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ON REGULARIZATION OF VARIATIONAL PROBLEMS IN FIRST-ORDER FIELD THEORY

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ABSTRACT. Standard Hamiltonian formulation of field theory is founded upon the Poincaré–Cartan form. Accordingly, a first-order Lagrangian L is called regular if $\det(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}) \neq 0$; in this case the Hamilton equations are equivalent with the Euler–Lagrange equations. Keeping the requirement on equivalence of the Hamilton and Euler–Lagrange equations as a (geometric) definition of regularity, and considering more general Lepagean equivalents of a Lagrangian than the Poincaré–Cartan equivalent, we obtain a regularity condition, depending not only on a Lagrangian but also on 2-contact parts of its Lepagean equivalents. In this way one gets a possibility to "regularize" many Lagrangian systems which are singular in the standard sense—this concerns e.g. all Lagrangians linear in the first derivatives of the field variables, among others the Dirac field Lagrangian. Also, with help of the present procedure, one can generate new regularity conditions for Lagrangians. Some examples of such regularity conditions, differing from the standard one, are stated explicitly.

1. INTRODUCTION

It is known that in field theory to a variational problem represented by a Lagrangian one can associate different Hamilton theories corresponding to different Lepagean equivalents of the Lagrangian (Dedecker [1], Gotay [3], Krupka [5]). Consequently, Hamilton equations depend upon a Lagrangian (resp. its Poincaré–Cartan form), *and* some *arbitrary* differential form corresponding to higher contact parts of the Lepagean equivalent of the Lagrangian. As pointed out by Dedecker [1], this admits a new approach to the problem of *regularity* in the calculus of variations, and leads to a *regularization procedure* based on a choice of an appropriate Lepagean equivalent of the given Lagrangian.

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Within the classical field theory, *regularity* of a variational problem associated with a first-order Lagrangian L depending on the "space-time variables" x^i, $1 \leq i \leq n$, "field variables" y^σ, $1 \leq \sigma \leq m$, and their "first derivatives" y_i^σ, is *identified* with the requirement

$$\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}\right) \neq 0\,, \tag{1.1}$$

ensuring the equivalence of the Euler-Lagrange and De Donder–Hamilton equations of L. Apparently, this condition is connected with the preference of a particular Lepagean form, the Poicaré–Cartan form of L. Unfortunately, it turns out that as a *definition* of regularity this condition is inappropriate, and its "direct" generalizations (e.g. to higher order variational problems on fibered manifolds, or to variational calculus on contact elements) can lead to confusion or even are impossible (cf. Dedecker [1], Krupka [6], Krupková [8], Saunders [10], and others). Moreover, almost all physically interesting Lagrangians in field theory do not satisfy the condition (1.1). The attempts to understand properly the concept of regularity resulted in different (non-equivalent) definitions, and, consequently, to different generalizations of the condition (1.1) (see e.g. Dedecker [1], Krupka [5], Krupka and Štěpánková [7], Saunders [9]).

In the present paper we follow the approach to Hamiltonian field theory due to Dedecker [1], Krupka [5] and Krupka and Štěpánková [7]. Namely, (1) Hamiltonian formulation is based upon the *Lepagean equivalents* of a Lagrangian, and (2) regulatity is understood to be a *bijective* correspondence between the set of extremals and Hamilton extremals.

It is known (Krupka [4]) that to every first-order Lagrangian there exists a family of Lepagean equivalents of the following form:

$$\rho = \theta + \nu\,, \tag{1.2}$$

where θ is the standard Poincaré–Cartan form, and ν is an (arbitrary) n-form of order of contactness ≥ 2. In particular, we study the case when the form ν is 2-contact, and arizing from a differential form defined on Y; in fibered coordinates the latter assumption means that the components $g^{ij}_{\sigma\nu}$ of ν are *independent of the* y^κ_l*'s*. Applying Krupka and Štěpánková definition of regularity [7], we find the following regularity condition

$$\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - 4g^{ij}_{\sigma\nu}\right) \neq 0\,, \tag{1.3}$$

involving both the Lagrangian and the 2-contact component of its Lepagean equivalent. This suggests a *regularization procedure*, based upon a proper choice of the g's such that the regularity condition (1.3) be satisfied. We apply this procedure to generate new regularity conditions for Lagrangians. Also, we investigate regularization of some interesting physical fields (the Dirac field, the electromagnetic field).

