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A NOTE ON n-ARY POISSON BRACKETS

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ABSTRACT. A class of n-ary Poisson structures of constant rank is indicated. Then, one proves that the ternary Poisson brackets are exactly those which are defined by a decomposable 3-vector field. The key point is the proof of a lemma which tells that an n-vector ($n \geq 3$) is decomposable iff all its contractions with up to $n-2$ covectors are decomposable.

In the last years, several authors have studied generalizations of Lie algebras to various types of n-ary algebras, e.g., [5, 12, 9, 11, 15]. In the same time, and intended to physical applications, the new types of algebraic structures were considered in the case of the algebra $C^\infty(M)$ of functions on a C^∞ manifold M, under the assumption that the operation is a derivation of each entry separately. In this way one got the Nambu-Poisson brackets, e.g., [12, 6, 1, 4, 7], and the generalized Poisson brackets [2, 3], etc. In this note, we write down the characteristic conditions of the n-ary generalized Poisson structures in a new form, and give an example of an n-ary structure of constant rank $2n$, for any n even or odd. Then, we prove that the ternary Poisson brackets are exactly the brackets defined by the decomposable 3-vector fields. The key point in the proof of this result is a lemma (that seems to appear also in [16]), which tells that an n-vector P is decomposable iff $i(\alpha_1)...i(\alpha_k)P$ is decomposable, for any choice of covectors $\alpha_1, ..., \alpha_k$, where k is fixed, and such that $1 \leq k \leq n-2$.

Our framework is the C^∞ category. If M is an m-dimensional manifold, an *n-ary Poisson bracket or structure* (called generalized Poisson structure in [2, 3]), with the *Poisson n-vector or tensor P*, is a bracket of the form

$$\{f_1, ..., f_n\} = P(df_1, ..., df_n) \quad (f_1, ..., f_n \in C^\infty(M)), \tag{1}$$

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where $P \in \Gamma \wedge^n TM$ is an n-vector (i.e., a completely skew-symmetric contravariant tensor) field, and the following *generalized Jacobi identity* of order n [11] is satisfied

$$\sum_{\sigma \in S_{2n-1}} (sign\,\sigma)\{\{f_{\sigma_1}, ..., f_{\sigma_n}\}, f_{\sigma_{n+1}}, ..., f_{\sigma_{2n-1}}\} = 0, \tag{2}$$

S_{2n-1} being the symmetric group.

For $n = 2$ the bracket is a usual Poisson bracket e.g., [14]. In this note, we always assume $n \geq 3$.

Proposition 1. *The n-vector field $P \in \Gamma \wedge^n TM$ defines an n-ary Poisson bracket iff either n is even and the Schouten-Nijenhuis bracket $[P, P] = 0$, or n is odd and P satisfies the conditions*

$$(i(\alpha)P) \wedge (i(\beta)P) = 0 \quad \forall \alpha, \beta \in T^*M, \tag{A}$$

$$\sum_{u=1}^{m} (i(dx^u)P) \wedge (L_{\partial/\partial x^u} P) = 0, \tag{D}$$

where (x^u) are local coordinates on M, and L denotes Lie derivative.

Proof. The left hand side of (2) contains only first and second order derivatives, and is skew symmetric in the arguments f_i. Hence, to ensure (2) it is enough to ask it to hold for the case of the local functions $f_i = x^{a_i}$, $(i = 1, ..., 2n - 1)$, and for the case of the functions $f_1 = x^u x^v$, $f_i = x^{a_i}$, $(i = 2, ..., 2n - 1)$, at $x = 0$. In the first case the result is

$$\sum_{u=1}^{n} P^{u[a_1 \ldots a_{n-1}} \frac{\partial}{\partial x^u} P^{a_n \ldots a_{2n-1}]} = 0, \tag{3}$$

and in the second case the result is

$$P^{v[a_2 \ldots a_n} P^{a_{n+1} \ldots a_{2n-1}]w} + P^{w[a_2 \ldots a_n} P^{a_{n+1} \ldots a_{2n-1}]v} = 0, \tag{4}$$

where square brackets denote index alternation. Now, (4) is equivalent to (A) if n is odd, and it is an identity if n is even. Then, (3) is equivalent to (D), and the use of the coordinate expression of the Schouten-Nijenhuis bracket (e.g., [14]) shows that, for n even, (D) is equivalent to $[P, P] = 0$. Q.e.d.

