *Fundamental theorem of calculus.* If $\omega : TM \to \mathfrak{g}$ is *any* 1-form on M satisfying the structural equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$, then there exists a locally defined function $f : M \to G$ such that $\omega = f^*\omega_G$. Moreover f is uniquely defined up to left translation by a constant element of G. (If M is simply connected, then f is globally defined.)
8. *Monodromy.* Let M and ω be as in 7. Let $\pi : \widetilde{M} \to M$ denotes the universal cover of M, with group of covering transformations Γ. Then $\pi^*\omega$ still satisfies the structural equation and we get a map $\tilde{f} : \widetilde{M} \to G$ satisfying $\tilde{f}^*\omega_G = \pi^*\omega$. If we fix a point $x_0 \in \widetilde{M}$ then the map $\Gamma \to G$ mapping $\gamma \mapsto \tilde{f}(\gamma x_0)\tilde{f}(x_0)^{-1}$ is a homomorphism of groups called the *monodromy representation.*

These facts lead to

Theorem 3.2 (Characterization of Lie groups). *Let P be a connected smooth manifold and let $\mathfrak{g}$ be a Lie algebra. Let ω be a $\mathfrak{g}$-valued 1-form on P satisfying the conditions*

(α) *$\omega : TP \to \mathfrak{g}$ is an isomorphism on each fibre.*
(β) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.
(γ) *ω is complete.*

Then

(a) *The universal cover $\pi : G \to P$ has, for any choice of $e \in G$, the structure of a Lie group with identity e, Lie algebra $\mathfrak{g}$, and Maurer-Cartan form $\pi^*\omega$.*
(b) *The group of covering transformations of $\pi : G \to P$ is the subgroup of G which is the image of the monodromy representation.*

In the spirit of this characterization of Lie groups we may roughly[5] describe a Klein geometry as consisting of:

(A) a principal bundle $H \to P \to M$, equipped with
(B) a $\mathfrak{g}$-valued 1-form ω on P

having the properties

(i) $\omega : TP \to \mathfrak{g}$ is an isomorphism on each fibre.
(ii) $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$
(iii) for each $v \in \mathfrak{h}$ we have $\omega(v^\dagger) = v$
(iv) $d\omega + \frac{1}{2}[\omega, \omega] = 0$

[4]A simpler, but probably more stringent, definition of completeness would say merely that $\omega^{-1}(v)$ is a complete vector field on G for every $v \in \mathfrak{g}$. This definition appears to be too weak to allow a proof of theorem 6.3.

[5]Compare this with theorem 6.3, which gives a complete characterization of locally homogeneous spaces.

(v) ω is complete.

Of course properties (ii) and (iii) follow from the others and so are redundant in this formulation. Note that property (iii) says that the form ω restricts to the Maurer-Cartan form of H on each fibre, and the property (ii), when evaluated on vectors tangent to the fibres, is a consequence of this.

4. Cartan Geometries

Cartan geometries are defined by throwing away conditions (iv) and (v) in the rough characterization of Klein geometries given in the last section. The effect of this is to remove the homogeneity except in the fibre direction.

Definition 4.1. A *Cartan geometry* (P, ω) *on* M *modeled on* $(\mathfrak{g}, \mathfrak{h})$ *with group* H consists of

- a principal bundle $H \to P \to M$.
- a $\mathfrak{g}$-valued 1-form ω on P called the Cartan connection.[6]

satisfying

(i) $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism for each $p \in P$.
(ii) $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$.
(iii) for each $v \in \mathfrak{h}$ we have $\omega(v^\dagger) = v$.

Associated to this data we have the following basic notions:

1. (*the curvature*) $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ a $\mathfrak{g}$-valued 2-form on P. This form is *semi-basic* in the sense that $\Omega_p(u, v) = 0$ if one of $u, v \in T_pP$ is tangent to the fibre pH. Under the right action of H on P it transforms according to $R_h^*\Omega = \mathrm{Ad}(h^{-1})\Omega$. By 3.2, the curvature form is the complete local obstruction to P being a Lie group.
2. (*the curvature function*) $K : P \to \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$ defined by

$$K(p)(u, v) = \Omega_p(\omega^{-1}(u), \omega^{-1}(v))$$

Note that $K \equiv 0 \Leftrightarrow \Omega \equiv 0$. The curvature function transforms according to $K(ph) = \mathrm{Ad}(h^{-1})K(p)$ where, if $\phi \in \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$, then

$$(\mathrm{Ad}(h)\phi)(u \wedge v) = \mathrm{Ad}(h)\,\phi\big(\mathrm{Ad}(h^{-1})u \wedge \mathrm{Ad}(h^{-1})v\big)$$

