Ercüment Ortaçgil

G-connections as twisted formal solutions of systems of PDE's related to geometric structures

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 19th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2000. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 63. pp. 177--186.

Persistent URL: http://dml.cz/dmlcz/701661

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 63 (2000) pp. 177-186

G-CONNECTIONS AS TWISTED FORMAL SOLUTIONS OF SYSTEMS OF PDE'S RELATED TO GEOMETRIC STRUCTURES

ERCÜMENT ORTAÇGIL

ABSTRACT. We study the relation between the theory of formal integrability developed by D.C. Spencer and the theory of higher order connections. We then define certain characteristic classes for G-structures of finite type as obstructions to integrability.

1. INTRODUCTION

The purpose of this note is to establish that the distinction between "a section of jets" and "jet of a section", the starting point of the formal theory of Spencer, plays a fundamental role in many important concepts and constructions in geometry like connection, curvature, torsion and characteristic classes.

This note consists of three parts. The first part is motivated by our attempt to understand [4]. We work in local coordinates, treat the group and the geometric object left invariant by the group simultaneously and do not need the use of differential forms on (co)frame bundles. In this respect, our approach differs from the ones in [4], [13] and is more in the spirit of [11].

In the second part, we clarify the passage from formal point– solutions in the first part to formal point–connections (we use these words somewhat unconventionally). A point–connection contains the same formal information as a point–solution. The main difference is that it is possible to choose global sections over the base manifold from point–connections, such sections being called connections. However, this global aspect of connections is not needed in this note. All concepts in the formal theory of Spencer now translate into the theory of connections. For instance, the structure function becomes the torsion.

In the last part, we first show that it is not necessary to work with a global object like a torsion–free connection to define the characteristic classes in [1] but it is sufficient to work pointwise with jets. We then describe a general method of producing closed forms from the structure functions of G–structures of finite type, again working pointwise

1991 *Mathematics Subject Classification.* Primary 53A55, 53C05, Secondary 58A20, 53A55.
Key words and phrases. Formal solutions, connections, characteristic classes.
This paper is in final form and no version of it will be submitted for publication elsewhere.

with jets. The nonvanishing of these classes is an obstruction to integrability. As a remarkable fact, Euler class arises as a particular case from the *torsion* of a *second order connection* Ω in our framework. However Ω projects to a first order connection $\overline{\Omega}$ and in this particular case the curvature of $\overline{\Omega}$ coincides with the torsion of Ω. We hope that this situation will be helpful for a better understanding of the relation between the torsion and curvature in general as studied in [10]. We also hope to study the above classes in the future in further detail together with some concrete examples and applications.

2. Formal solutions

We will start with some *transitive* action of the general linear group $G^1 = GL(n, \mathbf{R})$ on some manifold X with dim X = m. For simplicity, we will assume that X has some global coordinates $\xi^a, 1 \le a \le m$, which we fix once and for all. We will write this action as $\overline{\xi}^a \doteq \Phi^a(f^i_j, \xi^b)$ where the indices in f^i_j refer to the standard coordinates (z^i) in $\mathbf{R}^n$ which we also fix once and for all. We now choose some $\xi_0 \in X$ and let $S(\xi_0)$ be the subgroup of G^1 which stabilizes ξ_0. Let f be a local diffeomorphism of $\mathbf{R}^n$ fixing 0 such that $f^i_j(z) = \partial_{z^j} f^i(z) \in S(\xi_0)$ for all z near 0 (for instance, f could be identity). We now have

$$\xi^a_0 = \Phi^a(f^i_j(0), \xi^b_0)\,. \tag{2.1}$$

Replacing $f^i_j(0)$ in (2.1) with $f^i_j(z)$, differentiating $k-$times and evaluating at $z = 0$, we obtain equations of the form

$$0 = \Phi^{(j)}(f^i_\nu, \xi_0) \qquad 1 \le j \le k, \quad 1 \le |\nu| \le j + 1 \tag{2.2}$$

where $f^i_\nu = \partial_{z^\nu} f^i(0)$ and we have omitted in (2.2) some indices in (2.1). Let $S^k(\xi_0)$ denote the subset of the $k + 1$'th order jet group G^{k+1} consisting of (f^i_ν) satisfying (2.1) and (2.2). It is easily checked that $S^k(\xi_0)$ is a subgroup of $G^{k+1}$and is called the *k'th* prolongation of $S(\xi_0)$. We define $S^0(\xi_0) = S(\xi_0)$. To determine the Lie algebra $LS^k(\xi_0)$ of $S^k(\xi_0)$, we observe that the RHS of (2.2), that is

$$\Phi^{(j)}(f^i_\nu, \xi_0) = \frac{\partial \Phi^{(0)}(f, \xi_0)}{\partial f^a_b} f^a_{\mu+1_b} + \Pi \tag{2.3}$$

where $\Phi^{(0)}$ denotes the expression on the RHS of (2.1), $\Pi = 0$ for $j = 1$ and is a sum of terms which contain products of f^i_ψ 's for $2 \le |\psi| \le j$, and we have $1 \le |\nu| \le j+1, |\mu| = j$ in (2.3). If we assume that (f^i_ν) in (2.1) and (2.2) depend on t with $f^i_\nu(0)$ = identity and differentiate (2.1) and (2.2) with respect to t at $t = 0$, it follows from (2.3) that $LS^k(\xi_0)$ consists of those (g^i_ν) satisfying

$$0 = \frac{\partial \Phi^{(0)}(e, \xi_0)}{\partial f^a_b} g^a_{\mu+1_b} \qquad 0 \le |\mu| \le k\,. \tag{2.4}$$