It should be pointed out that though our regularity condition (1.3) looks formally be the same as Dedecker's [1], its range of applications and meaning are different. First of all, the underlying manifold structures used are different: Dedecker develops the theory on contact elements, while we use fibered manifolds. This enables us, among others, to consider Hamilton theory based upon Lepagean equivalents of order of contactness up to 2, which is the first natural step in generalizing the De Donder-Hamilton equations. Moreover, compared to Dedecker's, the definition of regularity we use is *different* (stronger). For more information and recent results in Hamiltonian field theory concerning the problem of regularity for higher order Lagrangian systems we refer to [9].

2. Lepagean equivalents and regularity

Let us consider a fibered manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = m+n$, and its first (resp. second) jet prolongation $\pi_1 : J^1Y \to X$ (resp. $\pi_2 : J^2Y \to X$). A fibered chart on Y (resp. associated chart on J^1Y) is denoted by (V, ψ), $\psi = (x^i, y^\sigma)$ (resp. (V_1, ψ_1), $\psi_1 = (x^i, y^\sigma, y^\sigma_i)$). We use the following notations:

$$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \quad \omega_i = i_{\partial/\partial x^i}\omega_0, \quad \omega_{ij} = i_{\partial/\partial x^j}\omega_i\,,$$

and

$$\omega^\sigma = dy^\sigma - y^\sigma_j dx^j\,. \tag{2.1}$$

A section δ of the fibered manifold π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

Recall that every q-form η on J^1Y admits a unique (canonical) decomposition into a sum of q-forms on J^2Y as follows:

$$\pi^*_{2,1}\eta = h(\eta) + \sum_{k=1}^{q} p_k(\eta)\,,$$

where $\pi_{2,1}$ is the canonical projection $J^2Y \to J^1Y$, $h(\eta)$ is a horizontal form, called the horizontal part of η, and $p_k(\eta)$, $1 \leq k \leq q$, is a k-contact form, called the k-contact part of η (see e.g. [4] for review).

By a *first order Lagrangian* we shall mean a horizontal n-form λ on J^1Y. A form ρ is called a *Lepagean equivalent* of a Lagrangian λ if (up to a projection) $h(\rho) = \lambda$, and $p_1(d\rho)$ is a $\pi_{2,0}$-horizontal form [4]. For a first order Lagrangian we have all its Lepagean equivalents of order 1 characterized by the following formula

$$\rho = \theta_\lambda + \nu\,, \tag{2.2}$$

where θ_λ is the Poincaré–Cartan equivalent of λ and ν is an arbitrary n-form of order of contacness ≥ 2, i.e., such that $h(\nu) = p_1(\nu) = 0$. With the help of Lepagean equivalents of a Lagrangian one obtains an intrinsic formulation of the *Euler–Lagrange* and *Hamilton equations*, respectively [4], [5]:

A section γ of π is an *extremal* of λ if and only if

$$J^1\gamma^* i_{J^1\xi} d\rho = 0 \tag{2.3}$$

for every π-vertical vector field ξ on Y.

A section δ of the fibered manifold π_1 is called a *Hamilton extremal* of ρ if

$$\delta^* i_\xi d\rho = 0\,, \tag{2.4}$$

for every π_1-vertical vector field ξ on J^1Y.

Notice that while the Euler–Lagrange equations (2.3) are uniquelly determined by the Lagrangian, Hamilton equations (2.4) depend on the choice of ν. Consequently, one has many different Hamilton theories associated to a given variational problem.

Clearly, if γ is an extremal then $J^1\gamma$ is a Hamilton extremal; conversely, however, a Hamilton extremal need not be holonomic, and thus a jet prolongation of some extremal. This suggests a definition of regularity as follows:

Definition [7]. A Lepagean form is called *regular* if every its Hamilton extremal is holonomic.

