Tomasz Urbański

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INTEGRATION OF A DENSITY AND THE FIBER INTEGRAL FOR REGULAR LIE ALGEBROIDS IN A NONORIENTABLE CASE

URBAŃSKI TOMASZ

Abstract

This paper splits into two parts. The first drives to integral of a density. The second part refers to Lie algebroids. I define an integration operator of A-differential forms with values in an orientation bundle over the bundle of isotropy Lie algebras in vertically oriented Lie algebroid A. I establish five basic properties of this operator, its commutation with an exterior and Lie derivations. Some of them are proved here.

1 Introduction

Basic facts and concepts with respect to Lie algebroids can be found in [2], [3], [1], [4]. Required results referring to vertically oriented Lie algebroids and the fiber integral of $\mathbb{R}$-valued forms are included in [3].

R.Bott, in the work [5], has defined an integration operation of differential forms on manifolds with values in an orientation bundle. This operation was a tool to cohomological researches of nonorientable manifolds. The aim of the presented work is to introduce an analogous fiber integral of or_M-valued forms on the ground of regular Lie algebroids with usage of ideas which comes from works [3] and [5].

In perspective, this work drives to an examination of a cohomology algebra of a regular Lie algebroid over a nonoriented base manifold.

In this paper we associate n-dimensional manifolds M and N with differential structures $\mathfrak{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ and $\mathfrak{B} = \{(V_\beta, y_\beta)\}_{\beta \in J}$ respectively.

2 Differential Forms with Values in an Orientation Bundle

2.1 Pullback of Forms with Values in an Orientation Bundle

Consider an orientation bundle or_N of the differential manifold N [5]. Let $\Omega\,(N; \mathrm{or}_N)$ be the vector space of differential forms on N with values in the orientation bundle or_N. So, k-form Φ is a global section of the vector bundle

$$\bigwedge^k T^*N \otimes \mathrm{or}_N\,.$$

Pointwise we have $\Phi_q : \bigwedge^k T_qN \to \mathrm{or}_N|_q$. In many sources an element of the $\Omega(N; \mathrm{or}_N)$ is also called a *density*.

To define a pullback operation assume that U is an open subset of M, V is an open subset of N and $T : U \to V$ is a diffeomorphism. Then, it is easy to show, that there exist an induced isomorphism of vector bundles $\tilde{T} : \mathrm{or}_M|_U \to \mathrm{or}_N|_V$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}[U] & \xrightarrow{\tilde{T}} & \pi^{-1}[V] \\ \pi \downarrow & & \downarrow \pi \\ U & \xrightarrow[T]{} & V \end{array}$$

commutes. Knowing that, for arbitrary $\Phi \in \Omega^k(N; \mathrm{or}_N)$ we define a form $T^*\Phi \in \Omega^k(U; \mathrm{or}_M|_U)$ by a formula

$$(T^*\Phi)_p(v_1 \wedge \ldots \wedge v_k) = \tilde{T}_p^{-1}\left(\Phi_{T(p)}(T_{*p}v_1 \wedge \ldots \wedge T_{*p}v_k)\right), \qquad p \in U.$$

If we suppose that $\omega \in \Omega^k(N)$, $e \in \mathrm{Sec}\,\mathrm{or}_N$, it is natural to define a form $\omega \otimes e \in \Omega^k(N; \mathrm{or}_N)$ by

$$\begin{array}{rcl} (\omega \otimes e)_q = \omega_q \otimes e_q : \bigwedge^k T_qN & \longrightarrow & \mathrm{or}_N|_q \\ v_1 \wedge \ldots \wedge v_k & \longmapsto & \omega_q(v_1 \wedge \ldots \wedge v_k) \cdot e_q. \end{array}$$

Than for any $p \in U$ holds an equality

$$(T^*(\omega \otimes e))_p = (T^*\omega)_p \otimes \tilde{T}_p^{-1}\left(e_{T(p)}\right). \tag{1}$$

For each $\alpha \in I$ denote by e_α the map given by

$$\begin{array}{rcl} e_\alpha : U_\alpha & \longrightarrow & \mathrm{or}_M \\ p & \longmapsto & [(\alpha, p, 1)]^{\cdot}. \end{array} \tag{2}$$

