Gerald Fischer

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A REPRESENTATION OF THE COALGEBRA OF DERIVATIONS FOR SMOOTH SPACES

GERALD FISCHER

Abstract

The generalized Leibniz-rule for higher derivations defines an abstract coalgebra. This can be represented for smooth spaces by a procedure of microlocalization.

1 Introduction

According to an idea of Gelfand there is a correspondence between commutative algebras and geometric spaces. This indeed follows the principle of quantum theory to describe everything by an algebra of observables. In order to allow a closer look on geometric spaces, what is necessary in the study of singularities for example, higher order geometric objects have to be involved. On the algebraic side this job could be done by what we call a higher order study of the algebra of observables. Higher order study here means an embedding into a $\mathbb{N}$-graded algebra where the degree of the grading gives the order. We do this for an arbitrary commutative K-algebra A with some field K and a $\mathbb{N}$-graded K-algebra

$$\dot{B} = \bigoplus_{n=0}^{\infty} B_n \tag{1}$$

which is the direct limit of the truncated algebras of finite order

$$tr_k B := \bigoplus_{n=0}^{k} B_n. \tag{2}$$

The multiplication in B

$$\sum_{n=0}^{k} a_n \cdot \sum_{m=0}^{k} b_m = \sum_{i=0}^{k} \sum_{l+s=i} a_l b_s \tag{3}$$

implies that the projection maps

*The paper is in final form and no version of it will be submitted elsewhere.

$$()_i : B \longrightarrow B_i \tag{4}$$

fulfill a generalized Leibniz-rule

$$(a \cdot b)_i = \sum_{l+s=i} (a)_l (b)_s \ , \ a, b \in B. \tag{5}$$

An embedding of the algebra A into the algebra B should be a special homomorphism from A to B which contains an isomorphism from A to B_0. Such a homomorphism is given by a sequence of maps $\phi = (\phi_0, \phi_1, \ldots)$ with

$$\phi_i = ()_i \circ \phi : A \longrightarrow B_i. \tag{6}$$

The homomorphism property implies the Leibniz-rule

$$()_i \circ \phi(a \cdot b) = (\phi(a) \cdot \phi(b))_i = \sum_{l+s=i} (\phi(a))_l (\phi(b))_s \ , \ a, b \in A \tag{7}$$

and the embedding ϕ is a derivation of higher order [Be] with $\phi_0 \cong id$ and $A \cong B_0$. One is lead to a natural choice for the algebra B by relying only on the multiplication map of A

$$\mu : A \otimes_K A \longrightarrow A \tag{8}$$

and its kernel $ker\mu \subset A \otimes_K A$. Because of the commutativity of A this is an ideal and the filtration which it defines in $A \otimes_K A$ gives rise to the associated graded algebra [La]

$$\bar{A} := ass_{ker\mu} A \otimes_K A = \bigoplus_{n=0}^{\infty} ker\mu^n / ker\mu^{n+1} \tag{9}$$

with $ker\mu^0 = A \otimes_K A$.
The first steps to an embedding into this algebra $\bar{A}$ are done by the wellknown construction of the universal derivation for the algebra A [La]

$$\begin{aligned} d: \ A \ &\longrightarrow \Omega_{A|K} \\ a \ &\mapsto da = 1 \otimes_K a - a \otimes_K 1 \, mod \, ker\mu^2 \end{aligned} \tag{10}$$

what gives for the universal differential modul

$$\Omega_{A|K} \cong ker\mu / ker\mu^2. \tag{11}$$

Together with the canonical isomorphism $A \cong A \otimes_K A / ker\mu$ the embedding up to first order (ϕ_0, ϕ_1) is fixed. In the case that $\bar{A}$ allows for a "developement"

$$p_k : A \otimes_K A / ker\mu^{k+1} \longrightarrow ker\mu^k / ker\mu^{k+1} \tag{12}$$

we can define the higher order maps ϕ_k by

$$\phi_k(a) = p_k(1 \otimes_K a - a \otimes_K 1 \, mod \, ker \mu^{k+1}). \tag{13}$$

This fulfills the Leibniz-rule of order k and the sequence ϕ of higher order differentials gives an embedding of A into $\bar{A}$.