In the sequel we shall consider Lepagean forms (2.2) where ν is 2-*contact*, and

$$\nu = p_2(\beta)\,,$$

where β is defined on Y and such that $p_i(\beta) = 0$ for all $i \geq 3$. In a fiber chart, where the Lagrangian λ is expressed by $\lambda = L\omega_0$, we can write

$$\rho = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j}\omega^\sigma \wedge \omega_j + g^{ij}_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}\,, \tag{2.5}$$

(summation over all sequences of indices) where the functions $g^{ij}_{\sigma\nu}$ do not depend on the y^κ_l's and satisfy the conditions

$$g^{ij}_{\sigma\nu} = -g^{ij}_{\nu\sigma}\,, \quad g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu}\,, \quad g^{ij}_{\sigma\nu} = g^{ji}_{\nu\sigma}\,. \tag{2.6}$$

Theorem. *Let λ be a first order Lagrangian, let $\lambda = L\omega_0$ be its expression in a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$ on Y. Let ρ be a Lepagean equivalent of λ of the form (2.5), (2.6). Assume that the matrix*

$$A^{ij}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma\nu}\right) \tag{2.7}$$

with rows (resp. columns) labelled by the pair (σ, i) (resp. (ν, j)), is regular.

Then ρ is regular, i.e., every Hamilton extremal δ of ρ is of the form $\delta = J^1\gamma$, where γ is an extremal of λ.

Proof. Expressing the Hamilton equations (2.4) in fiber coordinates we get along δ the following system of $m + mn$ first-order equations:

$$
\begin{aligned}
&\frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y_j^\sigma} + \Big(\frac{\partial^2 L}{\partial y^\sigma \partial y_j^\nu} - \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} + 4 d_k\, g_{\sigma\nu}^{jk}\Big)\Big(\frac{\partial y^\nu}{\partial x^j} - y_j^\nu\Big) \\
&\quad - \frac{\partial^2 L}{\partial y_k^\nu \partial y_j^\sigma}\Big(\frac{\partial y_k^\nu}{\partial x^j} - y_{kj}^\nu\Big) \\
&\quad + 2\Big(\frac{\partial g_{\nu\sigma}^{jk}}{\partial y^\varrho} + \frac{\partial g_{\varrho\nu}^{jk}}{\partial y^\sigma} + \frac{\partial g_{\sigma\varrho}^{jk}}{\partial y^\nu}\Big)\Big(\frac{\partial y^\varrho}{\partial x^j} - y_j^\varrho\Big)\Big(\frac{\partial y^\nu}{\partial x^k} - y_k^\nu\Big) = 0, \\
&A_{\sigma\nu}^{ij}\Big(\frac{\partial y^\nu}{\partial x^j} - y_j^\nu\Big) = 0 .
\end{aligned}
\tag{2.8}
$$

The matrix $A_{\sigma\nu}^{ij}$ is regular, hence the second set of the Hamilton equations gives

$$
\frac{\partial (y^\nu \circ \delta)}{\partial x^j} = y_j^\nu \circ \delta ,
\tag{2.9}
$$

i.e., $\delta = J^1\gamma$. Now the first set of Hamilton equations means that γ is an extremal. □

Taking into account the above theorem we can generalize the concept of a regular Lagrangian as follows:

Definition. Let $W \subset Y$ be open, $W \subset V$, where (V, ψ) is a fiber chart on Y, and let $\lambda = L\,\omega_0$ be a Lagrangian on $\pi_{1,0}^{-1}(V)$. We say that L is *regular* over W if there exist functions $g_{\sigma\nu}^{ij}$ on W satisfying (2.6) and the condition

$$
\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - 4 g_{\sigma\nu}^{ij}\right) \neq 0 .
\tag{2.10}
$$

If λ does not satisfy the standard regularity condition (1.1) but is regular in the sense of the above definition, we also say that λ is *regularizable* over W.

Corollary. *Let $m \geq 2$. Then every Lagrangian linear (affine) in the first derivatives is regularizable.*

3. Examples of regularity conditions for Lagrangians

If, in particular, in the condition (2.10) the functions $g_{\sigma\nu}^{ij}$ are expressed by means of the Lagrangian L, one obtains regularity conditions involving only the Lagrangian. We list some of these possibilities below; obviously many other conditions can be generated in a similar way.