It states the vector basis of a module $\mathrm{Sec}\,\mathrm{or}_M|_{U_\alpha}$. Assume in addition $x_\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$ is a local coordinate map of the manifold M corresponding to α and ω is a form given by $dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha$. Then we define a form $|dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha|$ with values in an orientation bundle or_M, by

$$|dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha| = (dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha) \otimes e_\alpha. \tag{3}$$

Now we can establish

Proposition 1 *Suppose (U_α, x_α), (V_β, y_β) are two charts on M and N respectively, and let $T : U_\alpha \to V_\beta$ be a diffeomorphism (not necessary orientation-preserving). Then we have a relation*

$$T^*\left|dy_1^\beta \wedge \ldots \wedge dy_n^\beta\right| = |\mathrm{J}(T_{\beta\alpha})| \cdot |dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha|.$$

Indeed, for each $p \in U_\alpha$, from (1) and the abvious equality $\operatorname{sgn} \mathrm{J}\left(T_{\alpha\beta}^{-1}(T(p))\right) = \operatorname{sgn} \mathrm{J}(T_{\beta\alpha}(p))$, we see that

$$\begin{aligned}
&\left(T^* \left|dy_1^\beta \wedge \ldots \wedge dy_n^\beta\right|\right)_p \\
= &\left(T^*\left(dy_1^\beta \wedge \ldots \wedge dy_n^\beta\right)\right)_p \otimes \tilde{T}_p^{-1}(e_\beta(T(p))) \\
= &\left(T^*\left(dy_1^\beta \wedge \ldots \wedge dy_n^\beta\right)\right)_p \otimes (\operatorname{sgn} \mathrm{J}(T_{\beta\alpha}(p)) \cdot e_\alpha(p)) \\
= &\left(\mathrm{J}(T_{\beta\alpha}(p)) \cdot (dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha)_p\right) \otimes (\operatorname{sgn} \mathrm{J}(T_{\beta\alpha}(p)) \cdot e_\alpha(p)) \\
= &(\operatorname{sgn} \mathrm{J}(T_{\beta\alpha}(p)) \cdot \mathrm{J}(T_{\beta\alpha}(p))) \cdot \left((dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha)_p \otimes e_\alpha(p)\right) \\
= &(|\mathrm{J}(T_{\beta\alpha})| \cdot |dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha|)_p .
\end{aligned}$$

In particular it follows, that if $g \in \Omega^0(N)$ is an arbitrary real function, then

$$T^*\left(g \cdot \left|dy_1^\beta \wedge \ldots \wedge dy_n^\beta\right|\right) = (g \circ T) \cdot |J(T_{\beta\alpha})|\, |dx_1^\alpha \wedge \ldots \wedge dx_n^\alpha| .$$

2.2 Integral of a Density

Let pair $(\mathbb{R}^n, y = id)$ be the canonical identity chart, U an open subset of $\mathbb{R}^n$, and $g \in \Omega^0(U)$ a measurable function on U. We define

$$\int_U g \cdot |dy_1 \wedge \ldots \wedge dy_n| = \int_U g\, dy_1 \ldots dy_n.$$

Suppose furthermore, that V is an open subset of $\mathbb{R}^n$, $T : U \to V$ is a diffeomorphism and let $\Phi \in \Omega_c^n(\mathbb{R}^n; \mathrm{or}_{\mathbb{R}^n})$ be such a form, that $\operatorname{supp} \Phi \subset V$. Then, by the classical change of variable formula

$$\begin{aligned}
\int_U T^*\Phi &= \int_U (g \circ T)\, |JT| \cdot |dy_1 \wedge \ldots \wedge dy_n| \\
&= \int_V g \cdot |dy_1 \wedge \ldots \wedge dy_n| \\
&= \int_V \Phi.
\end{aligned}$$

On arbitrary manifold M and a form $\Phi \in \Omega_c^n(M; \mathrm{or}_M)$ we define an integral

$$\int_M \Phi$$

in the following manner

- take an atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ (not necessary maximal),
- take a subordinate partition of unity $\{\rho_\alpha\}_{\alpha \in I}$,

- assume

$$\int_M \Phi = \sum_\alpha \int_{x_\alpha[U_\alpha]} (x_\alpha^{-1})^* (\rho_\alpha \cdot \Phi).$$

It can be easily shown, that above definition doesn't depend on choice of the atlas and the patition of unity.

