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ON LOCAL STABILITY OF DIFFERENTIAL $(n-1)$-FORMS ON AN n-MANIFOLD WITH BOUNDARY

WOJCIECH DOMITRZ

ABSTRACT. In this paper we consider classification of smooth differential $(n-1)$-forms on an n-manifold with boundary with respect to the equivalence defined by pullback via a diffeomorphism which preserves the manifold together with its boundary. We determine which smooth differential $(n-1)$-forms are locally stable on an n-manifold with boundary and find their local normal forms.

1. INTRODUCTION

Let M be a germ of a smooth manifold with boundary at $p \in \partial M$. For simplicity of notation, we take a germ at $0 \in R^n$ of the following set

$$M = \{(x_1, \ldots, x_n) \in R^n : x_1 \geq 0\}\,.$$

We will denote by G_M the group of diffeomorphism-germs $(R^n, 0) \to (R^n, 0)$ at 0, which preserve M. It is obvious that these diffeomorphism-germs preserve ∂M.

We define the natural equivalence of germs of differential k-forms on a manifold with boundary. Let α, β be germs of smooth k-forms on R^n at 0.

Definition 1. α, β are G_M-equivalent, if there exists $\Phi \in G_M$, such that

$$\Phi^*\alpha = \beta\,.$$

The analogue of Martinet's definition ([10]) for local stability of k-forms on a manifold with boundary is the following.

Definition 2. A differential k-form α is ∂M-stable at $x \in \partial M$, if for any neighbourhood U of x there is a neighbourhood V of α (in C^∞ topology of k-forms) such that if $\beta \in V$ then there are $y \in U$ and a diffeomorphism-germ

$$\Phi : (R^n, x) \to (R^n, y)\,,$$

which preserves M and

$$\Phi^*\beta = \alpha\,.$$

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If a k-form is ∂M-stable at $p \in \partial M$, then it is also stable at p in the sense of Martinet(on a manifold without boundary). Therefore, **there are no locally ∂M-stable germs of k-forms for $2 \leq k \leq \dim M - 2$ and there are no locally ∂M-stable germs of closed k-forms for $3 \leq k \leq \dim M - 2$** (see [10],[7],[9]).

Classification of differential k-forms is a classical problem (Darboux theorems). It was considered by many authors ([12], [10], [7], [8], [9], [13], [14], [15], [3], [4], [5]). Germs of smooth differential $(n-1)$-forms (on a manifold without boundary) were classified in [10], [7], [9]. Classification of germs of $(n-1)$-forms is given by the order of contact of the kernel $ker\alpha$ of the germ of the form α with its hypersurface of degeneration $\{d\alpha = 0\}$. In [3] germs of locally stable smooth differential $(n-1)$-forms on an n-dimensional manifold with boundary were classified provided the kernel of the $(n-1)$-form is transversal to the boundary. Now we also consider $(n-1)$-forms such that their kernels are tangent to the boundary. We show that classification of germs of locally stable $(n-1)$-forms are given by the order of contact of the kernel of the germ with its hypersurface of degeneration and the order of contact of the kernel of the germ with the boundary of the manifold. We also find normal forms of these germs.

In this paper all object are smooth (C^∞ differentiable).

The two following lemmas ([10]) will be useful.

Lemma 1. *Let τ be a k-form on R^n. If τ satisfies the following conditions : $\frac{\partial}{\partial x_1} \rfloor \tau = 0$, $\frac{\partial}{\partial x_1} \rfloor d\tau = 0$, then $\tau = \pi^* \iota^* \tau$, where*

$$\pi : R^n \to \{x_1 = 0\}, \quad \pi(x_1, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n),$$
$$\iota : \{x_1 = 0\} \to R^n, \quad \iota(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n).$$

Lemma 2. *Let τ be a k-form on R^n. If τ satisfies the following conditions: $\frac{\partial}{\partial x_1} \rfloor \tau = 0$, $\frac{\partial}{\partial x_1} \rfloor d\tau = \varphi\tau$, then $\tau = \zeta\pi^*\iota^*\tau$, where φ, ζ are smooth functions on R^n and $\zeta|_{\{x_1=0\}} = 1$.*