3. A Cartan geometry (P, ω) is called *complete* if, for any smooth function $f : \mathbb{R} \to \mathfrak{g}$, the vector field X_f on $\mathbb{R} \times P$ given by $(X_f)_{(t,g)} = (\partial_t, \omega_G^{-1} f(t)|_g)$ is complete.
4. The *torsion* of the Cartan geometry is the curvature form with values taken in $\mathfrak{g}/\mathfrak{h}$. Perhaps the most interesting cases of Cartan geometries are the ones for which the torsion vanishes, i.e., the curvature form takes values in $\mathfrak{h}$ (equivalently, the curvature function takes values in $\mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h})$.)

[6]The more familiar Ehresmann connection takes its values in the Lie subalgebra $\mathfrak{h}$.

5. The map $\varphi_p : T_{\pi(p)} \to \mathfrak{g}/\mathfrak{h}$ defined by the commutative diagram
$$\begin{array}{rcl} T_p(pH) & \underset{\approx}{\xrightarrow{\omega_H}} & \mathfrak{h} \\ \downarrow & & \downarrow \\ T_pP & \underset{\approx}{\xrightarrow{\omega_p}} & \mathfrak{g} \\ \pi_{*p}\downarrow & & \downarrow \\ T_{\pi(p)}M & \underset{\approx}{\xrightarrow{\varphi_p}} & \mathfrak{g}/\mathfrak{h} \end{array}$$
is an isomorphism satisfying $\varphi_{ph}(v) = \mathrm{Ad}(h^{-1})\varphi_p(v)$ for all $v \in T_{\pi(p)}M$.
6. Two Cartan geometries (P_i, ω_i) on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with group H are *geometrically isomorphic* if there is a bundle map $f : P_1 \to P_2$ covering the identity, i.e.
$$\begin{array}{rcl} P_1 & \underset{\approx}{\xrightarrow{f}} & P_2 \\ \pi_1\downarrow & & \downarrow\pi_2 \\ M & \overset{\mathrm{id}}{=} & M \end{array},$$
and satisfying $f^*\omega_2 = \omega_1$.
7. We call a Cartan geometry *reductive* if the model Klein pair $(\mathfrak{g}, \mathfrak{h})$ is reductive (i.e., there is an H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ given). In this case we call a vector-valued function f on P *covariant constant* if $v^\dagger f = 0$ for every $v \in \mathfrak{p}$.
8. A *gauge* on an open set $U \subset M$ is a section $\sigma : U \to \pi^{-1}(U)$. The $\mathfrak{g}$-valued 1-form $\eta = \sigma^*\omega$ on U is called an infinitesimal gauge, or simply a gauge, since it generally determines the original section[7]. The gauge σ determines a trivialization $\psi : U \times H \to \pi^{-1}(U)$ according to the formula $\psi(u,h) = \sigma(u) \cdot h$. In fact the formula
$$\psi^*\omega_{(x,h)} = \mathrm{Ad}(h^{-1})\pi_U^*\eta + \pi_H^*\omega_H$$
(where $\pi_U : U \times H \to U$ and $\pi_H : U \times H \to H$ are the canonical projections) shows that the infinitesimal gauge η determines Cartan connection on $\pi^{-1}(U)$. This gives an alternative way (the physicist's way) to describe a Cartan geometry.
9. If $v \in \mathfrak{g}$ then $\omega^{-1}(v)$ is a vector field on P (it corresponds to a left invariant field in the case of a homogeneous space). The projections to M of its integral curves are called *generalized circles* on M. If the geometry is reductive, say $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, the generalized circles corresponding to vectors $v \in \mathfrak{p}$ are called geodesics.
10. The Bianchi identity is $d\Omega = [\Omega, \omega]$. In the case of a reductive geometry with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ the Bianchi identity splits up into its $\mathfrak{h}$ component and its $\mathfrak{p}$ component. (In Riemannian geometry these are called the first and second Bianchi identities.)

Example 4.2. *Riemannian geometry.* Let M be a Riemannian manifold. The metric determines a canonical $O_n(\mathbb{R})$ bundle $P \to M$, the bundle of orthonormal frames on M. The relation between Riemannian geometry and the Cartan geometries with model $\mathbb{R}^n = \mathrm{Euc}_n(\mathbb{R})/O_n(\mathbb{R})$ is Levi-Civita's theorem, which may be read as saying:

> "Up to geometric isomorphism, there is a unique torsion free Cartan connection ω on the orthonormal frame bundle P over M such that φ_p is an isometry for all p. Conversely, the Cartan geometry determines the metric up to a constant scale factor."