3 Fiber Integral of a Density in a Vertically Oriented Lie Algebroid

3.1 Definition and Basic Properties

Let $\Omega_A(M; \mathrm{or}_M)$ denotes a vector space of A-differential forms with values in an orientation bundle or_M, where A is an arbitrary Lie algebroid over the manifold M, i.e. a space of all cross-sections of $\bigwedge A^* \bigotimes \mathrm{or}_M$.

Definition 1 *Suppose in addition, that A' is a second arbitrary Lie algebroid over the manifold N, and $H : A|_U \to A'|_{U'}$ is a homomorphism A in A' inducing a diffeomorphism $\hat{H}$ of open subsets $U \subset M$ on $V \subset N$. Let $\tilde{\hat{H}} : \mathrm{or}_M|_U \to \mathrm{or}_N|_V$ be the isomorphism of the vector bundles induced by $\hat{H}$. Then, for each form $\Phi \in \Omega^k_{A'}(N; \mathrm{or}_N)$ we define a form $H^*\Phi \in \Omega^*_A(U; \mathrm{or}_M|_U)$ by*

$$(H^*\Phi)_p(t_1 \wedge \ldots \wedge t_k) = \tilde{\hat{H}}_p^{-1}\left(\Phi_{\hat{H}(p)}(Ht_1 \wedge \ldots \wedge Ht_k)\right), \qquad p \in U.$$

Definition 2 *An ordered pair (A, ε) is called* vertically oriented Lie algebroid, *if A is a regular Lie algebroid of rank n over a foliated manifold $(M, \mathcal{F})$,*

$$0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} \mathcal{F} \to 0$$

is its Atiyah sequence, and ε is nowhere vanishing cross-section of the bundle $\bigwedge^n \boldsymbol{g}$.

Definition 3 *Let (A', ε') be one more vertically oriented Lie algebroid over a foliated manifold $(A', \mathcal{F}')$, and suppose that $\operatorname{rank} \boldsymbol{g} = \operatorname{rank} \boldsymbol{g}'$. A homomorphism of Lie algebroids $H : A \to A'$, which induces $\hat{H} : M \to M'$ and fulfils condition*

$$\left(\bigwedge^n H^+\right)(\varepsilon_p) = \varepsilon'_{\hat{H}p}, \qquad p \in M$$

is called homomorphism of vertically oriented Lie algebroids (A, ε) into (A', ε').

Since for any $\Phi \in \Omega^{n+k}_A(M; \mathrm{or}_M)$, $k \geq 0$, the form $\imath_\varepsilon \Phi \in \Omega^k_A(M; \mathrm{or}_M)$ defined by

$$(\imath_\varepsilon \Phi)_p(t_1 \wedge \ldots \wedge t_k) = \Phi_p(\varepsilon_p \wedge t_1 \wedge \ldots \wedge t_k), \qquad p \in M, t_i \in A|_p$$

is horizontal (i.e. $\imath_h(\imath_\varepsilon \Phi) = 0$ for $\eta \in \operatorname{Sec} \boldsymbol{g}$), there exists uniquely determined tangential differential form $\Psi \in \Omega^k_{\mathcal{F}}(M; \mathrm{or}_M)$ such that $\imath_\varepsilon \Phi = \gamma^* \Psi$. Assume furthermore, that if $\deg \Phi < n$, then $\imath_\varepsilon \Phi = 0$.