It is easy to prove the following (see [3])

Lemma 3. *Let α be germ of n-form on R^n at 0. If τ satisfies the following conditions:*

1. *$\alpha = g(x)dx_1 \wedge \cdots \wedge dx_n$, where g is a function-germ such that $g(0) = 0$, $dg_0 \neq 0$,*
2. *the germ of vector field $\frac{\partial}{\partial x_1}$ at 0 meets*

$$S = \{x \in R^n : g(x) = 0\}$$

transversally at 0,

then there exists $\Phi \in G_M$, $\Phi(x) = (\zeta(x), x_2, \ldots, x_n)$, such that

$$\Phi^*\alpha = \pm(x_1 - f(x_2, \ldots, x_n))dx_1 \wedge \cdots \wedge dx_n,$$

where ζ, f are function-germs on R^n.

2. Classification of n-forms on an n-manifold with boundary

In these section we present complete classification of germs of locally ∂M-stable smooth $(\dim M)$-forms. We find normal forms of these germs.

Let α be a germ of n-form on R^n at 0. It is easy to prove the two following propositions by standard Martinet's methods([10]).

Proposition 1. *If $\alpha_0 \neq 0$ then α is G_M-equivalent to $dx_1 \wedge \cdots \wedge dx_n$.*

Proposition 2. *If α satisfies the following conditions:*

1. $\alpha_0 = 0$,
2. $S = \{x \in R^n : \alpha_x = 0\}$ *is a germ of a regular hypersurface at* 0,
3. ∂M *meets S transversally at* 0,

then α is G_M-equivalent to

$$x_n dx_1 \wedge \cdots \wedge dx_n .$$

If α does not satisfy the assumption 3 of Proposition 2, then it is not ∂M-stable at 0.

Proposition 3. *If α satisfies the following conditions:*

1. $\alpha_0 = 0$,
2. $S = \{x \in R^n : \alpha_x = 0\}$ *is a germ of a regular hypersurface at* 0,
3. ∂M *is tangent to S at* 0,

then α is not ∂M-stable at 0.

Proof. From condition 3, $\frac{\partial}{\partial x_1}$ meets S transversally at 0. By Lemma 3 α is G_M-equivalent to

$$(x_1 - f(x_2, \ldots, x_n))dx_1 \wedge \cdots \wedge dx_n , \tag{1}$$

where f is a function-germ at 0, $f(0) = 0$ and 0 is a critical point of f.

By Sard's Theorem there is ϵ, which is regular value of f, in any neighbourhood of $f(0) = 0 \in R$. Let β denote the following n-form $\beta = \alpha + \epsilon dx_1 \wedge \cdots \wedge dx_n$. If α is ∂M-stable at 0 then there is a diffeomorphism Φ, which preserves M such that $\Phi^*\beta = \alpha$. The set $\{x \in R^n : \beta_x = 0\}$ is tangent to ∂M at $\Phi(0)$. This implies that $f(\Phi(0)) = \epsilon$ and $df_{\Phi(0)} = 0$, which is impossible. (see [3] for details). □

3. Classification of $(n-1)$-forms on an n-manifold with boundary

In this section we classify locally stable differential $(n-1)$-forms.

Let α be a germ of an $(n-1)$-form on R^n at 0. First we consider nondegenerate $(n-1)$-forms such that their kernels are transversal to the boundary.