(1) Suppose that $\partial g^{ij}_{\sigma\nu}/\partial y^\kappa_l = 0$, where

$$g^{ij}_{\sigma\nu} = \frac{1}{2}\Big(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - \frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_i}\Big).$$

Then the regularity condition for L is of the form

$$\det\left(2\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_i} - \frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j}\right) \neq 0. \tag{3.1}$$

(2) Suppose that $\partial g^{ij}_{\sigma\nu}/\partial y^\kappa_l = 0$, where

$$g^{ij}_{\sigma\nu} = \frac{1}{4}\Big(\frac{\partial^3 L}{\partial x^i \partial y^\sigma \partial y^\nu_j} - \frac{\partial^3 L}{\partial x^j \partial y^\sigma \partial y^\nu_i} - \frac{\partial^3 L}{\partial x^i \partial y^\nu \partial y^\sigma_j} + \frac{\partial^3 L}{\partial x^j \partial y^\nu \partial y^\sigma_i}\Big).$$

Then the regularity condition for L is of the form

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - \frac{\partial^3 L}{\partial x^i \partial y^\sigma \partial y^\nu_j} + \frac{\partial^3 L}{\partial x^j \partial y^\sigma \partial y^\nu_i} + \frac{\partial^3 L}{\partial x^i \partial y^\nu \partial y^\sigma_j} - \frac{\partial^3 L}{\partial x^j \partial y^\nu \partial y^\sigma_i}\right) \neq 0.$$

(3) Suppose that $\partial g^{ij}_{\sigma\nu}/\partial y^\kappa_l = 0$ and $\partial g^{ij}_{\sigma\nu}/\partial y^\kappa_{ls} = 0$, where

$$g^{ij}_{\sigma\nu} = \frac{1}{4}\Big(d_i\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_j} - d_j\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_i} - d_i\frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_j} + d_j\frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_i}\Big).$$

Then the regularity condition for L is of the form

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - d_i\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_j} + d_j\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_i} + d_i\frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_j} - d_j\frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_i}\right) \neq 0.$$

4. Regularizable Lagrangians in classical field theory

The following examples show that the Dirac field and the electromagnetic field, which are singular in the standard sense (i.e., not satisfying the regularity condition (1.1)), are regularizable. Hence, one obtains Hamilton equations which are equivalent with the Euler–Lagrange equations, without the need of using the theory of constraints (compare with Giachetta et al. [2]).

Dirac field in two dimensions. In this case the fiber dimension $m = 2$, and $\dim X = 2$. The Dirac Lagrangian is linear in the variables y^ν_i, hence *regularizable.* The conditions (2.6) on the functions $g^{ij}_{\sigma\nu}$ mean that we have only one independent free function g^{12}_{12}. Denote $u = u(x^j, y^\nu) = 4g^{12}_{12}$. Then for Lepagean equivalents of the Dirac Lagrangian of the form

$$\rho = L\,dx^1 \wedge dx^2 + \frac{\partial L}{\partial y^\sigma_i}\omega^\sigma \wedge \omega_i + u(x^j, y^\nu)\,\omega^1 \wedge \omega^2,$$

the regularity condition (2.10) reads

$$\det\begin{pmatrix} 0 & 0 & 0 & -u \\ 0 & 0 & u & 0 \\ 0 & u & 0 & 0 \\ -u & 0 & 0 & 0 \end{pmatrix} \neq 0 .$$

Consequently, for every function $u(x^j, y^\nu)$ (on an open subset W of Y) such that $u(x^j, y^\nu) \neq 0$ on W, the corresponding Hamilton equations are *equivalent* with the Euler–Lagrange equations.