Let $LS^{k,j}(\xi_0)$ denote those (g^i_ν) satisfying (2.4) with $|\nu| = j + 1$. Then $LS^{k,0}(\xi_0) = LS(\xi_0)$ and we define $LS^{k,-1}(\xi_0) = \mathbf{R}^n$. Now $LS^k(\xi_0) = LS^{k,0}(\xi_0) \oplus ... \oplus LS^{k,k}(\xi_0)$ as a vector space.

We now assume that M is a differentiable manifold together with some first order geometric object ξ defined on it whose components $\xi^a(x)$ transform under a coordinate change $(x^i) \to (y^i)$ by

$$\xi^a(y) = \Phi^a(\frac{\partial y^i}{\partial x^j}, \xi^b(x)). \tag{2.5}$$

For instance, let X = the set of $(0,2)$-tensors in $\mathbf{R}^n$ which are symmetric and positive definite, choose $\xi_0 = \delta^{ij}$ and consider the transitive action of G^1 on X defined by $(a^i_j)(g^{ij}) = a^i_s a^j_t g^{st}$. Then $S(\xi_0) = O(n)$ and $\xi^a(x) = g^{ij}(x)$ is a metric structure. Many important structures in geometry arise in this way, including conformal, almost complex, quaternionic, symplectic and polynomial structures (see [6]).

The problem is now to solve the system of $PDE's$ $\xi_0 = \Phi^{(0)}(\frac{\partial h^i}{\partial x^j}(x), \xi(x))$ for some diffeomorphism h, that is, to decide whether ξ is locally equivalent to the flat object ξ_0. This is the famous equivalence problem of Cartan and is also the starting point of the construction of the Janet sequence. We will refer to [3], [11] and the references therein for more details. To derive necessary conditions for the existence for such a diffeomorphism, we search for the $k+1'th$ order jet of h for some k. For this purpose, we fix some p in the coordinate system (x^i), differentiate $\xi_0 = \Phi^{(0)}(\frac{\partial h^i}{\partial x^j}(x), \xi(x))$ k-times and evaluate at p, arriving at *algebraic* equations of the form

$$\begin{aligned} \xi_0 &= \Xi^{(0)}(\frac{\partial h^i}{\partial x^j}(p), \xi(p)) \qquad (\Xi^{(0)} \doteq \Phi^{(0)}), \\ 0 &= \Xi^{(j)}(\frac{\partial^{|\nu|} h^i}{\partial x^\nu}(p), \frac{\partial^{|\mu|}\xi}{\partial x^\mu}(p)). \end{aligned} \tag{2.6}$$

In (2.6), we have $1 \le j \le k, 1 \le |\nu| \le j+1, 1 \le |\mu| \le j$. If (2.6) has a solution for $\frac{\partial^{|\nu|} h^i}{\partial x^\nu}(p) \doteq h^i_\nu(p)$, then ξ is called k-flat at p. ξ is called uniformly k-flat if it is k-flat everywhere (see [4]). Note that the first equation in (2.6) always has solutions by our assumption of transitivity, that is, ξ is always uniformly $0-flat$. Henceforth we will always assume that ξ is uniformly k-flat. Let $\Psi^{k+1}(p)$ denote the set of all solutions of (2.6), that is, the set of all $k+1$-jets (h^i_ν) of all local diffeomorphisms with source at $p \in M$ and target, say, at the origin of $\mathbf{R}^n$ satisfying (2.6). It is easily checked that $S^k(\xi_0)$ acts on $\Psi^{k+1}(p)$ on the left simply transitively and $\Psi^{k+1} = \bigcup_{p\in M} \Psi^{k+1}(p)$ is a principal $S^k(\xi_0)$-bundle over M. Note that the form of the expressions $\Xi^{(j)}$ in (2.6) do not depend on (x^i). It is important to observe that there may be no local diffeomorphism h such that $\frac{\partial^{|\nu|} h^i}{\partial x^\nu}(q)$ satisfies (2.6) for all q near p even though for any such q we can find a local diffeomorphism depending on q which satisfies (2.6) at q by k-flatness. Also note that if $s \in \Psi^{(k+1)}$ projects to $p \in M$, then s defines local coordinates near p, which we will call s-admissible, such that s = identity in (x^i). If (x^i) and (y^i) are s-and t-admissible respectively, then $\frac{\partial^{|\nu|} y^i}{\partial x^\nu}(p) \in S^k(\xi_0)$. Our local formulas below will be in arbitrary coordinates and we will indicate whenever we make use of admissible coordinates which will play an important role in our arguments. Note that if $P \to M$ is an *arbitrary* principal bundle, then $s \in P$ may not induce any particular coordinates near p because P may not be directly related to the geometry of M.