Definition 4 *By an integration operator of A-differential forms on M with values in an orientation bundle or_M over the bundle of isotropy Lie algebras g in the vertically oriented Lie algebroid (A,ε) we mean the operator*

$$\fint_A : \Omega_A^{\bullet}(M;\mathrm{or}_M) \longrightarrow \Omega_{\mathcal{F}}^{\bullet-n}(M;\mathrm{or}_M)$$

such that for each $\Phi \in \Omega_A^{n+k}(M;\mathrm{or}_M)$ the value $\fint_A \Phi \in \Phi_{\mathcal{F}}^k(M;\mathrm{or}_M)$ is the uniquely determined form defined by the formula

$$\gamma^*\left(\fint_A \Phi\right) = (-1)^{nk}\,\imath_\varepsilon \Phi.$$

Proposition 2 *Integration operator defined above has the following properties*

(a) *If $H : (A,\varepsilon) \to (A',\varepsilon')$ is a homomorphism of vertically oriented Lie algebroids inducing the diffeomorphism $\hat{H}$ of open subsets $U \subset M$ on $V \subset N$, then there is an equality*

$$\hat{H}^* \circ \fint_{A'} = \fint_A \circ H^*$$

on U.

(b) $\fint_A \circ \gamma^* = 0,$

(c) $\fint_A \gamma^*\Psi \wedge \phi = \Psi \wedge \fint_A \phi$ *for arbitrary forms $\Psi \in \Omega_{\mathcal{F}}(M;\mathrm{or}_M)$ and $\phi \in \Omega_A(M)$,*

(d) $\fint_A \phi \wedge \gamma^*\Psi = (-1)^{nk}\left(\fint_A \phi\right) \wedge \Psi$ *for arbitrary forms $\Psi \in \Omega_{\mathcal{F}}^k(M;\mathrm{or}_M)$, $\phi \in \Omega_A^{\geq n}(M)$,*

(e) $\fint_A$ *is an epimorphism.*

We will omit proofs of properties (a) and (b) because they are based on simple calculations. Now we will set to proving the formula (c).

Let k, q be arbitrary integer numbers and $\Psi \in \Omega_{\mathcal{F}}^k(M;\mathrm{or}_M)$, $\phi \in \Omega_A^q(M)$. Locally we can write

$$\Psi = \psi \otimes e_\alpha,$$

where $\psi \in \Omega_{\mathcal{F}}^k(M)$, and e_α is defined in (2). Consider two cases

- if $k+q<n$, then both sides of the proved formula are equal to zero,
- if $k+q \geq n$, then there are two possible situations

  1. $q<n$. Then $\fint_A \phi = 0$, so it should be proved, that

  $$\fint_A \gamma^*\Psi \wedge \phi = 0,$$

  but it is easy to see by a simple calculation.

2. $q \geq n$. To prove considered formula it is enough to show, that

$$\gamma^*\left(\Psi \wedge \fint_A \phi\right) = (-1)^{n(k+q-n)} \imath_\varepsilon (\gamma^*\Psi \wedge \phi).$$

To see it, let $p \in M$ be an arbitrary point and $t_i \in A|_p$, $i = 1, \ldots, k+q$ be such that $\varepsilon_p = t_1 \wedge \ldots \wedge t_n$. Then

$$\begin{aligned}
&\left(\gamma^*\left(\Psi \wedge \fint_A \phi\right)\right)_p (t_{n+1} \wedge \ldots \wedge t_{k+q}) \\
= &\left(\gamma^*\Psi \wedge \gamma^* \fint_A \phi\right)_p (t_{n+1} \wedge \ldots \wedge t_{k+q}) \\
= &\left(\gamma^*\Psi \wedge (-1)^{n(q-n)} \imath_\varepsilon \phi\right)_p (t_{n+1} \wedge \ldots \wedge t_{k+q}) \\
= &(-1)^{n(q-n)+k(q-n)} (\imath_\varepsilon \phi \wedge \gamma^*\Psi)_p (t_{n+1} \wedge \ldots \wedge t_{k+q}) \\
= &(-1)^{n(q-n)+k(q-n)} (\phi \wedge \gamma^*\Psi)_p (t_1 \wedge \ldots \wedge t_{k+q}) \\
= &(-1)^{n(q-n)+k(q-n)+qk} (\gamma^*\Psi \wedge \phi)_p (t_1 \wedge \ldots \wedge t_{k+q}) \\
= &(-1)^{n(k+q-n)} \imath_\varepsilon (\gamma^*\Psi \wedge \phi)_p (t_{n+1} \wedge \ldots \wedge t_{k+q}).
\end{aligned}$$

Property (d) is a simple corollary of the formula (c), which we have just proved.