[7]This is true if $\{h \in H \mid \mathrm{Ad}(h) = \mathrm{id}_{\mathfrak{g}}\} = \{e\}$.

The final statement may be seen as follows. Since $H = O_n(\mathbb{R})$ is compact and acts transitively on the one-dimensional subspaces of $\mathfrak{g}/\mathfrak{h}$, there is a unique (up to scale) positive definite quadratic form q on $\mathfrak{g}/\mathfrak{h}$ invariant under the adjoint action of H. Fix $x \in M$ and choose $p \in \pi^{-1}(x)$. It follows that the quadratic form $q \circ \varphi_p : T_xM \to \mathbb{R}$ is independent of the choice of $p \in \pi^{-1}(x)$ so we recover a unique Riemannian geometry from the Cartan geometry.

Remark 4.3. A consequence of this example is that Cartan geometries generalize Riemannian geometry.

We may also note that, once an orthonormal basis $\underline{e} = (e_1, \ldots, e_n)$ for $\mathfrak{g}/\mathfrak{h}$ is chosen, the prinicipal bundle P for a Cartan geometry (P, ω) on M modeled on Euclidean geometry may be canonically regarded as the bundle of orthonormal frames over M as follows. Fix $x \in M$. To every $p \in \pi^{-1}(x) \subset P$ we associate the orthonormal basis $\varphi_p^{-1}(\underline{e})$ for T_xM. Since $\varphi_{ph}^{-1}(\underline{e}) = \varphi_p^{-1}(Ad(h)\underline{e})$, every othonormal frame for T_xM arises in this manner, and the correspondence is bijective.

Let us see how this works in the case of a Riemannian surface M. The Cartan connection on P has the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega_1 & 0 & -\omega_{21} \\ \omega_2 & \omega_{21} & 0 \end{pmatrix}$$

If $x = \pi(p)$ then $\pi^*\varphi_p = (\omega_1, \omega_2)$. The Levi-Civita connection is ω_{12}. The curvature is

$$\Omega = d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & 0 & 0 \\ d\omega_1 - \omega_{21} \wedge \omega_2 & 0 & -d\omega_{21} \\ d\omega_2 + \omega_{21} \wedge \omega_1 & d\omega_{21} & 0 \end{pmatrix}$$

The torsion, which is $\Omega \pmod{\mathfrak{h}}$, is given by the two forms $d\omega_1 - \omega_{21} \wedge \omega_2$ and $d\omega_2 + \omega_{21} \wedge \omega_1$. These vanish in the Levi-Civita case, and then the curvature consists of the single form $d\omega_{12}$ which may be written as $d\omega_{21} = K\omega_1 \wedge \omega_2$, where $K : P \to \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}) = \mathbb{R}$ is the curvature function[8] Since the adjoint actions of $H = O_2(\mathbb{R})$ on $\mathfrak{h}$ and on $\lambda^2(\mathfrak{g}/\mathfrak{h})$ are trivial, the curvature function $K : P \to \mathbb{R}$ transforms trivially. Thus it is the lift of a function $K : M \to \mathbb{R}$, which is the Gauss curvature.

Example 4.4. *Projective geometry.* Even though projective geometry is not reductive, there is a notion of geodesic. The generalized circle corresponding to an element $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a geodesic if and only if $c \neq 0$ and c is an eigenvector for d. Suppose that M is a manifold with a collection of smooth paths $c : I \to M$ on it which are the solutions of an ODE having, in some coordinate system, the form:

$$\frac{\ddot{c}_1 + P_1(\dot{c})}{\dot{c}_1} = \frac{\ddot{c}_2 + P_2(\dot{c})}{\dot{c}_2} = \cdots = \frac{\ddot{c}_n + P_n(\dot{c})}{\dot{c}_n}$$

where the functions $P_j : TM \to \mathbb{R}$ are quadratic on each fibre. Then there is a unique torsion-free normal[9] projective geometry on M whose geodesics are the given curves.

There is a more general notion than geometric isomorphism by which two Cartan geometries may be related.

[8] We identify $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^2$ and $\mathfrak{h} = \mathbb{R}$ in the canonical ways.

[9] We do not give the definition for "normal" here, but see [4], p. 342.