We will call here an element of Ψ^{k+1}, somewhat unconventionally, a formal point-solution of order $k+1$. Let $s(x)$ be a local section of $\Psi^{k+1} \to M$. In view of (2.6), such a section satisfies

$$\begin{aligned} \xi_0 &= \Xi^{(0)}(s^i_j(x), \xi(x)), \\ 0 &= \Xi^{(j)}(s^i_\nu(x), \frac{\partial^{|\mu|}\xi}{\partial x^\mu}). \end{aligned} \tag{2.7}$$

Now let $h \in \Psi^{k+1}$ be represented by $h^i_\nu(p)$ satisfying (2.6) and let $s^i_\nu(x)$ be a section of $\Psi^{k+1} \to M$ passing through h, that is,

$$s^i_\nu(p) = h^i_\nu(p) \qquad 1 \le |\nu| \le k+1. \tag{2.8}$$

Clearly, $s^i_{r,\nu}(p) = \frac{\partial s^i_\nu}{\partial x^r}(p)$ may not be equal to $h^i_{\nu+1_r}(p) = \frac{\partial}{\partial x^r}(\frac{\partial^{|\nu|}h^i}{\partial x^\nu})(p)$ because, as indicated above, for q near p equality may not hold in (2.8) with the same h. This is one of the fundamental distinctions in jet theory. In particular, $s^i_{r,\nu}(p)$ may not be symmetric, that is, r may not commute with the indices in ν in $s^i_{r,\nu}(p)$. Differentiating the $j'th$ equation in (2.7), which we will denote by $(2.7)_j$, at $x = p$ for $0 \le j \le k$, we obtain

$$0 = \frac{\partial \Xi^{(j)}}{\partial s^a_\nu} s^a_{r,\nu}(p) + \frac{\partial \Xi^{(j)}}{\partial \xi^\mu} \frac{\partial^{|\mu+1|}\xi}{\partial x^{\mu+1_r}}(p) \qquad 0 \le j \le k. \tag{2.9}$$

The object $(s^i_\nu(p) \mid s^i_{r,\nu}(p))$ is geometrically the point $h = (s^i_\nu(p)) \in \Psi^{k+1}$ together with the tangent vector $(s^i_{r,\nu}(p))$ at h which projects to the coordinate vector in the direction of x^r. Let $J^1\Psi^{k+1}$ denote the set of all such objects with the obvious projection to M. We will not be interested here in the abstract principal bundle structure of $J^1\Psi^{k+1} \to M$ but certain properties of the transformation rules of $(s^i_\nu(p) \mid s^i_{r,\nu}(p))$ between coordinates, that is, certain elementary properties of the group operation of $J^1(S^k(\xi_0))$ (see [10]) will be implicitly used below.

As a very crucial observation, if we replace $s^i_{r,\nu}(p)$ in $(2.9)_j$ by $h^a_{\nu+1_r}(p)$ throughout, then the RHS of $(2.9)_j$ becomes identical with the RHS of $(2.6)_{j+1}$ for $0 \le j \le k-1$. Therefore we obtain a special tangent vector $(s^i_{r,\nu}(p))$ at h if we choose $s^a_{r,\nu}(p) = h^a_{\nu+1_r}(p) = s^a_{\nu+1_r}(p)$ for $0 \le |\nu| \le k$. However, for $|\nu| = k+1$, $s^a_{r,\nu}(p)$ in $(2.9)_k$ is not necessarily symmetric. We will denote this special object by $(s^a_\nu(p) \cup s^a_{r,\nu}(p))$. It is called a sesqui–holonomic jet (see [12], [13]) and will play a fundamental role in our construction below. We will denote the set of all sesqui–holonomic jets by $(sqh)J^1\Psi^{k+1} \subseteq J^1\Psi^{k+1}$. Since any horizontal space over M at $h \in \Psi^{k+1}$ is the tangent space of some section passing through h, we recapitulate as

Proposition 3. *Let $h \in \Psi^{k+1}$ project to $p \in M$ and be represented by $h^i_\nu(p) = \frac{\partial^{|\nu|}h^i}{\partial x^\nu}(p)$, $1 \le |\nu| \le k+1$, satisfying (2.6). Then there exists a section $s^i_\nu(x)$ of $\Psi^{k+1} \to M$ passing through h satisfying*

$$\frac{\partial s^i_\nu}{\partial x^r}(p) = s^i_{\nu+1_r}(p) = h^i_{\nu+1_r}(p) \qquad 1 \le |\nu| \le k. \tag{3.1}$$

Proposition 2.1 is contained in [4] (pg. 550, last paragraph).