Proposition 4. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 \neq 0$,
3. *a germ of a smooth vector field X at 0, which satisfies the following*

$$X \rfloor (d\alpha) = \alpha,$$

meets ∂M transversally at 0,

then α is G_M-equivalent to one and only one of the following two germs at 0

$$\alpha^{\pm} = (1 \pm x_1)dx_2 \wedge \cdots \wedge dx_n . \tag{2}$$

Proof. From condition 3 we can reduce X and α to the following form: $X = \pm\frac{\partial}{\partial x_1}$, $\iota_{\partial M}\alpha = dx_2 \wedge \cdots \wedge dx_n$, where $\iota_{\partial M} : \partial M \to R^n$. By the definition of X,

$$\frac{\partial}{\partial x_1} \rfloor d\alpha = \alpha . \tag{3}$$

According to Lemma 2, we have

$$\alpha = f(x)dx_2 \wedge \cdots \wedge dx_n\,, \tag{4}$$

where f is a smooth function on R^n such that $f|_{\partial M} = 1$. From Eq. 3 we have $\frac{\partial f}{\partial x_1} = \pm f$. Therefore $f = e^{\pm x_1}$ and it is easy to see that we can reduce α to Eq. 2.

Germs α^+ and α^- are not G_M-equivalent, because

$$d\alpha^+ = dx_1 \wedge \cdots \wedge dx_n\,,\ d\alpha^- = -dx_1 \wedge \cdots \wedge dx_n \tag{5}$$

and

$$\iota^*_{\partial M}\alpha^+ = dx_2 \wedge \cdots \wedge dx_n\,,\ \iota^*_{\partial M}\alpha^- = dx_2 \wedge \cdots \wedge dx_n \tag{6}$$

(see [3] for details). □

Now we consider nondegenerate $(n-1)$-forms such that their kernels are tangent to the boundary. We need the following lemma.

Lemma 4. *$\Phi : (R^n, 0) \to (R^n, 0)$ is a diffeomorphism-germ which preserves the germ of $(n-1)$-form $(1 + x_n)dx_1 \wedge \cdots \wedge dx_{n-1}$ at 0 if and only if Φ is a volume-preserving diffeomorphism of the form*

$$\Phi(x) = \left(\Psi(x_1, \ldots, x_{n-1}), \frac{1}{J(\Psi)(x_1, \ldots, x_{n-1})}x_n + \frac{1}{J(\Psi)(x_1, \ldots, x_{n-1})} - 1\right),$$

where $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ is a diffeomorphism-germ and $J(\Psi)$ is the determinant of its Jacobian.

Proof. Firstly we notice that $d\alpha = dx_n \wedge dx_1 \wedge \cdots \wedge dx_{n-1}$ must be preserved by Φ. So Φ is a volume-preserving diffeomorphism. A smooth vector field X such that $X\rfloor(d\alpha) = \alpha$, must be preserved by this diffeomorphism too. It is easy to see that $X = (1 + x_n)\frac{\partial}{\partial x_n}$. Therefore

$$\Phi(x) = (\Psi(x_1, \ldots, x_{n-1}), f(x)x_n + g(x_1, \ldots, x_{n-1}))\,,$$

where $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ is a diffeomorphism, $f : (R^n, 0) \to (R, 0)$ is a function such that $f(0) \neq 0$ and $g : (R^n, 0) \to (R, 0)$ is a function which does not depend on x_n. Therefore we get

$$(1 + f(x)x_n + g(x_1, \ldots, x_{n-1}))J(\Psi) = 1 + x_n\,.$$

and hence

$$g(x_1, \ldots, x_{n-1}) = 1/J(\Psi)(x_1, \ldots, x_{n-1}) - 1\,,\ f(x) = 1/J(\Psi)(x_1, \ldots, x_{n-1})\,. \quad \square$$

Let X be a vector field and let β be a germ of a function or a k-form. Let $\mathcal{L}^i_X\beta$ denote i Lie differentiations with respect to X. A vector field X is k-tangent at $y \in R^n$ to a regular hypersurface $\{x \in R^n : H(x) = 0\}$ if $\mathcal{L}^i_X H|_y = 0$ for $i = 1, \ldots, k$ and $\mathcal{L}^{k+1}_X H|_y \neq 0$.

We use the above lemma to prove the following result.