Dirac field in four dimensions. Now $m = 2$, $\dim X = 4$, and we get 6 independent functions $g^{ij}_{\sigma\nu}$. Denote

$$u_1 = 4g^{12}_{12}, \quad u_2 = 4g^{13}_{12}, \quad u_3 = 4g^{14}_{12}, \quad u_4 = 4g^{23}_{12}, \quad u_5 = 4g^{24}_{12}, \quad u_6 = 4g^{34}_{12}.$$

The matrix (2.7) takes the form

$$\begin{pmatrix} 0 & -\mathrm{M} \\ \mathrm{M} & 0 \end{pmatrix},$$

where M is the 4×4 matrix

$$\begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_4 & u_5 \\ -u_2 & -u_4 & 0 & u_6 \\ -u_3 & -u_5 & -u_6 & 0 \end{pmatrix}.$$

We can see that for any choice of functions $u_k(x^i, y^\sigma)$, $1 \leq k \leq 6$, such that $\det \mathrm{M} \neq 0$ we obtain a regular Hamilton theory for the Dirac field, based upon the Lepagean form

$$\begin{aligned} \rho = L\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 &+ \frac{\partial L}{\partial y^\sigma_i} \omega^\sigma \wedge \omega_i \\ &+ u_1\, \omega^1 \wedge \omega^2 \wedge \omega_{12} + u_2\, \omega^1 \wedge \omega^2 \wedge \omega_{13} + u_3\, \omega^1 \wedge \omega^2 \wedge \omega_{14} \\ &+ u_4\, \omega^1 \wedge \omega^2 \wedge \omega_{23} + u_5\, \omega^1 \wedge \omega^2 \wedge \omega_{24} + u_6\, \omega^1 \wedge \omega^2 \wedge \omega_{34}. \end{aligned}$$

A simple admissible choice is e.g.

$$\rho = L\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + \frac{\partial L}{\partial y^\sigma_i} \omega^\sigma \wedge \omega_i + f\, \omega^1 \wedge \omega^2 \wedge \omega_{14} + g\, \omega^1 \wedge \omega^2 \wedge \omega_{23}$$

with $f, g \neq 0$.

Electromagnetic field. For the electromagnetic field Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(y^\sigma_\nu y^\nu_\sigma - \mathrm{g}^{\sigma\nu} \mathrm{g}_{\mu\rho}\, y^\mu_\sigma y^\rho_\nu) \tag{4.1}$$

(where $(\mathfrak{g}_{\sigma\nu})$ is the Lorentz metric, and $y^\sigma_\nu = \partial A^\sigma/\partial x^\nu$), the condition (1.1) gives

$$\det\Big(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j}\Big) = 0.$$

However, this Lagrangian is regularizable, and admits many regularizations.

If, in particular, $\dim X = 2$, we have $m = 2$, and one independent parameter $u = u(x^j, y^\nu) = 4g^{12}_{12}$. Hence, the matrix (2.7) is of the form

$$\begin{pmatrix} 0 & 0 & 0 & -u \\ 0 & 1 & u+1 & 0 \\ 0 & u+1 & 1 & 0 \\ -u & 0 & 0 & 0 \end{pmatrix},$$

i.e., it is regular for every function $u \neq 0, -2$ (cf. Dedecker [1]).

If $\dim X = m = 4$, the computations are more complicated (since we have 36 independent functions $g^{ij}_{\sigma\nu}$), but completely analogous to the preceeding case. Again, one obtains many possibilities for regularization of the problem.

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ON RECURRENCES ON THE STREAMS

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ABSTRACT. New kind of recurrences implicit in constructions of [2] and [3] "for sequences on streams" is defined. A peculiar set of solutions of such recurrences are Tchebyshev-like special functions introduced in [2] and [3] which are polynomials in coordinates of a certain curve on hypersurface in R^m determined by quasi-numbers of determinant one [4]. These Tchebyshev-like special functions are shown [5] to satisfy an ordinary m-th order recurrence with parameter dependent coefficients. If $m = 2$ then one gets classical Tchebyshev polynomials of both kinds.

I. INTRODUCTION

Let $m \in \mathbf{N}$ and $m > 1$. Let $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$ denotes the cyclic group. Let $\{h_j(\alpha)\}_{j \in \mathbf{Z}_m}$ be the family of hyperbolic functions of the m-th order [1]

$$(1) \qquad R \ni \alpha \to h_j(\alpha) = \frac{1}{m} \sum_{k \in s_m} \omega^{-kj} \exp(\omega^k \alpha); \qquad j \in \mathbf{Z}_m \quad \omega = \exp\left(i\frac{2\pi}{m}\right).$$

It is not difficult to establish that

$$(2) \qquad \frac{1}{m} \sum_{k \in \mathbf{Z}_m} h_j(\omega^k \alpha + \beta) = h_0(\alpha) h_j(\beta); \qquad j \in \mathbf{Z}_m,$$

due to "hyperbolic-trigonometric" properties [4] of the set $\{h_j(\alpha)\}_{j \in \mathbf{Z}_m}$ of these fundamental solutions of $\frac{d^m}{d\alpha^m} y\alpha = y\alpha$.