Note that (2.10) is equivalent to $D(s(x))(p) = 0$, where D is the Spencer operator. Indeed, subtracting $(2.6)_{j+1}$ from $(2.9)_j$ we obtain

$$0 = \frac{\partial \Xi^{(j)}}{\partial s^a_\nu}(\frac{\partial s^a_\nu}{\partial x^r}(p) - s^a_{\nu+1_r}(p)) \qquad 0 \le j \le k-1 \tag{3.2}$$

and the RHS of (2.11) shows that D maps sections of Ψ^{k+1} to sections of $T^* \otimes V(\Psi^k)$, $1 \leq k$.

We now fix some $\alpha \in \Psi^{k+1}$ and choose some $\overline{\alpha} \in (sqh)J^1\Psi^{k+1}$ projecting to α. Now $\overline{\alpha} = (\alpha^i_\nu(p) \cup \overline{\alpha}^i_{r,\nu}(p))$, $1 \leq |\nu| \leq k+1$, and we are interested in the possibility of choosing $\overline{\alpha}^i_{r,\nu}(p)$ symmetric, that is, whether ξ is $k+1$-flat at p. First, note that $(2.9)_k$ is now of the form

$$0 = \frac{\partial \Xi^{(0)}}{\partial f^a_b} \overline{\alpha}^a_{r,\nu+1_b}(p) + \Theta + \frac{\partial \Xi^{(0)}}{\partial \xi} \frac{\partial^{|\mu|}\xi}{\partial x^\mu}(p) \tag{3.3}$$

where Θ is a sum of terms containing α_ψ as a factor for $2 \leq |\psi| \leq k+1$ and we have $|\nu|, |\mu| = k$ in (2.12).

We now view (2.12) in some α-admissible coordinate system near p. Then all middle terms in (2.12) vanish. Skewsymmetrizing, the last term also vanishes and we are left with

$$0 = \frac{\partial \Xi^{(0)}(e, \xi_0)}{\partial f^a_b} (\overline{\alpha}^a_{r,s\ldots b}(p) - \overline{\alpha}^a_{s,r\ldots b}(p)) \,. \tag{3.4}$$

Comparing (2.13) to (2.4), we see that $t_{\overline{\alpha}}(p) = \overline{\alpha}^a_{r,s\ldots b}(p) - \overline{\alpha}^a_{s,r\ldots b}(p)$ defines an element of $\Lambda^2(R^n) \otimes L^{k,k-1}S(\xi_0)$. If $\overline{\beta}^i_{r,\nu}(p)$ is another choice above $\alpha^i_\nu(p)$, subtracting (2.12) for $\overline{\alpha}$ and $\overline{\beta}$, we obtain

$$0 = \frac{\partial \Xi^{(0)}(e, \xi_0)}{\partial f^a_b} (\overline{\alpha}^i_{r,\nu}(p) - \overline{\beta}^i_{r,\nu}(p)) \qquad |\nu| = k+1 \,. \tag{3.5}$$

Comparing (2.14) to (2.4), we see that $\overline{\alpha}^i_{r,\nu}(p) - \overline{\beta}^i_{r,\nu}(p)$ is an element of $T^*(R^n) \otimes L^{k,k}S(\xi_0)$ which maps to $t_{\overline{\alpha}}(p) - t_{\overline{\beta}}(p)$ when skewsymmetrized. The kernel of this skewsymmetrization map is clearly $L^{k,k+1}S(\xi_0)$, that is, we have the exact sequence

$$L^{k,k+1}S(\xi_0) \longrightarrow T^*(R^n) \otimes L^{k,k}S(\xi_0) \longrightarrow \Lambda^2(R^n) \otimes L^{k,k-1}S(\xi_0) \,. \tag{3.6}$$

Consequently, defining the Spencer cohomology group $H^{k,2}(\xi_0) = \Lambda^2(R^n) \otimes L^{k,k-1}S(\xi_0)/\mathrm{Im}\,(\delta)$, $t_{\overline{\alpha}}(p)$ defines an element $t(p) = [t_{\overline{\alpha}}(p)] \in H^{k,2}(\xi_0)$ independent of $\overline{\alpha}$, called the structure function. Using (2.15), it is easy to check that $t(p)$ vanishes iff there exists some symmetric $\overline{\alpha}^i_{r,\nu}(p)$ over $\alpha^i_\nu(p)$. Finally, it follows easily from the group operation of $J^1S(\xi_0)$ that the definition of $t(p)$ depends only on the projection of $\alpha(p)$ in Ψ^1 and $t(p)$ transforms as a tensor on M.