Definition 4.5. Let $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h})$ be two models with group H. A *mutation* is an $\mathrm{Ad}(H)$ module isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ satisfying

(i) $\phi|\mathfrak{h} = \mathrm{id}_{\mathfrak{h}}$

(ii) for all $u, v \in \mathfrak{g}$ we have $[\phi(u), \phi(v)] = \phi([u, v]) \pmod{\mathfrak{h}}$

We say that the model $(\mathfrak{g}', \mathfrak{h})$ with group H is a mutation of the model $(\mathfrak{g}, \mathfrak{h})$ with group H. The mutation is *trivial*[10] if $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra isomorphism.

In the presences of a nontrivial mutation $\phi : (\mathfrak{g}, \mathfrak{h}) \to (\mathfrak{g}', \mathfrak{h})$ any Cartan geometry (P, ω) on M modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H determines a Cartan geometry (P, ω') (with $\omega' = \phi\omega$) on M modelled on $(\mathfrak{g}', \mathfrak{h})$ with group H. Moreover, since ϕ is a vector space isomorphism, we can make this change *without any loss of information.* However the curvature changes according to the formula

$$\Omega' = \phi\Omega + \tfrac{1}{2}([\omega', \omega'] - \phi[\omega, \omega])$$

Consider for example the mutations

$$\begin{array}{ccccc} \mathfrak{g}_{\text{hyperbolic}} & \to & \mathfrak{g}_{\text{Euclidean}} & \to & \mathfrak{g}_{\text{spherical}} \\ \begin{pmatrix} 0 & v^T \\ v & A \end{pmatrix} & \mapsto & \begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix} & \mapsto & \begin{pmatrix} 0 & -v^T \\ v & A \end{pmatrix} \end{array}$$

relating the hyperbolic $(O(n,1)/O(n))$, Euclidean $(\mathrm{Euc}(n)/O(n))$, and spherical $(O(n+1)/O(n))$ models. For example, in this context the unit sphere may be regarded as a Riemannian manifold of constant positive curvature, or equivalently, under mutation, as a flat spherical geometry.

Remark 4.6. In the reductive case $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ we may define $\mathfrak{g}' = \mathfrak{h} \oplus \mathfrak{p}'$, where $\mathfrak{p}' = \mathfrak{p}$ as an $\mathrm{Ad}H$ module, but the multiplication is changed on $\mathfrak{p}'$ to give zero (i.e., $[\mathfrak{p}', \mathfrak{p}'] = 0$). Then $\mathfrak{g}'$ is still a Lie algebra and the identity map $\mathfrak{g} \to \mathfrak{g}'$ is a mutation. This means that for reductive Cartan geometries modelled on Klein pairs $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{h} \oplus \mathfrak{p}, \mathfrak{h})$ one may as well assume that $[\mathfrak{p}, \mathfrak{p}] = 0$.

5. Development and Geometrically Oriented Cartan Geometries

Given a Cartan geometry (P, ω) on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with group H, we do not have the group G ($\mathfrak{g}$ only determines its identity component and then only up to covering) but we have a kind of substitute for G in $Gl(\mathfrak{g})$, with Lie algebra $\mathrm{End}(\mathfrak{g})$. Moreover the $\mathfrak{g}$-valued connection form ω determines the form $\mathrm{ad}\,\omega$, with values in $\mathrm{End}(\mathfrak{g})$, where for $v \in T_pP$,

$$(\mathrm{ad}\,\omega)_p(v) \in \mathrm{End}(\mathfrak{g}) \text{ maps } w \in \mathfrak{g} \mapsto [\omega_p(v), w].$$

Definition 5.1. Let $\sigma : [a, b] \to P$. The development of σ on $Gl(\mathfrak{g})$, via $\mathrm{ad}\,\omega$, starting at $g \in Gl(\mathfrak{g})$, is the unique curve[11] $\widehat{\sigma} : [a, b] \to Gl(\mathfrak{g})$ satisfying

- $\widehat{\sigma}(a) = g$
- $\widehat{\sigma}^* \omega_{Gl(\mathfrak{g})} = \sigma^* \mathrm{ad}\,\omega$

[10] DePaepe [2] has shown that primitive higher order models have no non-trivial mutations.

[11] Existence and uniqueness of $\widehat{\sigma}$ depends on the fundamental theorem of calculus.

Definition 5.2. A Cartan geometry (P,ω) on M modeled on $(\mathfrak{g},\mathfrak{h})$ with group H is called *geometrically oriented* if for every point $p \in P$ and every $h \in H$ there is a smooth path $\sigma : [a,b] \to P$ with $\sigma(a) = p, \sigma(b) = ph$ such that its development $\widehat{\sigma} : [a,b] \to Gl(\mathfrak{g})$ joins $e \in Gl(\mathfrak{g})$ to $\mathrm{Ad}(h) \in Gl(\mathfrak{g})$.