We call (A) and (D) the *algebraic* and the *differential condition*, respectively. The coordinate expressions (3), (4), and their equivalence with $[P, P] = 0$ in the n-even case, were also established in [2, 3]. In the n-odd case, the differential condition (D) has no independent invariant meaning, and it must be associated with the algebraic condition (A).

It is also important to notice that, since (A,D) always hold at the zeroes of P, P defines an n-ary Poisson bracket iff it does so on the subset $U \subseteq M$ where $P \neq 0$.

Before going on, we need some general facts about n-vectors $P \in \wedge^n L$, where L is an m-dimensional (e.g., real) linear space. First, P defines a linear mapping $\sharp_P : \wedge^{n-1} L^* \to L$ given by

$$\sharp_P(\lambda) = i(\lambda)P, \quad \lambda \in \wedge^{n-1} L^*. \tag{5}$$

We will say that $rank\,\sharp_P = dim\,im\,\sharp_P$ is the *rank* of P. (This definition is equivalent with the one used in older books on exterior algebra e.g., [10], which refered to L^*

rather than L, and where the vectors of $im\,\sharp_P$ were seen as the right hand side of the equations of the *adjoint system* of P.)

If $rank\,P = dim\,L$, we say that P is *non degenerate* (*regular* in [10]). On the other hand, if $rank\,P = n$, P is *decomposable* i.e., there are vectors $W_a \in L$ $(a = 1, ..., n)$ such that $P = W_1 \wedge ... \wedge W_n$. We recall the existence of classical decomposability conditions known as the *Plücker conditions* e.g. [8], which we will write down later, in the proof of Lemma 3.

The skew symmetry of P implies $im\,\sharp_P = Ann(A(P))$, where $A(P) := \{\alpha \in L^* \,/\, i(\alpha)P = 0\}$, hence, $rank\,P = m - dim\,A(P)$. Notice also that $P \in \wedge^n(im\,\sharp_P)$. Indeed, if $L = im\,\sharp_P \oplus K$, we have an expression

$$P = \sum_{u+v=n} Q_u \wedge S_v, \quad Q_u \in \wedge^u(im\,\sharp_P),\ S_v \in \wedge^v K,$$

and, since K^* may be identified with $A(P)$, the previous expression reduces to $P = Q_n$. On the other hand, if $P \in \wedge^n U$, where U is a subspace of L, $im\,\sharp_P \subseteq U$. Therefore, $im\,\sharp_P$ is the minimal subspace S of L such that $P \in \wedge^n S$.

The n-vector P will be called *irreducible* if there is no decomposition $im\,\sharp_P = S_1 \oplus S_2$ where $dim\,S_1 = n$, and where $P = P_1 + P_2$ with $0 \neq P_1 \in \wedge^n S_1$, $0 \neq P_2 \in \wedge^n S_2$. If such a decomposition exists, P is *reducible*, and, because $\forall \lambda \in \wedge^{n-1}L^*$, $i(\lambda)P = i(\lambda)P_1 + i(\lambda)P_2$, we have $S_1 = im\,\sharp_{P_1}$, $S_2 = im\,\sharp_{P_2}$. From these definitions, it follows that any n-vector P may be (non uniquely) written under the form

$$P = \sum_{i=0}^{s-1} V_{in+1} \wedge ... \wedge V_{in+n} + P', \tag{6}$$

where V_a $(a = 1, ..., sn)$ are independent vectors and $P' \in \wedge^n U$, where U is a complement of $span\,\{V_a\}$ in L, is irreducible with $rank\,P' = rank\,P - sn$.

An n-vector P, which satisfies condition (A) $\forall \alpha, \beta \in L^*$, must be irreducible since otherwise, and with the notation above, we have $(i(\alpha)P) \wedge (i(\beta)P) \neq 0$ if $i(\alpha)P_1 \neq 0$, $i(\alpha)P_2 = 0$, $i(\beta)P_1 = 0$, $i(\beta)P_2 \neq 0$. Of course, if $rank\,P < 2n - 2$, P is irreducible and (A) is an identity.

Now, we come back to the manifold M. Then, if $P \in \Gamma \wedge^n TM$, $rank\,P$ is a lower semicontinuous function on M. The following Proposition gives an interesting class of Poisson, but not Nambu-Poisson, generally, n-vectors, for an arbitrary even or odd order n. A first example of an n-ary Poisson structure of an even order n was given in [2], and it was a linear structure on the dual of a simple Lie algebra. We know of no previous examples of n-ary Poisson structures which are not Nambu-Poisson structures.