4. Connections

As we have seen above, a section of $(sqh)J^1\Psi^{k+1} \to M$ is locally of the form $(\alpha^i_\nu(x) \mid \alpha^i_{r,\nu}(x))$ where $0 \leq k$. Since $\alpha^i_j(x)$ defines n-independent 1-forms where $n = \dim M$, this bundle does not admit global sections in general for topological reasons. Therefore, given some point $h(p) \in (sqh)J^1\Psi^{k+1}$ lying above $p \in M$, we would like to define another object $\Gamma(h(p))$ above p in such a way that $h(p)$ and $\Gamma(h(p))$ would contain the same formal information and further it would be possible to choose a section Γ over M. For this purpose, we define $\Gamma(h(p))$ with local components $(\Gamma(h(p))^i_\nu \cup \Gamma(h(p))^i_{r,\nu})$

by the formulas

$$(4.1)\qquad \begin{aligned} \Gamma(h(p))^i_\nu &= \widetilde{h}(p)^i_a\, h(p)^a_\nu \qquad \widetilde{h}(p)^i_a h(p)^a_j = \delta^i_j\,, \\ \Gamma(h(p))^i_{r,\nu} &= \widetilde{h}(p)^i_a\, h(p)^a_{r,\nu} \qquad 1 \le |\nu| \le k+1\,. \end{aligned}$$

We will call $\Gamma(h(p))$ also a (sesqui-holonomic) formal point–connection induced by $h(p)$. Note that $\Gamma(h(p))$ determines $h(p)$ up to its projection in Ψ^1 in view of (3.1). The transformation rule of the above components are easily derived (see [7], [8]) and they are local coordinates on some geometric object bundle $(sqh)CH^{k+2}(\Psi) \to M$, where CH stands for Christoffel symbols. We have the obvious commutative diagram

$$(4.2)\qquad \begin{array}{ccc} (sqh)J^1\Psi^{k+1} & \xrightarrow{\Gamma} & (sqh)CH^{k+2}(\Psi) \\ \downarrow & & \downarrow \\ \Psi^{k+1} & \xrightarrow{\Gamma} & CH^{k+1}(\Psi) \end{array}$$

where the vertical maps are projections.

Our purpose is now to see how the concepts in the first section translate into connections. In view of (3.1), substituting $h(p)^i_j = \delta^i_j, h^i_\nu(p) = \Gamma(h(p))^i_\nu$, $2 \le |\nu| \le k+1$, and $h(p)^i_{r,\nu} = \Gamma(h(p))^i_{r,\nu}$, $|\nu| = k+1$, into $(2.6)_j$ for $1 \le j$ throughout, we obtain

$$(4.3)\qquad \begin{aligned} 0 &= \frac{\partial \Sigma^{(j)}}{\partial s^i_\nu}\Gamma(h(p))^i_{\nu+1_r} + \frac{\partial \Sigma^{(j)}}{\partial \xi^\mu}\frac{\partial^{|\mu+1|}\xi}{\partial x^{\mu+1_r}} \qquad 1 \le j \le k-1\,, \\ 0 &= \frac{\partial \Sigma^{(k)}}{\partial s^i_\nu}\Gamma(h(p))^i_{r,\nu} + \frac{\partial \Sigma^{(k)}}{\partial \xi^\mu}\frac{\partial^{|\mu+1|}\xi}{\partial x^{\mu+1_r}}\,. \end{aligned}$$

In $(3.3)_j$, the Christoffel symbols of top order $j+1$ appear quasi–linearly (see (2.12)). Therefore a standard partition of unity argument shows that the bundles on the RHS of (3.2) admit global sections and therefore $(sqh)CH^{k+2} \to M$ admits global sections. Let Ω be such a section. Now any geometric object of order $k+1$ can be covariantly differentiated with respect to Ω as in [8]. Let $E \to M$ denote the bundle determined by (2.5) so that ξ is a section of $E \to M$. Then $j^k\xi = \frac{\partial^{|\mu|}\xi}{\partial x^\mu}$, $0 \le |\mu| \le k$ is a section of $J^kE \to M$ and we have

Proposition 5. *$j^k\xi$ is parallel with respect to Ω. The formulas on the RHS of (3.3) give the covariant derivative of $j^k\xi$ with respect to Ω.*

Sections of $CH^{k+1}(\Psi) \to M$ are in one to one correspondence with torsion –free connections on $\Psi^k \to M$ (see [5], [10]). Another characterization of such sections are given in [4] (Proposition 8.4).

Let $\Omega = (\Omega^i_\nu(p) \cup \Omega^i_{r,\nu}(p)) \in (sqh)J^1\Psi^{k+1}$. The transformation rule of the components of Ω shows that $T(\Omega)(p) = \Omega^i_{r,s\ldots a}(p) - \Omega^i_{s,r\ldots a}(p)$ transforms as a tensor. If $\overline{\alpha} \in (sqh)J^1\Psi^{k+1}$ and $\Gamma(\overline{\alpha}) \in (sqh)CH^{k+2}(\Psi)$, then it is easy using the definitions that $t_{\overline{\alpha}}(p) = T(\Gamma(\overline{\alpha}))(p)$ where $t_{\overline{\alpha}}(p)$ is defined above. We therefore deduce

Proposition 6. *ξ is $k+1$–flat iff $j^k\xi$ is parallel with respect to some section of $CH^{k+2} \to M$.*