Theorem 1. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 \neq 0$,

3. *a germ of a smooth vector field X at 0, which satisfies the following*
$$X\rfloor(d\alpha) = \alpha\,,$$
is $(k-1)$-tangent to $\partial M = \{x \in R^n : x_1 = 0\}$ at 0 $(1 < k < n)$,
4. $dx_1 \wedge d(\mathcal{L}_X x_1) \wedge d(\mathcal{L}_X^2 x_1) \wedge \cdots \wedge d(\mathcal{L}_X^{k-3} x_1) \wedge d(\mathcal{L}_X^{k-2} x_1)|_0 \neq 0$

then α is G_M-equivalent to

(7) $(1 + x_n)\cdot$
$$\cdot\left(\pm dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} + \Big(x_n^{k-1} + \sum_{i=2}^{k-1} x_i x_n^{i-2}\Big) dx_n \wedge dx_2 \wedge \cdots \wedge dx_{n-1}\right).$$

Proof. By standard Martinet method we can find a diffeomorphism $\Phi : (R^n, 0) \to (R^n, 0)$ such that $\Phi^\star\alpha = (1 + x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and $\Phi^{-1}(\partial M) = \{x \in R^n : H(x) = 0\}$. It is easy to see that $X = (1 + x_n)\frac{\partial}{\partial x_n}$. From condition 3 we get $\frac{\partial^i H}{\partial x_n^i}|_0 = 0$ for $i = 1, \ldots, k-1$ and $\frac{\partial^k H}{\partial x_n^k}|_0 \neq 0$. By the Malgrange Preparation Theorem we have
$$\{x \in R^n : H(x) = 0\} = \left\{x \in R^n : x_n^k + \sum_{i=1}^{k} p_i(x_1, \ldots, x_{n-1}) x_n^{i-1} = 0\right\},$$
where p_i is a function-germ on R^n which does not depend on x_n for $i = 1, \ldots, k$.

Now we reduce the above hypersurface to the normal form by diffeomorphisms which preserve
$$(1 + x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}\,.$$
Firstly we reduce it to the form
$$\left\{x \in R^n : x_n^k + \sum_{i=1}^{k-1} q_i(x_1, \ldots, x_{n-1}) x_n^{i-1} = 0\right\}$$
by the diffeomorphism
$$\Theta(x) = \left(\int_0^{x_1} 1 - \frac{p_k(t, x_2, \ldots, x_{n-1})}{k} dt, x_2, \ldots, x_{n-1}, \frac{k}{k - p_k(x)}(x_n + 1) - 1\right).$$
By Lemma 4 this diffeomorphism preserves $(1 + x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$. From condition 4 we get that the mapping $Q = (q_1, \ldots, q_{k-1}) : (R^{n-1}, 0) \to (R^{k-1}, 0)$ is a submersion. If $k - 1 < n - 1$ then there exists a volume-preserving diffeomorphism $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ such that
$$Q \circ \Psi(x_1, \ldots, x_{n-1}) = \left(kx_1, kx_2, \frac{kx_3}{2}, \ldots, \frac{kx_{k-2}}{k-3}, \frac{kx_{k-1}}{k-2}\right).$$
Now by a diffeomorphism of the form $\Lambda(x_1, \ldots, x_n) = (\Psi(x_1, \ldots, x_{n-1}), x_n)$ which preserves $(1 + x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ we reduce the hypersurface to
$$\left\{x \in R^n : \frac{x_n^k}{k} + \sum_{i=2}^{k-1} \frac{x_i}{i-1} x_n^{i-1} + x_1 = 0\right\}.$$
Finally we use the diffeomorphism
$$\Gamma(x_1, \ldots, x_n) = \left(\frac{x_n^k}{k} + \sum_{i=2}^{k-1} \frac{x_i}{i-1} x_n^{i-1} \pm x_1, x_2, \ldots, x_n\right). \qquad \square$$

Now we prove that there are no more locally ∂M-stable nondegenerate $(n-1)$-forms.

Proposition 5. *If α satisfies assumptions 1, 2, 4 of Theorem 1 and a germ of a smooth vector field X at 0, which satisfies the following*

$$X\rfloor(d\alpha) = \alpha\,,$$

is $(n-1)$-tangent to $\partial M = \{x \in R^n : x_1 = 0\}$ at 0 then α is not ∂M-stable at 0.

