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ON SINGULARITIES OF ALMOST PERIODIC FUNCTIONS

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ABSTRACT. In this paper we consider nondegenerate critical points of almost periodic functions. We present a new explicit method of constructing almost periodic functions in order to show that their critical points need not in general have a well-defined distribution density in the domain.

1. INTRODUCTION

The links between crystallography and the theory of periodic functions are quite evident because every physical quantity that can be observed in a crystal must be translationally invariant – just in the same way as the crystal itself.

If we are not dealing with a perfect crystal it seems natural to consider functions that are 'not exactly periodic'. In fact many different classes of such functions have been introduced so far and it is not always quite clear which one is the most appropriate for a specific purpose. For instance several authors have investigated connections between quasicrystals and quasiperiodic functions (see [7] for some details and further references).

S.M. Gusein-Zade considered in [5] the metric density of critical points of quasiperiodic potentials, which are connected with chaotic behaviour leading to quasicrystallic structures. His main result can be outlined as follows.

Let $U : \mathbf{R}^k \to \mathbf{R}$ be a quasiperiodic function, that is $U = f \circ \rho$, where $\rho : \mathbf{R}^k \to \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$, $k < n$ is a linear function such that the image $\rho(\mathbf{R}^k)$ is dense in the torus $\mathbf{T}^n$, and $f : \mathbf{T}^n \to \mathbf{R}$ is analytic or smooth. Let $N_{i,r}(u)$ be the number of nondegenerate critical points of index i ($i = 0, 1, \dots, k$) with critical values less than u inside the ball $B_r = \{x \in \mathbf{R}^k : \|x\| < r\}$.

Proposition 1.1. *For every analytic (or almost every smooth) quasiperiodic function U the limit*

$$N_i(u) = \lim_{r \to +\infty} \frac{N_{i,r}(u)}{\mathrm{volume}(B_r)}$$

exists for all $u \in \mathbf{R}$.

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In this paper we shall discuss the behaviour of critical points in a broader class of functions, namely almost periodic functions. Although almost periodic functions have been studied very extensively in connection with partial differential equations and dynamical systems (and – to a much lesser extent – quasicrystals), it seems that not much is known about singularities of such functions. In the beginning we want to check if the derivative of an almost periodic function is always almost periodic. The answer is generally negative (finding a counterexample is one of the results of the present paper) but positive under some additional assumptions.

In Section 2 some classical facts concerning almost periodic functions are presented, in Section 3 a new method of constructing such functions is developed, and finally in Section 4 this method is applied twice: we construct the counterexample mentioned in the previous paragraph and an almost periodic function that does not have the density of critical points as defined in [5].

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2. Almost periodic functions

The notion of almost periodicity was originally introduced by H. Bohr for functions $f : \mathbf{R} \to \mathbf{C}$ (see [2]) and extended by S. Bochner to the class of functions from $\mathbf{R}^k$ into an arbitrary metric space (see [1]) but in the present paper we shall restrict ourselves to real functions of one real variable.

Definition 2.1. *A continuous function $f : \mathbf{R} \to \mathbf{R}$ is called* **almost periodic** *if for every $\varepsilon > 0$ there exists an $l = l(\varepsilon) > 0$ such that every interval $[a, b]$ of length $b - a = l$ contains a number τ for which*

$$\forall_{x \in \mathbf{R}} \, |f(x+\tau) - f(x)| \leq \varepsilon \, .$$

*Every τ for which the above condition holds is called an ε–***quasiperiod**.

Let us state several basic properties of almost periodic functions.

Proposition 2.1. *Every almost periodic function f is bounded and uniformly continuous.*

Proposition 2.2. *If f is an almost periodic function then for any $h \in \mathbf{R}$ the function $\tilde{f}(x) = f(x+h)$ is almost periodic.*

Proposition 2.3. *If f and g are almost periodic then so are $a \cdot f$ for any $a \in \mathbf{R}$ and $f + g$.*

Proposition 2.4. *If a sequence $f_n : \mathbf{R} \to \mathbf{R}$ of almost periodic functions converges uniformly on $\mathbf{R}$ then its limit f is almost periodic.*

Proofs of the above results can be found in many classical textbooks, e.g. [4] and [6]. Section I.2 of [4] presents also two different definitions of almost periodicity and shows their equivalence to Bohr's original Definition 2.1. The reader can also consult the article [3] for more details.