Proposition 2. *Assume that $P \in \Gamma \wedge^n TM$, and that $\forall x \in M$ there is an open neighbourhood U_x such that $P/_{U_x}$ can be written as*

$$P = \frac{1}{h!(n-h)!} \sum_{\sigma \in S^n} (sign\,\sigma) V_{\sigma_1} \wedge ... \wedge V_{\sigma_h} \wedge W_{\sigma_{h+1}} \wedge ... \wedge W_{\sigma_n}, \tag{7}$$

where S_n is the permutation group, (V_i, W_j) $(i, j = 1, ..., n)$ are independent vector fields on U_x, and h is a fixed integer such that $0 \leq 2h \leq n - 3$. Then P is a Poisson n-vector of constant rank, equal to $2n$ if $h \neq 0$, and to n if $h = 0$.

Proof. The condition on h was chosen such that the left hand sides of (A) and (D), as well as $[P, P]$, consist of sums of wedge products where at least one of the vectors W_j must be wedge multiplied by itself. If $h = 0$, $P = W_1 \wedge ... \wedge W_n$ i.e., P is decomposable and of rank n. If $h \neq 0$, by using a basis of vectors that starts with (V_i, W_j) and its dual cobasis in order to obtain generators of $im\,\sharp_P$, we see that $im\,\sharp_P = span\{V_i, W_j\}$ hence, $rank\,P = 2n$. (The particular case of an even order, decomposable n-vector P was noticed in [3].) Q.e.d.

We will say that a tensor P of form (7) with $h \neq 0$ is a *semi-decomposable n-vector*. An n-ary Poisson structure (bracket) defined by a (semi-) decomposable n-vector field will be called a *(semi-)decomposable n-ary Poisson structure (bracket)*.

Lemma 3. *Let L be an m-dimensional vector space, and $P \in \wedge^n L$. Then P is decomposable iff for any fixed number k such that $1 \leq k \leq n-2$, and any set of k covectors $\alpha_u \in L^*$ $(u = 1, ..., k)$, the $(n-k)$-vector $i(\alpha_1)...i(\alpha_k)P$ is decomposable.*

Appearently, this lemma is included in formula (4), page 116 of [16]. Our proof is different.

Proof. It is enough to prove the result for $k = 1$, and, on the other hand, the proof and the result do not hold for $k > n-2$.

As already recalled, decomposability of P is characterized by the Plücker conditions. These may be written in one of the following equivalent forms [8]: *i)* $\forall V \in im\,\sharp_P$, $V \wedge P = 0$, *ii)* $\forall \lambda \in \wedge^{n-1}L^*$, $(i(\lambda)P) \wedge P = 0$. (The first form of the conditions is rather obvious, and the second is equivalent since $i(\lambda)P$ are exactly the vectors of $im\,\sharp_P$.)

From

$$i(\alpha)[(i(\mu)i(\alpha)P) \wedge P] = -[i(\mu)(i(\alpha)P)] \wedge (i(\alpha)P \quad (\alpha \in L^*, \mu \in \wedge^{n-2}L^*),$$

we see that if P satisfies condition *ii)*, i.e., if P is decomposable, so are all $i(\alpha)P$, $\alpha \in L^*$.

Now, assume that all $i(\alpha)P$ are decomposable, and take $\epsilon^1 \in L^*$ such that $i(\epsilon^1)P \neq 0$. Then, $\exists e_a \in ker\,\epsilon^1 \subseteq L$ $(a = 2, ..., n)$ such that

$$i(\epsilon^1)P = e_2 \wedge ... \wedge e_n. \tag{8}$$

Let us also take $e_1 \in L$ such that $\epsilon^1(e_1) = 1$, and denote by L_1 the n-dimensional subspace $span\{e_1, ..., e_n\}$ of L, and by L_2 an arbitrary complement of $span\{e_2, ..., e_n\}$ in $ker\,\epsilon^1$. Then $L = L_1 \oplus L_2$, and we have an expression

$$P = \rho e_1 \wedge ... \wedge e_n + \sum_{i=1}^{n-1} P'_i \wedge P''_i + P''_n, \tag{9}$$

where $\rho \in \mathbf{R}$, $P'_i \in \wedge^{n-i}L_1$, $P''_i \in \wedge^i L_2$, $P''_n \in \wedge^n L_2$. Moreover, (8) implies $\rho = 1$ and

$$i(\epsilon^1)P'_i = 0 \quad (i = 1, ..., n-1). \tag{10}$$

(If some $P''_i = 0$ we will also assume $P'_i = 0$.)