Proof. We reduce α to the form $(1+x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and ∂M to the form

$$\left\{x \in R^n : x_n^n + \sum_{i=1}^{n-1} q_i(x_1, \ldots, x_{n-1})x_n^{i-1} = 0\right\}. \tag{8}$$

By Lemma 4 a diffeomorphism Φ which preserves $(1+x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and the form of ∂M (8) has the form

$$\Phi(x) = (\Psi(x_1, \ldots, x_{n-1}), x_n)\,,$$

where $\Psi : R^{n-1} \to R^{n-1}$ is a diffeomorphism which preserves $(n-1)$-form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$. It is easy to see that we cannot change the determinant of the Jacobian $J(q_1, \ldots, q_{n-1})$ by the action of these diffeomorphisms. □

Proposition 6. *If α satisfies assumptions 1, 2, 3 of Theorem 1 for $k > 2$ and α does not satisfy assumption 4 then α is not ∂M-stable at 0.*

Proof. We may assume that X is 2-tangent to ∂M at 0. Then we reduce α to the form $(1+x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and ∂M to the form

$$\left\{x \in R^n : x_n^3 + q_2(x_1, \ldots, x_{n-1})x_n + x_1 = 0\right\}, \tag{9}$$

where $\frac{\partial q_2}{\partial x_i}|_0 = 0$ for $i = 2, \ldots, n-1$, because α does not satisfy assumption 4. By Lemma 4 a diffeomorphism Φ which preserves $(1+x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and the form of ∂M (9) has the form

$$\Phi(x) = (x_1, \Psi(x_2, \ldots, x_{n-1}), x_n)\,,$$

where $\Psi : R^{n-1} \to R^{n-1}$ is a diffeomorphism which preserves $(n-2)$-form $dx_2 \wedge \cdots \wedge dx_{n-1}$. It is easy to see that we cannot change $Hess(q_2(0, x_2, \ldots, x_{n-1}))$-the determinant of the Hessian of $q_2(0, x_2, \ldots, x_{n-1})$ by the action of these diffeomorphisms. □

Now let us consider degenerate $(n-1)$-forms.

Proposition 7. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 = 0$,
3. $S = \{x \in R^n : d\alpha_x = 0\}$ *is a germ of a regular hypersurface at 0, which meets ∂M transversally at 0,*
4. *a germ of a smooth vector field X at 0, which satisfies the following*

$$X\rfloor\alpha = 0\,,\ X(0) \neq 0\,,$$

meets ∂M and S transversally at 0, *then α is G_M-equivalent to one and only one of the following two germs at* 0

$$(\pm\frac{1}{2}(x_1 - x_2)^2 + 1)dx_2 \wedge \cdots \wedge dx_n .$$

Proof. From condition 4 we can reduce X and S to the following forms $X = \pm\frac{\partial}{\partial x_1}$,

$$S = \{x \in R^n : x_1 = g(x_2, \dots, x_n)\} ,$$

where g is a smooth function on ∂M. Then α can be reduced to such form that $\iota_S\alpha = dx_2 \wedge \cdots \wedge dx_n$, where $\iota_S : S \hookrightarrow R^n$ by diffeomorphism $\Phi(x, y) = (x_1, \phi(x_2, \dots, x_n))$, which preserved the form of X. Hence by Lemma 2 we deduce that $\alpha = h(dx_2 \wedge \cdots \wedge dx_n)$, where h is a smooth function on R^n, $h|_S = 1$. On the other hand

$$(d\alpha) = \pm(x_1 - f)dx_1 \wedge \cdots \wedge dx_n ,$$

which follows from Lemma 3. Therefore $h = (\pm\frac{1}{2}(x_1 - f)^2 + 1)$.

Then α is G_M-equivalent to

$$\Big(\pm \frac{1}{2}(x_1 - f(x_2, \dots, x_n))^2 + 1\Big)dx_2 \wedge \cdots \wedge dx_n . \tag{10}$$

From condition 3 one can easily reduce α to the following form

$$\Big(\pm \frac{1}{2}(x_1 - x_2)^2 + 1\Big)dx_2 \wedge \cdots \wedge dx_n$$

(see [3] for details).