Proposition 2.5. *Let f be a differentiable almost periodic function. If its derivative f' is uniformly continuous then f' is also almost periodic.*

Proof. Fix $\varepsilon > 0$. Then the uniform continuity of f' means that for some positive δ we have $|f'(x+\tau) - f'(x)| \le \varepsilon$ whenever $|\tau| \le \delta$. Thus for $n \ge \frac{1}{\delta}$ we obtain

$$|n[f(x+\frac{1}{n}) - f(x)] - f'(x)| = |n\int_0^{\frac{1}{n}}[f'(x+\tau) - f'(x)]d\tau| \le$$
$$\le n\int_0^{\frac{1}{n}}|f'(x+\tau) - f'(x)|d\tau \le n\int_0^{\frac{1}{n}}\varepsilon d\tau = \varepsilon\,.$$

This inequality means that the sequence $n[f(x+\frac{1}{n}) - f(x)]$ converges uniformly to f'. The result follows immediately from Propositions 2.1 to 2.4. □

Theorem 4.1 below shows that the uniform continuity condition on f' is essential even if f is smooth. Since a function with bounded derivative must be uniformly continuous, we get at once

Corollary 2.1. *If f is an almost periodic function with bounded second derivative then f' is almost periodic.*

By obvious induction we prove the following

Corollary 2.2. *If f is almost periodic and all derivatives of f are bounded then they are almost periodic functions.*

3. Construction of composite almost periodic functions

In this section we present a method enabling us to construct an almost periodic function with prescribed properties by glueing parts defined on bounded intervals (for this reason we use the term 'composite').

First we shall define a function $s : \mathbf{Z} \to \mathbf{N}$ or in other words a sequence of natural numbers infinite in both directions. The construction relies on the binary representation of integers. For this purpose observe that if we try to subtract formally a natural number n written in the binary system from 0, we can obtain a binary representation of a negative number $-n$ (e.g. $-1 = \ldots 1111, -2 = \ldots 1110, -3 = \ldots 1101, -4 = \ldots 1100, -5 = \ldots 1011$, where dots stand for an infinite sequence of digits 1). In other words we get $-n$ from $n-1$ by replacing each binary digit with its negation. For any $n \in \mathbf{Z}$ we define s_n in the following way:

$s_n = m$ whenever the binary representation of n coincides with $\ldots 01010$ at last $m-1$ positions, but differs at m-th (here dots mean the infinite alternating sequence of digits, which obviously does not represent any integer).

More explicitly:

- $s_n = 1$ whenever n ends with 1, i.e. for $n = 2k+1, k \in \mathbf{Z}$
- $s_n = 2$ whenever n ends with 00, i.e. for $n = 4k, k \in \mathbf{Z}$
- $s_n = 3$ whenever n ends with 110, i.e. for $n = 8k+6, k \in \mathbf{Z}$
- $s_n = 4$ whenever n ends with 0010, i.e. for $n = 16k+2, k \in \mathbf{Z}$, etc.

For instance the part of this sequence from s_{-8} to s_7 looks as follows:

$$(\ldots, 2, 1, 5, 1, 2, 1, 3, 1, s_0 = 2, 1, 4, 1, 2, 1, 3, 1, \ldots).$$

Remark 3.1. *We shall further need the observation that in the set of 2^{l+1} elements ranging from s_{-2^l} to s_{2^l-1} there are 2^l equal to 1, 2^{l-1} equal to 2, ..., two numbers l, one occurrence of $l+1$ and one of $l+2$.*

Proposition 3.1. *Let k be divisible by $2^l, l \in \mathbf{N}$. Then for any $n \in \mathbf{Z}$ if $s_{n+k} \neq s_n$ then $s_{n+k} > l$ and $s_n > l$.*

Proof. Only the last m digits of the binary representation of n are important if we want to know whether $s_n = m$ or not. On the other hand n and $n+k$ have at least l common final digits, hence for $m \leq l$ the statement $s_n = m$ is equivalent to $s_{n+k} = m$. □

Theorem 3.1. *Let $h_n : [0,1] \to \mathbf{R}$ be a fundamental (Cauchy) sequence of continuous functions satisfying $h_n(0) = h_n(1) = 0$. Then the function $H : \mathbf{R} \to \mathbf{R}$ defined by:*

$$H(x) = h_{s_n}(x-n) \text{ for } x \in [n, n+1)$$

is almost periodic.