Proposition 5.3. *If H is connected then (P,ω) is geometrically oriented.*

6. Complete Flat Cartan Geometries

Definition 6.1. A Cartan geometry (P,ω) on M modelled on $(\mathfrak{g},\mathfrak{h})$ with group H is *flat* if the curvature form Ω vanishes or, equivalently, if the curvature function K vanishes.

Example 6.2. Klein geometries G/H, and more generally locally Klein geometries $\Gamma\backslash G/H$, are examples of flat Cartan geometries. These examples are also complete. If G is connected they are also geometrically oriented.

Theorem 6.3. *Let ξ be a complete, flat, geometrically oriented Cartan geometry (P,ω) on M modeled on $(\mathfrak{g},\mathfrak{h})$ with group H. Then there is a connected Lie group G with Lie algebra $\mathfrak{g}$ such that*

(a) *G contains H as a closed subgroup,*

(b) *G contains a discrete subgroup Γ acting by left mutiplication as a group of covering transformations on G/H.*

(c) *$(\Gamma\backslash G, \omega_{\Gamma\backslash G})$ is a locally Klein geometry on $\Gamma\backslash G/H$ geometrically isomorphic to ξ.*

In particular $M = \Gamma\backslash G/H$.

This theorem, together with remark 4.3, may be regarded as a full justification for the the statement made in the introduction that Cartan geometries constitute a common generalization of Riemannian and Klein geometries. Thus a Cartan geometry may be regarded as a curved version of its model Klein geometry, or as non-Euclidean analogue of Riemannian geometry.

7. Cartan Space Forms

Is it possible for a Cartan geometry to have a constant non-zero curvature function? The answer depends on the nature of the H module $\mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$ whose module structure $\phi \mapsto h\cdot\phi$ is given by

$$(h\cdot\phi)(u\wedge v) = \mathrm{Ad}(h)\phi(\mathrm{Ad}(h^{-1})u \wedge \mathrm{Ad}(h^{-1})v)$$

Thus

$$\left\{ \begin{array}{c} h\cdot\phi = \phi \\ \text{for all} \quad h \in H \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \mathrm{Ad}(h)(\phi(u\wedge v)) = \phi(\mathrm{Ad}(h)u \wedge \mathrm{Ad}(h)v) \\ \text{for all} \quad h \in H \end{array} \right\}$$

We denote by $\mathrm{Hom}_H(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$ the subspace of all $\phi \in \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$ satisfying this condition. Because K transforms according to the formula $K(ph) = h^{-1}\cdot K(p)$ a necessary condition for a non-zero, torsion free, constant curvature is that $\mathrm{Hom}_H(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h}) \neq 0$.

Example 7.1. In Euclidean geometry, $\dim \mathrm{Hom}_H(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h}) = 1$. On the other hand, for projective geometry, $\mathrm{Hom}_H(\lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h}) = 0$.

Definition 7.2. A Cartan geometry (P, ω) on M modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H has *constant curvature* if the curvature function

$$K : P \to \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$$

is constant.

Definition 7.3. A *Cartan space form* is Cartan geometry which is complete, torsion-free, geometrically oriented, and of constant curvature.

Of course the locally Klein geometries $\Gamma\backslash G/H$, with vanishing curvature, are trivial examples of Cartan space forms. The following result shows that, up to mutation, these examples consititute all possibilities (up to some technical restrictions which may, in fact, be unnecessary.)

Theorem 7.4. *Let ξ be a Cartan space form (P, ω) on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with group H. Let $K \in Hom(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$ be the value of the constant curvature function and define $[u,v]' = [u,v] - K(u \wedge v)$. Then*

(a) *The bracket $[-,-]'$ equips $\mathfrak{g}$ with the stucture of a Lie algebra, which we denote by $\mathfrak{g}'$.*
(b) $\mathrm{Ad}(h)[u,v]' = [\mathrm{Ad}(h)u, \mathrm{Ad}(h)v]'$ *for all $u, v \in \mathfrak{g}'$*

Now assume that either (i) *H is connected or* (ii) *M is simply connected. Then*

(c) *There is a realization G' of $\mathfrak{g}'$ such that $H \subset G'$ is a closed subgroup and ξ is a mutation of a locally Klein geometry $\Gamma\backslash G'/H$.*

Remark 7.5. If geometric orientability is invariant under mutation then the conditions (i) and (ii) are unnecessary in this theorem.