It was observed in [2], [3] that Tchebyshev-like special functions $T_{\vec{n}}^{(j)}(x)$; $j \in \mathbf{Z}_m$:

$$(3) \qquad T_{\vec{n}}^{(j)}(x) = h_j(\vec{n}\alpha)\,;\ j \in \mathbf{Z}_m\,;\ x = h_0(a)$$

where $\vec{n} \in \Lambda_m = \{n, n+\omega^1, n+\omega^2, \ldots, n+\omega^{m-1}; n \in \mathbf{Z}\}$ do satisfy the equation

$$(4) \qquad \frac{1}{m} \sum_{s \in \mathbf{Z}_m} T_{n+\omega^s}(x) = x T_n(x)\,; \qquad n \in \mathbf{Z}.$$

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For $m = 2$ equations (3) and (4) lead to classical Tchebyshev polynomials of the first and second type respectively: $T_n^{(0)}(x)$, $\frac{T_n^{(1)}(x)}{\sqrt{x^2-1}}$.

The case of $T_{\vec{n}}^{(0)}$; $\vec{n} \in \Lambda_m$ was considered in quite a detail in [2], [3], [1] where relation (4) was treated implicitly as supplemented by the de Moivre group [4] property of its solutions i.e. the following convolution formula was used throughout:

$$h_k(\alpha+\beta) = \sum_{j\in \mathbf{Z}_m} h_j(\alpha)h_{k-j}(\beta)\,; \qquad k \in \mathbf{Z}_m\,. \tag{5}$$

In view of (3) and with $\beta \in \Lambda(\mathbf{Z}_m)$ we may write the requirement of de Moivre property of the set $T_n^{(j)}(x)$; $j \in \mathbf{Z}_m$ of solutions of (4) in extended form as follows:

$$T_{\vec{n}+\vec{m}}^{(j)}(x) = \sum_{k\in \mathbf{Z}_m} T_{\vec{n}}^{(k)}(x)T_{\vec{m}}^{(j-k)}(x)\,; \qquad j \in \mathbf{Z}_m \qquad \vec{m},\vec{n} \in \Lambda(\mathbf{Z}_m) \tag{6}$$

where

$$\Lambda(\mathbf{Z}_m) = \left\{\vec{n}(\vec{k}) = \sum_{s\in \mathbf{Z}_m} k_s\omega^s; k_s \in \mathbf{Z}; s \in \mathbf{Z}_m\right\} \tag{7}$$

with $\vec{k} = (k_0, k_1, \ldots, k_{m-1})$ (see Figure 1).