As a final point in this section, recall the section $s(x) = (s^i_\nu(x))$ of $\Psi^{k+1} \to M$ passing through $h(p) \in \Psi^{k+1}$ in Proposition 2.1 and consider the local section $\Gamma(s(x))$ of $CH^{k+1}(\Psi) \to M$. The Spencer operator $D : \Psi^{k+1} \to T^* \otimes V(\Psi^k)$ is defined by

(2.11) (we use the same notation for bundles and their sheaves of local sections) and we have seen that $D(s)(p)$ vanishes due to $k-$flatness. We also have the covariant differentiation operator $C : CH^{k+1}(\Psi) \to T^* \otimes V(CH^k(\Psi))$ constructed in [9] and studied further in [8], [10]. In fact, replacing $\frac{\partial s^i_\nu}{\partial x^r}(p)$ and $s^i_{\nu+1_r}(p)$ in (2.11) by $d_r(\Omega^i_\nu)(p)$ (see [7], [9] and (4.2) below for d_r) and $\Omega^i_{\nu+1_r}$ respectively, we obtain the local formulas describing C. Therefore, if $s(x)$ denotes the section in Proposition 2.1, then C vanishes on $\Gamma(s(x)) = \Omega(x)$ at p. In abstract terms, the map $\Gamma : \Psi^{k+1} \to CH^{k+1}(\Psi)$ induces a map $V(\Gamma)$ on the vertical level and we have the commutative diagram

$$\begin{array}{ccc} \Psi^{k+1} & \xrightarrow{\Gamma} & CH^{k+1}(\Psi) \\ \downarrow & & \downarrow \\ T^* \otimes V(\Psi^k) & \xrightarrow{V(\Gamma)} & T^* \otimes V(CH^k(\Psi)) \end{array} \tag{6.1}$$

In short, we have

Proposition 7. *When we pass from formal solutions to connections, Spencer operator D is replaced by the covariant differentiation operator C.*

8. Closed forms

We will first interpret the construction in [1] in our framework. Let $h \in \Psi^1$ project to $p \in M$ and let Q be a polynomial invariant under the standard action of $S(\xi_0)$ on R^n. Now $(h^i_j(x))$ transforms by $h^i_a(x)\frac{\partial x^a}{\partial y^j} = h^i_j(y)$ and $(a^i_j) \in S(\xi_0)$ acts on (h^i_j) by $a^i_s h^s_j = \overline{h}^i_j$, that is, j is the $1-$form index and i is the R^n-index in h^i_j. Let (x^i) be some $h-$admissible coordinates near p so that $h^i_j(x) = \delta^i_j$ at $x = p$. Consider $\omega(x,p) \doteq Q(\delta^i_j)$ with the obvious meaning for $Q(\delta^i_j)$. The invariance of Q shows that $\omega(x,p)$ does not depend on (x^i) and therefore we obtain a *global* form ω on M of degree = degree (Q). If ξ is further $1-$flat at p, then Proposition 2.1 gives a local section $s^i_j(x)$ of $\Psi^1 \to M$ passing through h such that $s^i_{r,j}(p)$ is symmetric, that is,

$$\frac{\partial s^i_j}{\partial x^r}(p) - \frac{\partial s^i_r}{\partial x^j}(p) = s^i_{r,j}(p) - s^i_{j,r}(p) = 0\,. \tag{8.1}$$

Since $\omega(x) = Q(s(x))$ for x near p, differentiating the last equality at $x = p$, we deduce $d\omega(p) = 0$ in view of (4.1). Since p is arbitrary, ω is closed. In fact, we have $d\sigma = \omega$ for some *global* form σ on $\Psi^1$and it is easy to describe σ explicitly in local coordinates (compare to [2]). It seems to us that the above construction works also if ξ is $k-$flat for $2 \leq k$ but this time we obtain closed forms on Ψ^1 as in [2]. We hope to clarify such constructions in some future work.

We will now describe a method of producing closed forms of even degree. Let ξ be $k-$flat for $1 \leq k$ and let $s(x)$ be a local section of $(sqh)J^1\Psi^{k+1} \to M$. Then $s(x) = (s^i_\nu(x) \cup s^i_{r,\nu}(x))$, $1 \leq |\nu| \leq k+1$, but we may *not* have $s^i_{r,\nu}(x) = \frac{\partial s^i_\nu}{\partial x^r}(x)$ *for all* x even though we will have such an equality holding pointwise as in Proposition 2.1. This is again the fundamental distinction between a section of jets and jet of a section. However, there is a natural condition which makes the last equality hold locally. For this purpose, we assume that $L^{k,k}S(\xi_0) = 0$, that is, $S(\xi_0)$ is of type k. For instance, $O(n)$ is of type 1. More generally, any compact subgroup of G^1 is of type 1.