Proof. $H(x)$ is certainly continuous. Next fix a real number $\varepsilon > 0$. By our assumption there exists a number $l \in \mathbf{N}$ such that

$$\forall_{p,q>l} \forall_{x \in [0,1]} \, |h_p(x) - h_q(x)| \leq \varepsilon.$$

Now H is almost periodic because every integer k divisible by 2^l is an ε−quasiperiod. Indeed, if $x \in [n, n+1)$ then $x + k \in [n+k, n+k+1)$ and

$$|H(x+k) - H(x)| = |h_{s_{n+k}}(x-n) - h_{s_n}(x-n)| \leq \varepsilon,$$

since by Proposition 3.1 if $s_{n+k} \neq s_n$ then $s_{n+k} > l$ and $s_n > l$. □

It is easy to observe that the above procedure can be generalized to the case of k−dimensional domain. We can define $S : \mathbf{Z}^k \to \mathbf{N}$ as

$$S(n_1, \ldots, n_k) = \min(s_{n_1}, \ldots, s_{n_k})$$

and prove in a straightforward way analogues of Proposition 3.1 and Theorem 3.1 in order to obtain a method of constructing almost periodic functions $H : \mathbf{R}^k \to \mathbf{R}$.

4. Examples

Consider the function $f_1(x) = \exp(-1/x^2)\exp(-1/(1-x)^2)$. Assuming in addition that $f_1(x) = 0$ if $x \notin (0,1)$ we can easily see that $f_1 : \mathbf{R} \to \mathbf{R}$ is a smooth function with the property that $f_1^{(i)}(0) = f_1^{(i)}(1) = 0$ for all natural numbers i. Moreover, it has no critical points in $(0,1)$ beside a maximum at $\frac{1}{2}$ and its graph forms a bell-shaped curve. Since all derivatives of f_1 vanish at 0 and 1, we can safely glue two copies of this curve. Strictly speaking, we define by induction an infinite sequence of smooth functions:

$$f_{n+1}(x) = \begin{cases} f_n(2x) & \text{for } x \leq \frac{1}{2}, \\ f_n(2x-1) & \text{for } x > \frac{1}{2}. \end{cases}$$

For $i = 0, 1, \ldots$ we put

$$M_i = \sup_{x \in [0,1]} |f_1^{(i)}(x)|.$$

All numbers M_i are of course strictly positive finite reals.

Remark 4.1. *By simple induction we can check that*

$$\sup_{x \in [0,1]} |f_n^{(i)}(x)| = 2^{(n-1)i} M_i,$$

and in particular

$$\sup_{x\in[0,1]} |f_n(x)| = M_0.$$

Theorem 4.1. *There exists a smooth almost periodic function whose derivative is not almost periodic.*

Proof. To construct such a function it is sufficient to apply Theorem 3.1 to functions $h_n(x) = \frac{f_n(x)}{n}$. The sequence h_n is fundamental (by Remark 4.1 it converges to 0 uniformly on $[0,1]$) and Theorem 3.1 gives us an almost periodic function $H : \mathbf{R}\to\mathbf{R}$. To investigate its derivative observe that according to Remark 4.1

$$\sup_{x\in[0,1]} |h_n'(x)| = \frac{2^{n-1}M_1}{n},$$

which can be arbitrarily large as n grows. We have thus shown that H' is unbounded, which means that H' cannot be almost periodic by Proposition 2.1. □

Theorem 4.2. *There exists a smooth almost periodic function $H : \mathbf{R}\to\mathbf{R}$ for which there exists no density of critical points in the sense of Gusein-Zade even though all derivatives of H are almost periodic.*

Proof. We shall again apply Theorem 3.1 with $h_n(x) = \frac{f_n(x)}{n!}$ (the assumptions can be checked as above). By Remark 4.1

$$\sup_{x\in\mathbf{R}} |H^{(i)}(x)| = \sup_{n\in\mathbf{N}} \frac{2^{(n-1)i}}{n!} M_i$$

and this is finite because $\lim_{n\to+\infty} \frac{a^n}{n!} = 0$ for any positive real a. By Corollary 2.2 all derivatives of H are almost periodic. To count critical points observe that inside $(0,1)$ there are exactly 2^{n-1} points where f_n attains a local maximum. Therefore by Remark 3.1 total number of maximal points of H in the interval $[-2^l, 2^l]$ equals $2^l \cdot 1 + 2^{l-1} \cdot 2 + \ldots + 1 \cdot 2^l + 1 \cdot 2^{l+1} = (l+3)2^l$. Dividing this number by the length of $[-2^l, 2^l]$ we get $\frac{l+3}{2}$ which has no finite limit as l tends to infinity. □

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