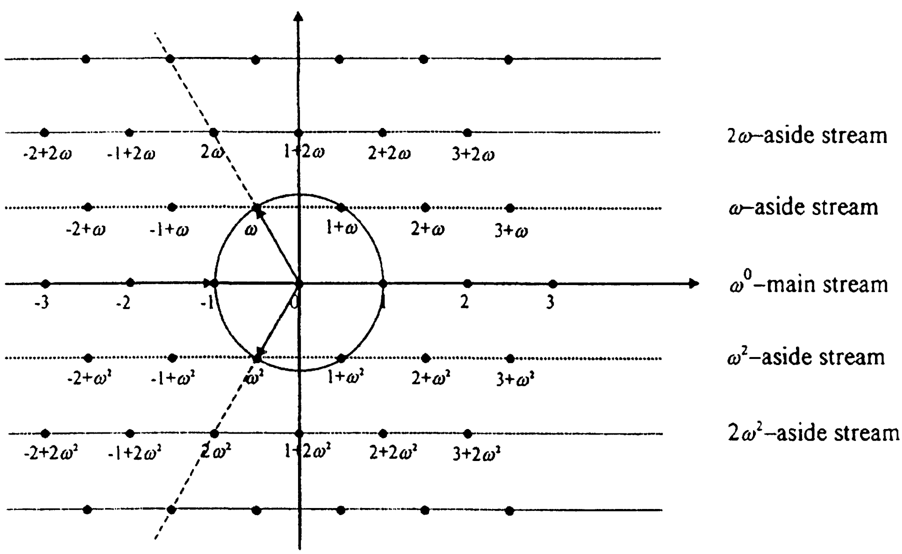

Figure 1
Neighbour streams for $m = 3$

Then in particular

$$T_{n+m}^{(j)}(x) = \sum_{k\in \mathbf{Z}_m} T_n^{(k)}(x)T_m^{(j-k)}(x)\,; \qquad j \in \mathbf{Z}_m \qquad m,n \in \mathbf{Z} \tag{8}$$

for the "main stream" (i.e. $\vec{n} = n \in \mathbf{Z}$) solution of (4).

Observation 1

Equation (4) solutions of the form $\{T_n^{(j)}(x)\}_{j\in\mathbf{Z}_m}$ are not the only one solutions unless the requirement (8) is added to (4). This observation is implicit in constructions of [2] and [3]. □

It is not difficult to recognize that

Observation 2

The solutions $\{T_n^{(j)}(x)\}_{j\in\mathbf{Z}_m}$ of equation (4) result from pseudo-characteristic equation of equation (4)

$$x = \frac{1}{m}\sum_{s\in\mathbf{Z}_m} \lambda^{\omega^s}(x) \tag{9}$$

(we are looking for some solutions of the form $T_{\vec{n}}(x) = \lambda^{\vec{n}}(x)$; $\vec{n} \in \Lambda_m$). □

Remark 1. It is easy to see that $\left\{\exp\{\omega^s\alpha\}\right\}_{s\in\mathbf{Z}_m}$ under the identification $x = h_0(\alpha)$.

Remark 2. We call (9) the "pseudo-characteristic" equation of equation (4) because of Observation 1.

Example. Consider $m = 3$ case and equation (4) without any additional requirements imposed on the set of its solutions - as for example the requirement (8). It is clear that in order to determine - say - $\{T_n(x)\}$ sequence ("the main stream sequence") two other: $\{T_{n+\omega}(x)\}$ and $\{T_{n+\omega^2}(x)\}$ sequences ("aside neighborhood stream sequences") must be given apriori (see - Fig.1).

Conclusion 1. Equation (4) is a relation resulting from a certain recurrence relation among "sequences on streams" $\{T_{\vec{n}}(x)\}_{\vec{n}\in\Lambda(\mathbf{Z}_m)}$ - the relation obtained by restriction of a recurrence to the main and "aside neighbor stream sequences". This is to be defined rigorously in what follows.

II. Recurrences on the streams- an example of Tchebyshev type

In this section we define a recurrence relation for sequences of objects which are sequences "on the streams" in $\Lambda(\mathbf{Z}_m)$ thus giving to the equation (4) a proper setting -see-conclusion at the end of this section.

Let us consider the set $\Lambda(\mathbf{Z}_m)$ defined by (7). We may define the following mappings

Definition 1. For $j \in \mathbf{Z}_m$ let $T^{(j)} : \Lambda(\mathbf{Z}_m) \times \mathbf{R} \to \mathbf{C}$; $T^{(j)}(\vec{n}, x) = h_j(\vec{n}\alpha)$ where $x = h_0(a)$.
We shall call $\{T^{(j)}\}$; $j \in \mathbf{Z}_m$ the Tchebyshev mappings.

Notation.

$$T^{(j)}(\vec{n}, x) = T_{\vec{n}}^{(j)}(x)\,; \qquad j \in \mathbf{Z}_m\,.$$

It is easy to see that

Observation 3

The special functions $\{T^{(j)}_{\vec{n}}(x)\}_{\vec{n}\in\Lambda(\mathbf{Z}_m)}$; $j \in \mathbf{Z}_m$ form the set of particular solutions of the equation

$$\frac{1}{m}\sum_{s\in\mathbf{Z}_m} T_{\vec{n}+\omega^s}(x) = xT_{\vec{n}}(x)\,; \quad \vec{n}(\vec{k}) \in \Lambda(\mathbf{Z}_m) \tag{10}$$

The linear equation (10) is a kind of recurrence for sequences on streams what we are now going to make precise. Let $\vec{n}(\vec{k}) \in \Lambda(\mathbf{Z}_m)$.