8. Locally Symmetric Spaces

In this section we take M to be a manifold and $\xi = (P, \omega)$ to be a *reductive* Cartan geometry on M modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H. We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ is an H submodule.

Definition 8.1. ξ is called *locally symmetric* if the curvature function

$$K : P \to \mathrm{Hom}(\lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$$

is covariant constant.

This definition includes what are usually called symmetric spaces, but is more general since, for example, there is no requirement for the group H to be compact.

Theorem 8.2. *Suppose that ξ is complete, torsion free and locally symmetric, and that M is a connected and simply-connected manifold. Fix $p_0 \in P$ and set $K_0 = K(p_0)$.*

- *Let H_0 be the identity component of the group $\{h \in H \mid K(p_0 h) = K_0)\}$. Then H_0 is a Lie subgroup of H (with Lie algebra $\mathfrak{h}_0 \subset \mathfrak{h}$, say.)*
- *Let P_0 be the path component of $\{p \in P \mid K(p) = K_0\}$ containing p_0. Then P_0 is a H_0-principal bundle over M.*
- *Let $\omega_0 = \omega | P_0$. Then ω_0 takes values in $\mathfrak{h}_0 \oplus \mathfrak{p}$*
- *(P_0, ω_0) is a Cartan space form on M.*

In particular, by theorem 7.4, $M = G'/H_0$ where G' has Lie algebra $\mathfrak{g}' = \mathfrak{h}_0 \oplus \mathfrak{p}$ as an H module, and the Lie bracket on $\mathfrak{g}'$ is given by $[u,v]' = [u,v] - K_0(u \wedge v)$

From this it is clear that the universal cover of a complete, torsion free, locally symmetric space is always a Klein geometry. It would be interesting to base a discussion of Riemannian symmetric spaces on the point of view given here, but we have not done this.

9. Stäckel Metrics

A classical question in Riemannain geometry is

> "Is it possible for a Riemannian metric g to have the same (unparametrized) geodesics as some other metric $\overline{g}$?"[12]

The first examples were given by U. Dini in 1869 on surfaces. A study of this question in the surface case appears as chapter 3 in Darboux's treatise of 1894 [1]. Dini's work was generalized to the case of arbitrary dimensions by Stäckel in 1893 [5]. A 20th century treatment of this may be found in [3].

Definition 9.1. A Stäckel metric is a Riemannian metric which, in some coordinate system $x = (x_1, \dots, x_n)$ assume the form

$$g = \sum_{i=1}^{n} A_i \, dx_i^2$$

where $A_i = \prod_{j \neq i} |f_j^2 - f_i^2|$ and f_i is a function of the x_i coordinate only.

Our treatment of this classical question is based on the comparison of the Riemannian and projective geometries which we now consider.

n dim. Euclidean model	n dim. projective model
$\mathfrak{g} = \left\{\begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix}\right\}, H = \left\{\begin{pmatrix} 1 & 0 \\ 0 & O_n \end{pmatrix}\right\}$ with $A^T = -A$ and $v \in \mathbb{R}^n$	$\widetilde{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{R}), \widetilde{H} = \left\{\begin{pmatrix} \Delta & b \\ 0 & A \end{pmatrix}\right\}$ with $\Delta \det A = 1$
n dimensional Riem. geometry	n dimensional proj. geometry
Given an orthonormal coframe θ on $U \subset M$, we get a unique torsion-free gauge on U of the form $\eta = \begin{pmatrix} 0 & 0 \\ \theta & \alpha \end{pmatrix}$.	Given an arbitrary coframe $\overline{\theta}$ on $U \subset M$, we get a unique torsion -free normal gauge on U of the form $\widetilde{\eta} = \begin{pmatrix} 0 & \widetilde{v} \\ \widetilde{\theta} & \widetilde{\alpha} \end{pmatrix}$

The path geometry on the Riemannian M arising from its geodesics determines a canonical torsion-free normal projective geometry on M according to the correspondence[13]

$$\eta = \begin{pmatrix} 0 & 0 \\ \theta & \alpha \end{pmatrix} \mapsto \widetilde{\eta} = \begin{pmatrix} 0 & \widetilde{v} \\ \theta & \alpha \end{pmatrix}$$

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[12] Of course we exclude the trivial case of the metric $\overline{g} = cg$ for c constant.

[13] See [4], p 350.

Definition 9.2. Quadratic forms Q and $\overline{Q}$ on the real vector space W are called *strongly distinct* if there is no 2-plane $V \subset W$ such that $\overline{Q}|V = cQ|V$ for $c \in \mathbb{R}$.

This definition is associated with the following obvious lemma.