Conformal group $CO(n)$ is of type 2 if $3 \leq n$. It is known that any reductive group of finite type is either of type 1 or 2. Now it follows from (2.4) that $L^{s,t}S(\xi_0) = 0$ for $k \leq t \leq s$. Now the crucial fact is that the kernels of the projection homomorphisms $S^j(\xi_0) \to S^{k-1}(\xi_0)$ are trivial for $k \leq j$. This fact is not difficult to show using (2.3) and (2.4). Seperating out $s^i_\nu(x)$ for $|\nu| = k+1$ and writing the above section now with the new notation as $s(x) = (s^i_\mu(x) \cup s^i_\nu(x) \cup s^i_{r,\nu}(x))$, $1 \leq |\mu| \leq k$, $|\nu| = k+1$, it follows that $s^i_\nu(x)$ is unique once $s^i_\mu(x)$ is choosen. Now the middle term in (2.15) vanishes. It follows that the choice of $s^i_\mu(x)$ also uniquely determines $s^i_{r,\nu}(x)$. Fixing some $p \in M$, Proposition 2.1 gives a section $o(x)$ of $\Psi^{k+1} \to M$ passing through $s(p)$ with $\frac{\partial o^i_\nu}{\partial x^r}(p) = s^i_{r,\nu}(p)$ for $|\nu| = k+1$. If $\overline{o}(x)$ is another such section, uniqueness of $s^i_{r,\nu}(p)$ implies $\frac{\partial \overline{o}^i_\nu}{\partial x^r}(p) = \frac{\partial o^i_\nu}{\partial x^r}(p) = s^i_{r,\nu}(p)$. In particular, since $s^i_\nu(x)$ is such a section passing through $s(p)$, we obtain $\frac{\partial s^i_\nu}{\partial x^r}(p) = s^i_{r,\nu}(p)$. Since p is arbitrary, we have proved

Proposition 9. *If ξ is k-flat and $S(\xi_0)$ is of type k, then for any local section $s(x)$ of $(sqh)J^1\Psi^{k+1} \to M$, we have $\frac{\partial s^i_\nu}{\partial x^r}(x) = s^i_{r,\nu}(x)$ for $|\nu| = k+1$.*

At this stage, it is immediate to show that if ξ is further $k+1$-flat then it is integrable, that is, it is locally equivalent to the flat object ξ_0. This fact is the main result of [4] (Theorem 5.1).

Henceforth we will assume that ξ is k-flat and $S(\xi_0)$ is of type k. Our purpose is to define certain characteristic classes from $t_{\overline{\alpha}} = T(\Gamma(\overline{\alpha}))$. Consider the standard action of $S(\xi_0)$ on $(1,k)$-symmetric tensors and let Q be an invariant polynomial. Let $p \in M$ and choose some $\overline{\alpha} = (\alpha(p) \cup \overline{\alpha}(p)) \in (sqh)J^1\Psi^{k+1}$ projecting to $p \in M$. In the α-admissible coordinate sytem (x^i) near p denote the value of $t_{\overline{\alpha}}(p)$ by $t(x,p)$. We define $\omega(x,p) = Q(t(x,p))$. The invariance of Q shows that $\omega(x,p)$ does not depend on the choice of the admissible (x^i) and we obtain a global form ω on M of degree = 2(degree Q). To see that ω is closed, we choose a local section $s(x)$ of $(sqh)J^1\Psi^{k+1} \to M$ passing through $\alpha(p)$. It follows from Proposition 4.1 that $d\omega(p) = 0$. Clearly, we could argue also with point $-$connections as we will do now. This will make clear that the Euler class is a particular case of the above construction. Indeed, let $s(x)$ be the section in Proposition 4.1 and consider $\Gamma(s(x)) = \Omega(x) = (\Omega^i_\nu(x) \cup \Omega^i_{r,\nu}(x))$, $2 \leq |\nu| \leq k+1$. Proposition 4.1 together with (3.1) now gives (see the argument before (3.4))

Proposition 10. *We have*

$$d_r\Omega^i_\nu(x) = \frac{\partial \Omega^i_\nu}{\partial x^r}(x) + \Omega^i_{ra}\Omega^a_\nu = \Omega^i_{r,\nu} \qquad |\nu| = k+1\,. \tag{10.1}$$

Now it easily follows from definitions that

$$t_{s(x)}(x) = d_r\Omega^i_{s\dots t}(x) - d_s\Omega^i_{r\dots t}(x) = T(\Gamma(s(x))(x)\,. \tag{10.2}$$

Note that $\Omega^i_{ra}(x)\Omega^a_\nu(x)$ vanishes at p if (x^i) is admissible and then $d_r = \frac{\partial}{\partial x^r}$ at p.