Definition 2. Let

$$S(k_1, k_2, \ldots, k_{m-1}) = \{\vec{n}(\vec{k}); k_0 \in Z\} \subset \Lambda(\mathbf{Z}_m)\,; \qquad k_1, k_2, \ldots, k_{m-1} \in \mathbf{Z}\,.$$

Then $S(k_1, k_2, \ldots, k_{m-1})$ is called a stream. $S(0,0,\ldots,0)$ is called the main stream. A stream $S(k_1, k_2, \ldots, k_{m-1}) \neq S(0,0,\ldots,0)$ is called an aside stream (see [2] and see Fig.1 for $m = 3$).

Definition 3. Let

$$r(m) = \begin{cases} m-4 & \text{for } m = 6k; k \in N\,; \\ m-2 & \text{for } m = 6k+2 \vee m = 6k+4; k \in N \cup \{0\}\,; \\ m-1 & \text{for } m = 2k+1; k \in N\,. \end{cases}$$

We shall call this $r : N \cup \{0\} \to N$ function the rank function of the equation (10).

It is not difficult to see (consult Figure 1 and draw corresponding pictures for other values of m) that the following holds:

Observation 4

For any given $m > 2$ and for any different $r(m)$-neighbour to the main one - aside streams $S_k; k = 1, 2, \ldots, r(m)$ with the sequences $T_1, T_2, \ldots, T_{r(m)}$ on them being given ("initial sequences") the solution of equation (10) is uniquely determined.

Name. We call the recurrence (10) the rank $r(m)$ recurrence on the streams as it is recurrence relation between sequences defined on streams i.e. objects indexed by streams.

Remark 3. The set of streams $S(k_1, k_2, \ldots, k_{m-1}) \equiv S(\vec{\kappa})$; $\vec{\kappa} = (k_1, k_2, \ldots, k_{m-1}) \in Z^{m-1}$; is linearly ordered by the following order relation $\leq$:

$$S(\vec{\kappa}) \leq S(\vec{\chi}) \equiv \vec{\kappa} \leq \vec{\chi} \;\text{ iff either }\; \mathrm{Im}\left\{\vec{n}(\vec{\kappa}) = \sum_{s\in\mathbf{Z}_m} \kappa_s\omega^s\right\} < \mathrm{Im}\left\{\vec{n}(\vec{\chi}) = \sum_{s\in\mathbf{Z}_m} \chi_s\omega^s\right\}$$

or - in the case of equality of the above imaginary parts - the order relation $\vec{\kappa} \leq \vec{\chi}$ is defined lexicographically. Because of this the phrase "$r(m)$ neighbour stream sequences" has appropriately established meaning.

Conclusion 2. Summarizing: the relation (4) is not a recurrence relation for infinite sequence of objects. It is a relation between $r(m)$ streams.

This relation results from the recurrence relation (10) via restricting this relation to the subset $\Lambda_m \subset \Lambda(\mathbf{Z}_m)$ i.e. via restricting relation to $r(m)$ neighbour stream sequences.

For $m = 2$ and only for $m = 2$ $\Lambda_m = \Lambda(\mathbf{Z}_m)$, which is the Tchebyshev polynomials standard case.

Information. Other "non-trigonometric" extensions of Tchebyshev polynomials to polynomials of several independent variables are known from rather scattered literature.

A summarizing paper on that subject is [6], which is recommended here. The author of [7] seems to rediscover the family of hyperbolic functions of the m-th order $\{h_j(\alpha)\}_{j\in\mathbf{Z}_m}$ and made a lot of new observations relevant to our approach. However the author of [7] had decided to pursue the "non-trigonometric" approach to extend the definition of standard Tchebyshev polynomials onto specific polynomials of several independent variables.

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