To show the property (e), let consider a section $\sigma \in \mathrm{Sec} \bigwedge^n g^*$ such that $\imath_\varepsilon \sigma = 1$ and a form of the connection $\kappa : A \to g$ (than $\kappa|_g = id$). Than for arbitrary $\Psi \in \Omega_{\mathcal{F}}(M; \mathrm{or}_M)$ there holds an equality

$$\fint_A \gamma^*\Psi \wedge \kappa^*\sigma = \Psi.$$

Indeed,

$$\fint_A \gamma^*\Psi \wedge \kappa^*\sigma = \Psi \wedge \fint_A \kappa^*\sigma,$$

but

$$\gamma^*\left(\fint_A \kappa^*\sigma\right) = (-1)^{n \cdot 0} \imath_\varepsilon (\kappa^*\sigma) = \sigma(\kappa \circ \varepsilon) = \sigma(\varepsilon) = 1.$$

4 Commutation of the Integration Operator with Derivatives

4.1 Construction of exterior derivatives and Lie derivatives

Let $X \in \mathfrak{X}(M)$ be an arbitrary vector field, $\alpha \in I$ be any index, $f \in C^\infty(U_\alpha)$ and $e_\alpha : U_\alpha \to \mathrm{or}_M$. Then the formula

$$\nabla_X (f e_\alpha) = X(f) \cdot e_\alpha,$$

and

$$
\begin{aligned}
&(d_{\mathcal{F}}^{\mathrm{or}})(\Psi)(X_0 \wedge \ldots \wedge X_k) = \\
&\sum_{i=0}^{k} (-1)^i \mathcal{L}_{\lambda^{\nabla}|_{\mathcal{F}} \circ X_i} \left(\Psi \left(X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_k \right) \right) \\
&+ \sum_{i<j}^{k} (-1)^{i+j} \Psi \left([X_i, X_j] \wedge X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_k \right), \\
&(d_A^{\mathrm{or}})(\Phi)(\eta_0 \wedge \ldots \wedge \eta_k) = \\
&\sum_{i=0}^{k} (-1)^i \mathcal{L}_{L \circ \eta_i} (\Phi (\eta_0 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_k)) \\
&+ \sum_{i<j}^{k} (-1)^{i+j} \Phi \left([\![\eta_i, \eta_j]\!] \wedge \eta_0 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_k \right).
\end{aligned}
$$

Remark 1 *Incidentally, there is an exterior derivation operator* $d : \Omega^* (M; \mathrm{or}_M) \to \Omega^{*+1} (M; \mathrm{or}_M)$ *locally defined, over* U_α *by*

$$d (\omega \otimes e_\alpha) = (d\omega) \otimes e_\alpha$$

in the R. Bott's book [5, p. 80]. See, that operator $d_{\mathcal{F}}^{\mathrm{or}}$ *states its interpretation in the names of algebroids.*

Proposition 3 *There holds an equality*

$$\gamma^* \circ d_{\mathcal{F}}^{\mathrm{or}} = d_A^{\mathrm{or}} \circ \gamma^*, \tag{4}$$

where $\gamma : A \to \mathcal{F}$ *is an anchor.*

4.2 Theorems of a Destination

Theorem 4 *The integration operator* $⨏_A$ *of A-differential forms on M with values in an orientation bundle* or_M *over the bundle of isotropy algebras* g *in the vertically oriented Lie algebroid* (A, ε) *commutes with exterior derivatives*

$$d_{\mathcal{F}}^{\mathrm{or}} \circ ⨏_A = ⨏_A \circ d_A^{\mathrm{or}} \tag{5}$$

if and only if

(a1) *the isotropy Lie algebras* $g|_p$ *are unimodular, and*

(a2) *the cross-section* ε *is invariant with respect to the adjoint representation of A on* $\bigwedge^n g$.

defines in a proper way the covariant derivative

$$\nabla : \mathfrak{X}(M) \times \operatorname{Sec} \mathrm{or}_M \longrightarrow \operatorname{Sec} \mathrm{or}_M .$$

Hence, the map

$$\lambda^{\nabla} : TM \longrightarrow A(\mathrm{or}_M)$$

defined by

$$\lambda^{\nabla}(v) = \nabla_v(\cdot), \qquad v \in TM$$

is a connection in the regular Lie algebroid $A(\mathrm{or}_M)$. Since ∇ is a flat connection, λ^{∇} is a homomorphism of the Lie algebroids, whence the map

$$L : A \to A(\mathrm{or}_M)$$

defined by the formula

$$L = \lambda^{\nabla} \circ \gamma$$

states a representation of the Lie algebroid A in the orientation bundle or_M (for the definition of a representation see [4]).