By the standard argument([10]), it is easy to prove that germs of the following $(n-1)$-forms at 0

$$\Big(\frac{1}{2}(x_1 - x_2)^2 + 1\Big)dx_2 \wedge \cdots \wedge dx_n ,$$

$$\Big(- \frac{1}{2}(x_1 - x_2)^2 + 1\Big)dx_2 \wedge \cdots \wedge dx_n$$

are not G_M-equivalent. □

If S does not meet ∂M transversally at 0, then α is not ∂M-stable at 0, because $d\alpha$ is not ∂M-stable at 0 (see Proposition 3).

Now we prove

Lemma 5. *$\Phi : (R^n, 0) \to (R^n, 0)$ is a diffeomorphism-germ which preserves the germ of $(n-1)$-form $(1 \pm x_n^2)dx_1 \wedge \cdots \wedge dx_{n-1}$ at 0 if and only if $\Phi(x) = (\Psi(x_1, \dots, x_{n-1}), \pm x_n)$, where $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ is a volume-preserving diffeomorphism-germ.*

Proof. Firstly we notice that

$$Ker\,\alpha = \{X : X \text{ is a smooth vector field and } X \rfloor \alpha = 0\}$$

is $\Big\{f\frac{\partial}{\partial x_n} : f \text{ is a smooth function }\Big\}$. $Ker\,\alpha$ must be preserved by this diffeomorphism. Therefore $\Phi(x) = (\Psi(x_1, \dots, x_{n-1}), \psi(x))$, where $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ is a diffeomorphism and $\psi : (R^n, 0) \to (R, 0)$ is a function such that $\frac{\partial \psi}{\partial x_n}|_0 \neq 0$. The hypersurface $S = \{x \in R^n : d\alpha_x = 0\} = \{x \in R^n : x_n = 0\}$ and the pullback of α to S must be preserved by Φ too. From this we get that Ψ preserves $dx_1 \wedge \cdots \wedge dx_{n-1}$ and finally that $\psi(x) = \pm x_n$. □

We need the above lemma to prove the following result of local ∂M-stability of degenerate $(n-1)$-forms such that their kernels are tangent to the boundary.

Theorem 2. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 = 0$,
3. $S = \{x \in R^n : d\alpha_x = 0\}$ *is a germ of a regular hypersurface at* 0,
4. *a germ of a smooth vector field X at 0, which satisfies the following*

$$X \rfloor \alpha = 0\,,\ X(0) \neq 0\,,$$

is $k-1$-tangent to ∂M at 0 $(1 < k < n-1)$,

5. $dx_1 \wedge d(\mathcal{L}_X x_1) \wedge d(\mathcal{L}_X^2 x_1) \wedge \cdots \wedge d(\mathcal{L}_X^{k-2} x_1) \wedge d(\mathcal{L}_X^{k-1} x_1)|_0 \neq 0$

then α is G_M-equivalent to

$$(\pm\frac{1}{2}(x_n)^2 + 1)(\pm dx_1 \wedge \cdots \wedge dx_{n-1} + (x_n^{k-1} + \sum_{i=2}^{k} x_i x_n^{i-2}) dx_n \wedge dx_2 \wedge \cdots \wedge dx_{n-1})\,.$$

Proof. By standard Martinet method we can find a diffeomorphism $\Phi : (R^n, 0) \to (R^n, 0)$ such that $\Phi^* \alpha = (1 \pm x_n^2/2) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and $\Phi^{-1}(\partial M) = \{x \in R^n : H(x) = 0\}$. It is easy to see that $X = f(x)\frac{\partial}{\partial x_n}$, where f is a function-germ at 0 such that $f(0) \neq 0$. From condition 3 we get $\frac{\partial^i H}{\partial x_n^i}|_0 = 0$ for $i = 1, \ldots, k-1$ and $\frac{\partial^k H}{\partial x_n^k}|_0 \neq 0$. By the Malgrange Preparation Theorem we have

$$\{x \in R^n : H(x) = 0\} = \left\{x \in R^n : x_n^k + \sum_{i=1}^{k} p_i(x_1, \ldots, x_{n-1}) x_n^{i-1} = 0\right\}\,,$$

where p_i is a function-germ on R^n which does not depend on x_n for $i = 1, \ldots, k$.