Let us now consider the above construction in the case of a metric where $S(\xi_0) = O(n)$ is of type 1 and $\xi = g$ is 1-flat. Choosing $h = (h^i_j \cup h^i_{jk} \cup h^i_{r,jk}) \in (sqh)J^1\Psi^2$ with some arbitrary h^i_j, consider $\Gamma(h) = (\widetilde{h}^i_a h^a_{jk}, \widetilde{h}^i_a h^a_{r,jk}) = (\Omega^i_{jk}, \Omega^i_{r,jk}) \in (sqh)CH^3(\Psi^2)$, where $\widetilde{h}_a^i h_j^a = \delta_j^i$. If we choose some other $\overline{h} = (\overline{h}_j^i \cup \overline{h}_{jk}^i \cup \overline{h}_{r,jk}^i)$, then $\overline{h}_j^i = a_s^i h_j^s$ for some $(a_j^i) \in O(n)$. It follows that $\overline{h}_{jk}^i = a_s^i h_{jk}^s$ and therefore $\overline{\Omega}_{jk}^i = \Omega_{jk}^i =$ the Christoffel symbols of the Levi–Civita connection. Proposition 3.1 expresses the well known fact that the metric is parallel with respect to this connection. Now (4.2) and (4.3) for $k = 1$ show that $t(x)$ is identical with the *curvature* of the Levi–Civita connection and if Q is the Pfaffian, we obtain the Euler class. Observe the remarkable fact that we are defining Euler class from the *torsion* of some *second order point–connection* Ω. However, Ω projects to a first order point–connection $\overline{\Omega}$ and in this particular case the torsion of Ω coincides with the curvature of $\overline{\Omega}$. For the relation between torsion and curvature in general we will refer to [10]. Note also that the uniqueness of the Levi–Civita connection will not hold if $2 \leq k$ but all these connections will have the same torsion.

Now let $\overline{\xi}$ be another k–flat object on M and let $\overline{\omega}$ be the closed form defined by the same polynomial Q above. We have

Proposition 11. *We have $[\overline{\omega}] = [\omega]$ where $[\ \ ]$ denotes the class in de Rham cohomology $H^*(M, \mathbf{R})$.*

We will not give here the easy proof of Proposition 4.3 which is however interesting as it shows the quite natural counterparts in jet theory of some well known concepts from general bundle theory like bundle morphisms and the pullbacks of connections and their curvatures by such morphisms. It follows from Proposition 4.3 that the nonvanishing of $[\omega]$ is an obstruction to the existence of some uniformly $k+1$–flat ξ (hence integrable by [4]) on M.

A careful examination of the above arguments shows that the main point in the above construction is *not* uniform k–flatness but k–uniformness. Let $\Delta^k(M, \xi)$ denote the moduli space of all such k–uniform ξ–structures on M with a natural definition of equivalence. We can now prove

Proposition 12. *Any invariant polynomial Q induces a map*

$$\widetilde{Q} : \ \Delta^k(M, \xi) \longrightarrow H^*(M, \mathbf{R}) . \tag{12.1}$$

In the light of the general considerations above, one is now left with the problem of computing the above classes in some concrete situations that we hope to pursue in some future work. In conclusion, we hope to have convinced the reader that the distinction between a section of jets and jet of a section is a fundamental fact which underlies many important constructions in geometry.

REFERENCES

[1] D. V. Alekseevsky, P. W. Michor, *Characteristic classes for G-structures*, Differential Geometry and its Applications **3** (1993), 323–329.

[2] S. S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Annals of Math. **99** (1974), 48–69.

[3] R. B. Gardner, *The Method of Equivalence and its Applications*, CBMS-NSF Reginal Conference Series in Applied Mathematics **58**, SIAM, 1989.

[4] V. Guillemin, *The integrability problem for G-structures*, Trans. A.M.S. **116** (1965), 544–560.

[5] I. Kolar, *Torsionfree connections on higher order frame bundles*, New developments in differential geometry, Debrecen, 1994, 233–241.
[6] M. Kon, K. Yano, *Structures on Manifolds*, World Scientific Publishing Co. Pte. Ltd., Series in pure Math., Vol.3, 1984.
[7] E. Ortacgil, *Some remarks on the Christoffel symbols of Ehresman e-connections*, Proc. Conf. "Third Meeting on current ideas in mechanics and related fields", Segovia (Spain), June 19-23, 1995, Extracta Mathematicae, Vol. 11, Num.1, 172–180, 1996.
[8] E. Ortacgil, *Classical differential geometry with Christoffel symbols of Ehresmann e-connections*, Arch. Math.(Brno), **34** (1998), 229–237.
[9] E. Ortacgil, *On a differential sequence in geometry*, Tr. J. Math. **20** (1996), 473–478.
[10] E. Ortacgil, *On a diagram in the theory of connections*, Proc. Conf."Differential Geometry and its Applications", 1998, Brno, Czech Republic, 1999, 273–283.
[11] J. F. Pommaret, *Lie Pseudogroups and Mechanics*, Gordon&Breach, 1988.
[12] J. F. Pommaret, *Sesqui-holonomic jets and Spencer operator*, C.R.Acad. SC. Paris, t. 299, Serie 1, no. 4, 1984.
[13] P. C. Yuen, *Higher order frames and linear connections*, Cahiers Topol.Geom. Diff. **12** (1971), 333–371.

Boğaziçi University
Mathematics Department
Bebek, 80815 Istanbul
TURKEY
E-mail: ortacgil@boun.edu.tr