Lemma 9.3. If Q and $\overline{Q}$ are strongly distinct then there are numbers $0 < q_1 < q_2 < \cdots < q_n$ and vectors $e_1, \cdots, e_n \in W$ such that

- $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis of W for Q.
- $\{q_1 e_1, q_2 e_2, \cdots, q_n e_n\}$ is an orthonormal basis of W for $\overline{Q}$.

Moreover the q_i are unique and the e_i are unique up to independent changes of sign.

Definition 9.4. We say that Riemannian metrics g and $\overline{g}$ on M are *strongly distinct* if they are strongly distinct at each point of M.

Theorem 9.5. If g and $\overline{g}$ are strongly distinct metrics on M, with the same (unparametrized) geodesics, then each point $p \in M$ lies in a coordinate neighborhood (U, x) on which the metrics assumes the form:

$$g = \sum_{i=1}^{n} A_i \, dx_i^2, \quad \overline{g} = \sum_{i=1}^{n} A_i \left(\frac{dx_i}{f_i \det f} \right)^2$$

where f_i is a function of x_i alone, $f = \prod_{i=1}^{n} f_i$, and $A_i(x) = \prod_{j \neq i} |f_j^2 - f_i^2|$. Conversely, any metrics g and $\overline{g}$ of this form have the same unparametrized geodesics.

Proof. We omit the proof of the converse, showing only that if g and $\overline{g}$ have the same geodesics then they have the given form. By lemma 9.3 we may choose (uniquely, up to signs) orthonormal frames θ and $\overline{\theta}$, for g and $\overline{g}$ on some neighborhood V_1 of the given point p such that $\overline{\theta}_i = q_i^{-1} \theta_i$ for some unique smooth functions q_i satisfying $q_1 < q_2 < \cdots < q_n$. The corresponding Riemannian torsion-free gauges

$$\eta = \begin{pmatrix} 0 & 0 \\ \theta & \alpha \end{pmatrix} \text{ and } \overline{\eta} = \begin{pmatrix} 0 & 0 \\ \overline{\theta} & \overline{\alpha} \end{pmatrix}$$

(with α and $\overline{\alpha}$ skew-symmetric) determine, respectively, torsion-free normal projective gauges

$$\Phi = \begin{pmatrix} 0 & v \\ \theta & \alpha \end{pmatrix} \text{ and } \overline{\Phi} = \begin{pmatrix} 0 & \overline{v} \\ \overline{\theta} & \overline{\alpha} \end{pmatrix}$$

which are gauge equivalent since the geodesics are the same. This means that there is a smooth map

$$h = \begin{pmatrix} \Delta & b \\ 0 & f \end{pmatrix} : U \to H \text{ such that } \overline{\Phi} = \mathrm{Ad}(h^{-1})\Phi + h^* \omega_H$$

or, in more detail,

$$\left.\begin{aligned} \begin{pmatrix} 0 & \overline{v} \\ \overline{\theta} & \overline{\alpha} \end{pmatrix} &= \begin{pmatrix} \Delta^{-1} & -b\Delta^{-1}f^{-1} \\ 0 & f^{-1} \end{pmatrix} \begin{pmatrix} 0 & v \\ \theta & \alpha \end{pmatrix} \begin{pmatrix} \Delta & b \\ 0 & f \end{pmatrix} \\ &\quad + \begin{pmatrix} \Delta^{-1} & -\Delta^{-1}bf^{-1} \\ 0 & f^{-1} \end{pmatrix} \begin{pmatrix} d\Delta & db \\ 0 & df \end{pmatrix} \end{aligned}\right\} \qquad (*)$$

Equating the (2,1) blocks of $(*)$ gives $\overline{\theta} = f^{-1}\theta\Delta$ which, since $\overline{\theta}_i = q_i^{-1}\theta_i$, implies that f is diagonal. We may write $f = \operatorname{diag}(f_1, f_2, \ldots, f_n)$, so that $f_i = \Delta q_i$. Since $\Delta \det f = 1$, the previous equation implies

$$q_i = f_i \det f \tag{1}$$

In particular, $q_1 < q_2 < \cdots < q_n \Rightarrow f_1 < f_2 < \cdots < f_n$. Equating the (2,2) blocks of $(*)$ gives $\overline{\alpha} = f^{-1}\theta b + f^{-1}\alpha f + f^{-1}df$. Considering the diagonal and off diagonal entries separately, we get

$$0 = f_i^{-1}\theta_i b_i + f_i^{-1}df_i, \text{ for } 1 \le i \le n \tag{2}$$

$$\overline{\alpha}_{ij} = f_i^{-1}\theta_i b_j + f_i^{-1}\alpha_{ij} f_j, \text{ for } 1 \le i \ne j \le n \tag{3}$$