Let $\epsilon^i \in L^*$ be covectors which vanish on L_2, and are such that $\epsilon^i(e_j) = \delta^i_j$ $(i, j = 1, ..., n)$. According to our hypothesis, the $(n-1)$-vectors

$$i(\epsilon^a)P = (-1)^{a-1} e_1 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_n + \sum_{i=1}^{n-1} (i(\epsilon^a)P'_i) \wedge P''_i$$

$(a = 2, ..., n)$, where the hat denotes the absence of the factor, must also be decomposable. In view of (10), for

$$\lambda = \epsilon^1 \wedge ... \wedge \hat{\epsilon}^a \wedge ... \wedge \hat{\epsilon}^b \wedge ... \wedge \epsilon^n \quad (b \neq a),$$

we have $i(\lambda)i(\epsilon^a)P = \pm e_b$, where $b = 2, ..., n$, and the sign depends on whether $a < b$ or $b < a$, and the Plücker condition *ii)* yields

$$(11) \qquad e_b \wedge (i(\epsilon^a)P) = \sum_{i=1}^{n-1} e_b \wedge (i(\epsilon^a)P'_i) \wedge P''_i = 0.$$

This implies $e_b \wedge (i(\epsilon^a)P'_i) = 0$, $i = 1, ..., n-1$, and the $(n-i-1)$-vector $i(\epsilon^a)P'_i$ belongs to the ideal generated by $e_2 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_n$. Therefore, $i(\epsilon^a)P'_i = 0$, except for $i = 1$, and, using again (10),

$$i(\epsilon^a)P'_1 = \kappa e_2 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_n \quad (\kappa \in \mathbf{R}).$$

Accordingly,

$$P'_1 = (-1)^{a-1}\kappa e_2 \wedge ... \wedge e_a \wedge ... \wedge e_n,\ P'_2 = 0,\ ...,\ P'_{n-1} = 0,$$

and we deduce

$$(12) \qquad P = e_2 \wedge ... \wedge e_n \wedge ((-1)^{n-1}e_1 + (-1)^{a-1}P''_1) + P''_n.$$

In other words, P is reducible. But, then, if we take $\alpha = \beta + \gamma \in L^*$, where β vanishes on the second term of (12) but not on the first, and γ vanishes on the first term but not on the second, we see that $i(\alpha)P$ is not decomposable unless $P''_n = 0$. Hence, our P must be decomposable. Q.e.d.

The decomposable n-vectors $P \neq 0$ are important because they define the n-planes, via $im\,\sharp_P$. $\forall \alpha \in L^*$, one has

$$(13) \qquad im\,\sharp_{i(\alpha)P} \subseteq (ker\,\alpha) \cap (im\,\sharp_P),$$

while, if $i(\alpha)P \neq 0$, the subspaces in the right hand side of (13) are transversal in L, and the intersection has the dimension $rank\,P-1$. Hence, if $rank\,(i(\alpha)P) = rank\,P-1$ one has

$$(14) \qquad im\,\sharp_{i(\alpha)P} = (ker\,\alpha) \cap (im\,\sharp_P),$$

In particular, this is always true if P is decomposable.

Notice that, because of (6), if P is reducible, $\exists \alpha \in L^*$ such that $i(\alpha)P \neq 0$ and $rank\,(i(\alpha)P) \leq rank\,P - n < rank\,P - 1$. Therefore, if (14) holds $\forall \alpha \in L^*$ with $i(\alpha)P \neq 0$ hence, $rank\,(i(\alpha)P) = rank\,P - 1$, P is irreducible.

Now, as a particular case of Lemma 3 we get

Lemma 4. *Let L be an m-dimensional linear space, and $P \in \wedge^3 L$. Then, if P satisfies condition (A), $\forall \alpha, \beta \in L^*$, P is decomposable.*

Proof. For any $\alpha \in L^*$, the bivector $Q = i(\alpha)P$ is decomposable, since condition (A) implies $Q \wedge Q = 0$, and in the case of a bivector this is equivalent with the Plücker decomposability condition *ii)* above. Indeed, if Q is decomposable, obviously $Q \wedge Q = 0$. Conversely, $Q \wedge Q = 0$ implies

$$i(\alpha)(Q \wedge Q) = 2i(\alpha)Q \wedge Q = 0, \quad \forall \alpha \in L^*.$$

Therefore, the result follows from Lemma 3. □

From Proposition 2 and Lemma 4 we get

Theorem 5. *A 3-vector field P defines a ternary Poisson bracket on the manifold M iff, around every point $x \in M$, P is decomposable.*

In particular, there is no differential condition to be imposed on a Poisson trivector, since condition (D) is a consequence of decomposability.