Now we have operators

$$\begin{aligned} (\theta^{\mathrm{or}}_{\mathcal{F}})_X &: \Omega_{\mathcal{F}}(M; \mathrm{or}_M) \longrightarrow \Omega_{\mathcal{F}}(M; \mathrm{or}_M), \\ (\theta^{\mathrm{or}}_{A})_{\eta} &: \Omega_{A}(M; \mathrm{or}_M) \longrightarrow \Omega_{A}(M; \mathrm{or}_M), \end{aligned}$$

and

$$\begin{aligned} d^{\mathrm{or}}_{\mathcal{F}} &: \Omega_{\mathcal{F}}(M; \mathrm{or}_M) \longrightarrow \Omega_{\mathcal{F}}(M; \mathrm{or}_M), \\ d^{\mathrm{or}}_{A} &: \Omega_{A}(M; \mathrm{or}_M) \longrightarrow \Omega_{A}(M; \mathrm{or}_M) \end{aligned}$$

called Lie derivatives (with respect to the $\mathcal{F}$-tangent field X, and the cross-section $\eta \in \operatorname{Sec} A$ respectively), and exterior derivatives respectively, described by the formulae

$$\begin{aligned} (\theta^{\mathrm{or}}_{\mathcal{F}})_X(\Psi)(X_1 \wedge \ldots \wedge X_k) &= \mathcal{L}_{\lambda^{\nabla}|_{\mathcal{F}} \circ X}(\Psi(X_1 \wedge \ldots \wedge X_k)) + \\ &\quad - \sum_{i=1}^{k} \Psi(X_1 \wedge \ldots \wedge [X, X_i] \wedge \ldots \wedge X_k), \\ (\theta^{\mathrm{or}}_{A})_{\eta}(\Phi)(\eta_1 \wedge \ldots \wedge \eta_k) &= \mathcal{L}_{L \circ \eta}(\Phi(\eta_1 \wedge \ldots \wedge \eta_k)) + \\ &\quad - \sum_{i=1}^{k} \Phi(\eta_1 \wedge \ldots \wedge [\![\eta, \eta_i]\!] \wedge \ldots \wedge \eta_k), \end{aligned}$$

It is easy to see that γ^* is a monomorphism. So, we can express proved equality in a form

$$\gamma^*\left(d^{\mathrm{or}}_{\mathcal{F}}\int_A \Phi\right) = \gamma^*\left(\int_A (d^{\mathrm{or}}_A \Phi)\right), \qquad \Phi \in \Omega_A(M; \mathrm{or}_M).$$

Next, from (4) we obtain the following appearance of the proved formula

$$d^{\mathrm{or}}_A\left(\gamma^*\left(\int_A \Phi\right)\right) = \gamma^*\left(\int_A (d^{\mathrm{or}}_A \Phi)\right), \qquad \Phi \in \Omega_A(M; \mathrm{or}_M).$$

Finally, from the definition of an operator $\int_A$ we see, that we can focus below on the equality

$$d^{\mathrm{or}}_A \circ \imath_\varepsilon(\Phi) = (-1)^n \imath_\varepsilon \circ d^{\mathrm{or}}_A(\Phi), \qquad \Phi \in \Omega_A(M; \mathrm{or}_M). \tag{6}$$

It ensue from the definition of the operator $\imath$, that if $\deg\Phi < n-1$, then both sides of (6) are permanent equal to zero. The same argument proves, that when $\deg\Phi = n-1$, then equality (6) refines to formula

$$\imath_\varepsilon \circ d^{\mathrm{or}}_A(\Phi) = 0, \qquad \Phi \in \Omega^{n-1}_A(M; \mathrm{or}_M).$$

Further, we have to give two technical lemmas.

