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DISCONNECTIONS OF PLANE CONTINUA

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ABSTRACT. The existence of an arc which disconnects a locally connected plane continuum X and extension of this arc to a simple closed curve which disconnects X in the same way as given arc is a subject of this paper.

1. At first we remind some known facts about disconnection of continua in the Euclidean plane E^2 ([3]). It is obvious that if empty set disconnects continua $A, B \subset E^2$ (i.e. A, B are disjoint), then there exists a simple closed curve $S \subset E^2$ such that $S \cap (A \cup B)$ is empty set and S disconnects E^2 between the sets A and B. It means that the empty disconnector of $A \cup B$ can be extended to a simple closed curve, which disconnects $A \cup B$ in the same way as empty set. Property of extension of disconnector to a simple closed curve is true for 0-dimensional disconnectors [3]:

Let A, B be continua such that $A \backslash B$ and $B \backslash A$ are connected and $\dim (A \cap B) = 0$. Then there exist a simple closed curve $S \subset E^2$ such that $S \cap (A \cup B) = A \cap B$ and S disconnects E^2 between sets $A \backslash B$ and $B \backslash A$.

For connected disconnectors of dimension 1 we must introduce new conception:

Definition. Let A be a continuum included in topological space X. A disconnects X **irreducibly with respect to subcontinua** between points $x, y \in X$, if A disconnects X between x, y and any proper subcontinuum of A of X has not this property.

Due to this definition we can introduce new fact about extension of disconnector:

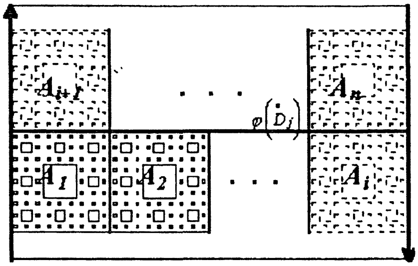
Theorem 1. *Let $X \subset E^2$ be a locally connected continuum. Let $L \subset X$ be a point or an arc which irreducibly with respect to subcontinua disconnects X between points $x, y \in X$. Then there exists a simple closed curve S such that $S \cap X = L$ and the points x, y belong to different components of $E^2 \backslash S$.*

Proof. Let U_x, U_y be the components of $X \backslash L$ containing x, y respectively. Let G_y be a component of $E^2 \backslash (U_x \cup L)$ containing U_y. If L is an arc – the end points of L are elements of $\mathrm{Cl}U_x \cap \mathrm{Cl}U_y$ since L irreducibly disconnects X. Therefore $\dot{L}$ is included in the boundary $\mathrm{Bd}_{E^2} G_y$ of G_y in the Euclidean plane and hence $L \subset \mathrm{Bd}_{E^2} G_y$ and $A_x = \mathrm{Bd}_{E^2} G_y \backslash \mathring{L}$ is connected. Then $A_x \cup L$ disconnects E^2 such that G_y is included in one of components of $E^2 \backslash (A_x \cup L)$ and U_x is included in the closure of the other components of $E^2 \backslash (A_x \cup L)$.

The paper is in final form and no version of it will be submitted elsewhere.

Observe that A_x is included in the boundary of one of components of $E^2\backslash X$, which in turn is included in G_y (since U_x is the component of $X\backslash L$). Denote this component as G.

In the boundary of G can be found (exactly one) a component V (it can occur the equation $V = U_y$) of $X\backslash L$ such that for $B = \mathrm{Cl}_X V \cap \mathrm{Cl}_{E^2} G$ holds: $\dot{L} \subset B$ and B disconnects G_y such that U_y is included in the closure of one of components of $G_y\backslash B$ and A_x is included in the closure of the other one (see the picture below).



Now:

- when L is an arc we can find an arc $L' \subset \mathrm{Cl}_{E^2} G$ such that $\dot{L'}=\dot{L}$ and for which B and A_x are included in closures of different components of $G\backslash L'$. Hence the simple closed curve $S = L \cup L'$ fulfills the conditions of the above theorem;
- when L is a point we can find a simple closed curve $S \subset \mathrm{Cl}_{E^2} G$ such that $L \subset S$ and $S\backslash L \subset G$ for which B and A_x are included in closures of different components of $G\backslash S$ and hence this simple closed curve satisfies the conditions of the theorem;

It completes the proof of Theorem 1.

Theorem 2. *Let A, B be closed and connected subsets of the Euclidean plane E^2 and let $C \subset E^2$ be a locally connected continuum such that $A \cap B = \emptyset$ and $C\backslash (A \cup B)$ is connected. Then there exists a simple closed curve $S \subset E^2\backslash (A \cup B)$ which disconnects E^2 between A and B, and $S \cap C$ is connected.*

Proof. Denote $H = C\backslash (A \cup B)$. Let S_1 be a simple closed curve which disconnects E^2 between A and B. If $S_1 \cap H$ is empty, then $S = S_1$ is a needed curve. Therefore we can assume that $S_1 \cap H \neq \emptyset$.

Let $\delta < \min\{\mathrm{d}(A \cup B, S_1), \mathrm{d}(A, B)\}$.There are only finitely many components of $H \setminus \mathrm{B}\left(A \cup B, \frac{\delta}{2}\right)$, which are not included in $\mathrm{B}(A \cup B, \delta)$. Let $H_1, H_2, ..., H_n$ be an order all of these components into a sequence. Then $S_1 \cap H \subset \bigcup_{i=1}^{n} H_i$.

Since H is connected and open in locally connected continuum C, there exist arcs $L_1, L_2, ..., L_n$ in H such that L_i connects components H_1 and H_i. Therefore the set $P = \bigcup_{i=1}^{n} (H_i \cup L_i)$ is connected and $S_1 \cap H \subset P$. Let S' be a minimal subset of S_1 such that $S' \cup P$ disconnects E^2 between A and B. Then S' is connected (maybe it is the empty set), since P is connected. Let S be a simple closed curve in $S' \cup P$ which disconnects E^2 between A and B. Evidently $S' \subset S$ and $S \cap C \subset P$ and hence the theorem is proved.

Directly from this theorem it follows:

Corollary. *Let A, B be closed and connected subsets of the Euclidean plane E^2 and let $C \subset E^2$ be a locally connected continuum such that $A \cap B = \emptyset$ and both $A \cup B \cup C$ and $C \setminus (A \cup B)$ are connected. Then there exists a locally connected continuum $L \subset C$ which disconnects $A \cup B \cup C$ irreducibly with respect to subcontinua between A and B, and L is a subset of a simple closed curve (i.e. A is a simple closed curve or an arc or a point).*

2. In 1966 K. Borsuk presented a construction of locally plane and locally connected curve which was supposed to be not embedded in any surface ([2], theorem 6.1, pp. 79-81). The Borsuk's example relied on a misconviction that the curve under construction stays to be locally plane after each step of the construction. However this is not the case. As a result the opposite might be true. By using the above theorems it can be proved that ([1], theorem 3.1)

For each locally plane Peano curve X there exists a closed surface such that X is embeddable in this surface.

As the result the only continua for which a homeomorphic embedding into a topological surface does not exist are those continua which are not locally plane or which are not locally connected.

Finally, locally plane Peano continua which appeared to be regular deserve to be investigated further in detail with the well established topological surfaces methods at hand.

References

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