Let us also notice

Corollary 6. *A ternary Poisson bracket is a Nambu-Poisson bracket iff the distribution $im\,\sharp_P$ is involutive.*

This follows from the well known fact that, where non zero, a Nambu-Poisson bracket is a Jacobian determinant e.g., [1, 6].

We finish by a few more related remarks.

First, it is known that all the Nambu-Poisson tensors are decomposable. This follows from the fact that they must satisfy the algebraic condition [12]

$$(N1) \qquad \sum_{k=1}^{n}[P^{b_1\dots b_{k-1}ub_{k+1}\dots b_n}P^{va_2\dots a_{n-1}b_k} + P^{b_1\dots b_{k-1}vb_{k+1}\dots b_n}P^{ua_2\dots a_{n-1}b_k}] = 0.$$

In [1] there is an algebraic proof of the fact that (N1) implies decomposability. Lemma 3 above allows for a very short proof of the same result. Namely, (N1) is equivalent to

$$(N2) \qquad i(\alpha)P \wedge i(\Phi)i(\beta)P + i(\beta)P \wedge i(\Phi)i(\alpha)P = 0,$$

$\forall \alpha, \beta \in T^*M$ and $\forall \Phi \in \wedge^{n-2}T^*M$, and (N2) is the polarization of the, once more equivalent, condition

$$(N3) \qquad i(\alpha)P \wedge i(\Phi)i(\alpha)P = 0 \quad \forall \alpha \in T^*M, \forall \Phi \in \wedge^{n-2}T^*M.$$

By the Plücker relations this means that $i(\alpha)P$ is decomposable for all $\alpha \in T^*M$, which by Lemma 3 is equivalent to P being decomposable.

The above proof clarifies the relation between the Plücker and the Nambu decomposability conditions.

Second, an n-ary Poisson structure P of constant rank defines a tensorial G-structure on the manifold M, and it is natural to ask what are the integrability conditions of this structure. Following are two examples of integrable structures, written by means of the corresponding systems of local coordinates:

$$(15) \qquad P = \frac{1}{h!(n-h)!}\sum_{\sigma \in S_n}(sign\,\sigma)\frac{\partial}{\partial y^{\sigma_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{\sigma_h}} \wedge \frac{\partial}{\partial x^{\sigma_{h+1}}} \wedge \dots \wedge \frac{\partial}{\partial x^{\sigma_n}},$$

$$(16) \qquad P = \sum_{i=0}^{s-1}\frac{\partial}{\partial x^{2iu+1}} \wedge \dots \wedge \frac{\partial}{\partial x^{2(iu+u)}}.$$

In (16), we have an integrable Poisson n-vector where $n = 2u$ is even, and the n-vector is reducible and does not satisfy condition (A). We may say that P of (16) is the generalization of a symplectic structure since the latter can be defined by the same formula for $n = 2$.

The last remark is that, if a tensor field P defines a (semi-) decomposable n-ary Poisson structure on a manifold M, there is an interesting Grassmann subalgebra on M, namely, $\Sigma M = \Sigma^q M$ where

$$(17)\qquad \Sigma^q M := \{U \in \Gamma \wedge^q TM \;/\; \forall \alpha \in T^*M,\ (i(\alpha)P) \wedge (i(\alpha)U) = 0\}.$$

On ΣM, a differential operator $\delta : \Sigma^q M \to \Sigma^{q+n-1} M$ may be defined by

$$(18)\qquad \delta U = \sum_{u=1}^{m} ((i(dx^u)P) \wedge (L_{\partial/\partial x^u} U) + (i(dx^u)U) \wedge (L_{\partial/\partial x^u} P)),$$

where (x^u) are local coordinates on M. The fact that δ is well defined follows by the same argument as in the proof of Proposition 2, while using the polarization of condition (17) with respect to α.

In particular, $P \in \Sigma^n M$ and $\delta P = 0$, and for $\forall f \in C^\infty(M)$, $f \in \Sigma^0 M$ and $\delta f = i(df)P$. Generally, we do not have $\delta^2 = 0$, and only the *twisted cohomology* $ker\delta/(im\delta \cap ker\delta)$ [13] can be considered.

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