2 The coalgebra of derivations

The homomorphism property of an embedding results in the general Leibniz-rule, which plays the central part in what follows. In a more abstract form the Leibniz-rule for a higher order derivation $D = (d_0, \ldots, d_k) : A \to B$ looks as

$$d_i \circ \mu_A = \mu_B \circ \Delta(d_i) : A \otimes_K A \longrightarrow B \tag{14}$$

with $\Delta(d_i) = \sum_{l+s=i} d_l \otimes d_s$, $i \in \{0, \ldots, k\}$. Hence the Leibniz-rule is a comultiplication map of a coalgebra. So we define the abstract coalgebra of K-derivations up to order k:

Definition: 1 (Coalgebra of Derivations) *The K-linear space*

$$\mathcal{D}_K^k = span_K(d_0, \ldots, d_k)$$

with abstract basis elements $d_0, \ldots, d_k$ equipped with the counit ϵ

$$\begin{aligned} \epsilon \ : \ & \mathcal{D}_K^k \longrightarrow K \\ & x \mapsto \epsilon(x) = \epsilon(\sum_{i=0}^{k} \alpha_i d_i) := \alpha_0 \end{aligned} \tag{15}$$

and the comultiplication

$$\begin{aligned} \Delta \ : \ & \mathcal{D}_K^k \longrightarrow \mathcal{D}_K^k \otimes_K \mathcal{D}_K^k \\ & d_i \mapsto \Delta(d_i) := \sum_{l+s=i} d_l \otimes d_s \end{aligned} \tag{16}$$

is called the coalgebra of K-derivations up to order k.

The embedding ϕ in the higher order study of an algebra arises as a representation of this coalgebra. Therefor we give the definition:

Definition: 2 (Representation of a Coalgebra) *A representation of a K-coalgebra $\mathcal{C}$ on a K-algebra C is a K-linear map $\rho : \mathcal{C} \to End_K(C)$ with*

$$i) \quad \rho(x)1_C = \epsilon(x)1_C \tag{17}$$

$$ii) \quad \rho(x)\mu_C(a \otimes b) = \mu_C\left((\rho \otimes \rho)(\Delta(x))a \otimes b\right), \ a, b \in C. \tag{18}$$

The second property is the implementation of the comultiplication, in the case $\mathcal{C} = \mathcal{D}$ the implementation of the Leibniz-rule. In this sense the embedding of an algebra in the introduction is a representation of $\mathcal{D}_K$. In the following we construct another such representation for $K = \mathbb{R}$ which comes more or less canonically along with the differential structure of $\mathbb{R}$.

3 Smooth spaces

A higher order study for smooth manifolds means taking into account higher order geometric objects [F1] or in the language of this note a higher order study of the algebra of observables which then is $C^\infty(M,\mathbb{R})$ the algebra of smooth functions. Choosing the more general definition of smooth structures due to Froehlicher and Kriegl one is brought very easily to a procedure providing a representation of $\mathcal{D}_{\mathbb{R}}$ on the smooth functions.

Definition: 3 (Froehlicher–Kriegl Smooth Spaces [FK]) *A smooth space in the sense of Froelicher and Kriegl is a triple $(M, \Gamma_M, \mathcal{C}_M)$ of a set M, a family Γ_M of smooth curves in M and a family $\mathcal{C}_M$ of smooth functions on M. The two families of maps from and to $\mathbb{R}$ determine each other in the way that a map belongs to one of these sets if and only if its composites with all the maps in the other family are in $C^\infty(\mathbb{R},\mathbb{R})$.*