Equation (2) is equivalent to

$$df_i = -b_i\theta_i, \text{ for } 1 \le i \le n \tag{4}$$

Symmetrizing equation (3) yields

$$\overline{\alpha}_{ij} + \overline{\alpha}_{ji} = f_i^{-1}\theta_i b_j + f_j^{-1}\theta_j b_i + f_i^{-1}\alpha_{ij} f_j + f_j^{-1}\alpha_{ji} f_i$$

which is

$$0 = f_i^{-1}\theta_i b_j + f_j^{-1}\theta_j b_i + (f_i^{-1} f_j - f_j^{-1} f_i)\alpha_{ij}$$

yielding

$$\alpha_{ij} = \frac{f_j b_j \theta_i + f_i b_i \theta_j}{f_i^2 - f_j^2} \tag{5}$$

The condition that Φ is torsion-free is $d\theta_i + \sum_{j \ne i} \alpha_{ij} \wedge \theta_j = 0$ for $1 \le i \le n$. Substituting (5) into this equation yields

$$d\theta_i + \theta_i \wedge \left(\sum_{j \ne i} \frac{f_j b_j}{f_i^2 - f_j^2} \theta_j \right) = 0 \tag{6}$$

which shows that the distributions $\theta_i = 0$ are integrable for $i = 1, 2 \ldots, n$. Thus there are functions $y_1, y_2, \ldots, y_n, r_1, r_2, \ldots, r_n$ defined on some neighborhood V_2 of our point p such that $\theta_i = r_i dy_i$. Since the θ_i form a coframe, it follows that the r_i are never zero and that the y_i constitute a coordinate system on some neighborhood $U \subset V_2$ of our point p. Substituting $\theta_i = r_i dy_i$ and $b_j\theta_j = -df_j$ into equation (6) yields

$$dr_i \wedge dy_i + r_i dy_i \wedge \left(\sum_{j \ne i} \frac{f_j}{f_j^2 - f_i^2} df_j \right) = 0$$

or

$$dy_i \wedge \left(-\frac{dr_i}{r_i} + \frac{1}{2} \sum_{j \ne i} \frac{d(f_j^2)}{f_j^2 - f_i^2} \right) = 0$$

Since $dy_i \wedge df_i = dy_i \wedge (-b_i\theta_i) = -b_i dy_i \wedge r_i dy_i = 0$, we see that f_i is a function of y_i alone. So the equation above may be written

$$dy_i \wedge \left(-d \ln |r_i| + \frac{1}{2} \sum_{j \ne i} d \ln |f_j^2 - f_i^2| \right) = 0$$

or

$$d\ln|r_i| - \frac{1}{2}\sum_{j\neq i} d\ln|f_j^2 - f_i^2| = 0 \ (\text{mod } dy_i)$$

or more simply as $d\ln(|r_i|A_i^{-\frac{1}{2}}) = 0$ (mod dy_i), where $A_i = \prod_{j\neq i}|f_j^2 - f_i^2|$. Define $u_i = |r_i|A_i^{-\frac{1}{2}}$. Then $r_i^2 = A_i u_i^2$, u_i never zero and, since $d\ln u_i = 0$ (mod dy_i), u_i is a function of y_i alone. Thus we may make choose new coordinates x, satisfying $dx_i = u_i\, dy_i$, and f_i is a function of x_i alone. Now we may express the metrics as

$$g = \sum_{i=1}^{n}\theta_i^2 = \sum_{i=1}^{n} r_i^2 dy_i^2 = \sum_{i=1}^{n} A_i u_i^2 dy_i^2 = \sum_{i=1}^{n} A_i\, dx_i^2$$

$$\overline{g} = \sum_{i=1}^{n}\overline{\theta}_i^2 = \sum_{i=1}^{n} q_i^{-2}\theta_i^2 = \sum_{i=1}^{n} q_i^{-2} A_i\, dx_i^2 = \sum_{i=1}^{n} A_i\left(\frac{dx_i}{f_i \det f}\right)^2 \qquad \square$$

Remark 9.6. In the special case of surfaces these metrics are called Liouville metrics and they assume the attractive form

$$g = (U - V)(dx^2 + dy^2), \quad \overline{g} = \left(\frac{1}{V} - \frac{1}{U}\right)\left(\frac{dx^2}{U} + \frac{dy^2}{V}\right)$$

where $U = U(x)$ and $V = V(y)$.

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