Now we reduce the above hypersurface to the normal form by diffeomorphisms which preserve

$$(1 \pm x_n^2/2) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}\,.$$

From condition 5 we get that the mapping $P = (p_1, \ldots, p_k) : (R^{n-1}, 0) \to (R^k, 0)$ is a submersion. If $k < n-1$ then there exists a volume preserving diffeomorphism $\Psi : (R^{n-1}, 0) \to (R^{n-1}, 0)$ such that

$$P \circ \Psi(x_1, \ldots, x_{n-1}) = (kx_1, kx_2, \frac{kx_3}{2}, \ldots, \frac{kx_{k-1}}{k-2}, \frac{kx_k}{k-1})\,.$$

Now by a diffeomorphism of the form $\Lambda(x_1, \ldots, x_n) = (\Psi(x_1, \ldots, x_{n-1}), x_n)$ which preserves $(1 \pm x_n^2/2) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ we reduce the hypersurface to

$$\left\{x \in R^n : \frac{x_n^k}{k} + \sum_{i=2}^{k} \frac{x_i}{i-1} x_n^{i-1} + x_1 = 0\right\}\,.$$

Finally we use the diffeomorphism

$$\Gamma(x_1, \ldots, x_n) = (\frac{x_n^k}{k} + \sum_{i=2}^{k} \frac{x_i}{i-1} x_n^{i-1} \pm x_1, x_2, \ldots, x_n)\,. \qquad \square$$

Now we show that assumptions 4 and 5 of Theorem 2 are necessary for local ∂M-stability of degenerate $(n-1)$-form.

Proposition 8. *If α satisfies assumptions 1, 2, 3, 5 of Theorem 2 and a germ of a smooth vector field X at 0, which satisfies the following*

$$X\rfloor(d\alpha) = \alpha\,,$$

is $(n-2)$-tangent to $\partial M = \{x \in R^n : x_1 = 0\}$ at 0 then α is not ∂M-stable.

Proof. We reduce α to the form $(1 \pm x_n^2/2)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and ∂M to the form $\left\{x \in R^n : x_n^{n-1} + \sum_{i=1}^{n-1} q_i(x_1, \ldots, x_{n-1})x_n^{i-1} = 0\right\}$. By Lemma 5 a diffeomorphism Φ which preserves $(1 + x_n^2)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ has the form

$$\Phi(x) = (\Psi(x_1, \ldots, x_{n-1}), \pm x_n)\,,$$

where $\Psi : R^{n-1} \to R^{n-1}$ is a diffeomorphism which preserves $(n-1)$-form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$. It is easy to see that we cannot change the determinant of the Jacobian $J(q_1, \ldots, q_{n-1})$ by the action of these diffeomorphisms. □

Proposition 9. *If α satisfies assumptions 1, 2, 3, 4 of Theorem 2 for $k > 1$ and α does not satisfy assumption 5 then α is not ∂M-stable at 0.*

Proof. We may assume that X is 1-tangent to ∂M at 0. Then we reduce α to the form $(1 \pm x_n^2/2)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and ∂M to the form

$$\left\{x \in R^n : x_n^2 + q_2(x_1, \ldots, x_{n-1})x_n + x_1 = 0\right\}\,. \tag{11}$$

$\frac{\partial q_2}{\partial x_i}|_0 = 0$ for $i = 2, \ldots, n-1$, because α does not satisfy assumption 5. By Lemma 5 a diffeomorphism Φ which preserves $(1 + x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$ and the form of ∂M (11) has the form

$$\Phi(x) = (x_1, \Psi(x_2, \ldots, x_{n-1}), \pm x_n)\,,$$

where $\Psi : R^{n-1} \to R^{n-1}$ is a diffeomorphism which preserves $(n-2)$-form $dx_2 \wedge \cdots \wedge dx_{n-1}$. It is easy to see that we cannot change $Hess(q_2(0, x_2, \ldots, x_{n-1}))$- the determinant of the Hessian of $q_2(0, x_2, \ldots, x_{n-1})$ by the action of these diffeomorphisms. □

Now we present classification of locally ∂M-stable degenerated $(n-1)$-forms such that their kernels are transversal to the boundary ∂M.