Lemma 5 *Equality (6) takes place for each form $\Phi \in \Omega^{n+k}_A(M; \mathrm{or}_M)$ ($k \geq 0$ is fixed, and $n+k \leq \operatorname{rank} A$) if and only if for arbitrary sections $\xi_1, \ldots, \xi_{k+1} \in SecA$ and for arbitrary chosen neighbourhood $U \subset M$ on which $\varepsilon = \sigma_1 \wedge \ldots \wedge \sigma_n$ for some $\sigma_i \in \mathrm{Sec}\,\boldsymbol{g}$ (each point $p \in M$ has a neighbourhood U, for which ε is in such a form), holds the following equality*

$$0 = \left(\sum_{i<j}(-1)^{i+j}[\sigma_i,\sigma_j] \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \hat{\sigma}_j \wedge \ldots \wedge \sigma_n\right) \wedge \wedge\xi_1 \wedge \ldots \wedge \xi_{k+1} + \sum_j (-1)^{j+n}\left(\sum_i \sigma_1 \wedge \ldots \wedge [\![\xi_j, \sigma_i]\!] \wedge \ldots \wedge \sigma_n\right) \wedge \wedge\xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_{k+1}.$$

Lemma 6 *Equality (6) takes place for each form $\Phi \in \Omega^{n-1}_A(M; \mathrm{or}_M)$ if and only if*

$$\left.\sum_{i<j}(-1)^{i+j}\left([\sigma_i,\sigma_j] \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \hat{\sigma}_j \wedge \ldots \wedge \sigma_n\right)\right|_U = 0$$

(when U and σ_i are such, as in the above lemma) or equivalently, when $\boldsymbol{g}|_p, p \in M$ is unimodular.

Now, suppose that holds the equality (5). Considering forms of degree $n-1$, we obtain from the first lemma, unimodularity of $g|_p, p \in M$. Taking it under consideration, for arbitrary form of degree n (i.e. $k=0$), in view of the second lemma, we get the equality

$$\sum_i \sigma_1 \wedge \ldots \wedge [\eta, \sigma_i] \wedge \ldots \wedge \sigma_n \Big|_U = 0, \qquad \eta \in \operatorname{Sec} A,$$

which implies, that

$$\bigwedge^n \mathrm{ad}_A(\eta)(\varepsilon) \Big|_U = 0,$$

i.e. the invariance of the section ε with respect to adjoint representation A in $\bigwedge^n g$.

Assume now, that there hold conditions (a1) and (a2). The unimodularity of $g|_p, p \in M$ gives, according to the second lemma, equality (6) for forms of degree $n-1$. For forms of degree $\geq n$, the equality (6) follows from the first lemma.

Corollary 7 *The integration operator $\fint_A$ in unimodular invariantly oriented Lie algebroid (A, ε) of A-differential forms on M with values in an orientation bundle* or_M *over the bundle of isotropy algebras g induces morphism*

$$\fint_A^{\#} : H_A^{*}(M; \mathrm{or}_M) \longrightarrow H_{\mathcal{F}}^{*-n}(M; \mathrm{or}_M),$$

where $H_A(M;\mathrm{or}_M)$, $H_{\mathcal{F}}(M;\mathrm{or}_M)$ are cohomology algebras in the algebra $\Omega_A(M;\mathrm{or}_M)$ with respect to the operator d_A^{or}, and the algebra $\Omega_{\mathcal{F}}(M;\mathrm{or}_M)$ with respect to the operator $d_{\mathcal{F}}^{\mathrm{or}}$ respectively.

Theorem 8 *The integration operator $\fint_A$ in unimodular invariantly oriented Lie algebroid (A, ε) of A-differential forms on M with values in an orientation bundle* or_M *over the bundle of isotropy algebras g commutes with Lie derivatives*

$$(\theta_{\mathcal{F}}^{\mathrm{or}})_X \circ \fint_A = \fint_A \circ (\theta_A^{\mathrm{or}})_\eta,$$

where $\eta \in \operatorname{Sec} A$ and $X \in \mathfrak{X}(M)$ are γ-related.

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