Theorem 3. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 = 0$,
3. $S = \{x \in R^n : d\alpha_x = 0\}$ *is a germ of a regular hypersurface at* 0,
4. *a germ of a smooth vector field X at* 0, *which satisfies the following*

   $$X\rfloor\alpha = 0\,,\ X(0) \neq 0\,,$$

   meets ∂M transversally at 0 and is tangent to S at 0,
5. $j^{k+1}\alpha$ *meets the following submanifold of the manifold of $(k+1)$-jets of $(n-1)$-forms*

   $$\bigcup_{x \in \partial M} \Big\{j^{k+1}\beta : \beta \text{ satisfies at } x \text{ the assumptions } 1-4,$$
   $$(\mathcal{L}_X^i\beta)_x = 0 \text{ for } i = 1, 2, \ldots, k-1,\ (k < n),\ (\mathcal{L}_X^k\beta)_x \neq 0\Big\}$$

   transversally at 0,

then α is G_M-equivalent to

$$(1+\sum_{i=2}^{k} x_i x_1^{i-1} \pm x_1^k) dx_2 \wedge \cdots \wedge dx_n \,. \tag{12}$$

Proof. We use method described in [7], [8]. From condition 4, X and α can be reduced to the forms $\pm\frac{\partial}{\partial x_1}$ and $f(x)dx_2 \wedge \cdots \wedge dx_n$, where f is a smooth function, such that $f(0) \neq 0$ and

$$\frac{\partial^i f}{\partial x_1^i}(0) = 0\,,\ i = 1, 2, \cdots, k-1\,, \frac{\partial^k f}{\partial x_1^k}(0) \neq 0\,. \tag{13}$$

Hence f is a universal deformation of the function $f(x_1, 0, \ldots, 0)$ on the manifold with boundary and it can be reduced to the following form ([2])

$$f(x) = 1 + \sum_{i=2}^{k} x_i x_1^{i-1} \pm x_1^k\,, \tag{14}$$

Consequently, it is easy to see that α is G_M-eqivalent to the form (12) (see [3] for details). □

A germ, which satisfies assumptions 1-3 and 5 (for $k = n$) of Theorem 3 is stable at 0 (on a manifold without boundary) in the sense of Martinet's standard definition ([7],[9]). It is not true in the case of ∂M-stability.

Proposition 10. *If α satisfies the following conditions:*

1. $\alpha_0 \neq 0$,
2. $d\alpha_0 = 0$,
3. $S = \{x \in R^n : d\alpha_x = 0\}$ *is a germ of a regular hypersurface at* 0,
4. *a germ of a smooth vector field X at 0, which satisfies the following*

$$X \rfloor \alpha = 0\,,\ X(0) \neq 0\,, \tag{15}$$

meets ∂M transversally at 0 and is tangent to S at 0,

5. $(\mathcal{L}_X^i \alpha)_0 = 0$ *for* $i = 1, 2, \ldots, n-1$ *and* $(\mathcal{L}_X^n \alpha)_0 \neq 0$,

then α is not stable at 0.

Proof. By the same method as in the proof of Theorem 3 α can be reduced to the following form

$$g(x_2, \ldots, x_n)\Big(1 + \sum_{i=2}^{n} x_i x_1^{i-1} \pm x_1^n\Big) dx_2 \wedge \cdots \wedge dx_n\,,$$

where g is a smooth function, $g(0) \neq 0$. Let $\beta = b\alpha$, where $b \in R, b \neq 0$. One can show that α and β are G_M-equivalent if and only if $b = 1$ (see [3] for details). □

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