A morphism in the category of smooth spaces is a map $g : M \to N$ such that

$$g^*(\mathcal{C}_N) \subset \mathcal{C}_M. \tag{19}$$

The subset relation defines a partial order in the set of smooth structures on the set M. This allows to assign to every set $\mathcal{F}$ of functions on M a unique differential structure $gen(\mathcal{F})$ which is the smallest one containing $\mathcal{F}$.
The definition of smooth structures with the help of curves has an interesting quantum theoretic aspect. The algebra of smooth functions is the algebra of observables for a smooth space. Dual to the observables are the states, the extreme positive functionals on the algebra of observables [Ha]. Here they are given by the evaluation functionals ev_x and so correspond to the points x in M. More information about the system than the states provides the consideration of the dynamics of the states $x(t)$, what in our case is the study of curves in M. So the set Γ_M can be considered as the set of all possible dynamices for M.
With the help of the observables $\mathcal{C}_M$ we introduce equivalence relations on this set Γ_M:

Definition: 4 *We say that $\gamma_1, \gamma_2 \in \Gamma_M$ have contact of order k, denoted by $\gamma_1 \sim_k \gamma_2$, if*

$$\frac{d^l}{dt^l}\left(f(\gamma_1(t)) - f(\gamma_2(t))\right)|_{t=0} = 0\,,\ \forall f \in \mathcal{C}_M, l \leq k. \tag{20}$$

$[\gamma]_k$ denotes the equivalence classes along these relations for a representant γ. In analogy to the embedding of algebras in the introduction we do now an embedding of spaces. The analog of the degree in the $\mathbb{N}$-graded algebra here is the order of contact which we use for the construction of a sequence of smooth spaces. As the underlying sets we take the set of equivalence classes

$$T^kM := \{[\gamma]_k | \gamma \in \Gamma_M\}\,. \tag{21}$$

In oder to provide these sets with suitable smooth structures we introduce the microlacalization map:

Definition: 5 *Let $\mathcal{F}(T^kM)$ be the set of real functions on T^kM. The microlocalization map*

$$m_k : \mathcal{C}_M \longrightarrow \mathcal{F}(T^kM) \tag{22}$$

is defined by

$$m_k(f)([\gamma]_k) := \frac{1}{k!}\frac{d^k}{dt^k} f\left(\gamma(t)\right)|_{t=0}. \tag{23}$$

The image $m_k(\mathcal{C}_M) = \{m_k(f) | f \in \mathcal{C}_M\}$ of $\mathcal{C}_M$ generates the smooth structure $\mathcal{C}_{T^kM} := gen(m_k(\mathcal{C}_M))$ on T^kM and brings us to the sequence of smooth spaces

$$(T^kM, \mathcal{C}_{T^kM}),\ k \in \mathbb{N}. \tag{24}$$

Canonically with this sequence appear the surjective maps

$$\begin{aligned} \pi^k_l \ &: \ T^kM \longrightarrow T^lM \\ &[\gamma]_k \mapsto [\gamma]_l\ ,\ l \leq k. \end{aligned} \tag{25}$$

Composition with π^k_l makes functions on T^lM into functions on T^kM constant along $(\pi^k_l)^{-1}([\gamma]_l)$. So π^k_l is smooth and induces an inclusion

$$\pi^{k*}_l : \mathcal{C}_{T^lM} \hookrightarrow \mathcal{C}_{T^kM}. \tag{26}$$

As a smooth surjection π^k_l is the projection in the fibre space (T^kM, π^k_l, T^lM) and our sequence of smooth spaces becomes a sequence of fibrations

$$\ldots \overset{\pi^{k+1}_k}{\to} T^kM \overset{\pi^k_{k-1}}{\to} T^{k-1}M \overset{\pi^{k-1}_{k-2}}{\to} \ldots \tag{27}$$

For the special ($l = 0$) fibre space (T^kM, π^k_0, M) there is a canonical section, the "zero section" embedding of M

$$\begin{aligned} s^k_0 \ &: \ M \longrightarrow T^kM \\ &x \mapsto [\gamma^0_x]_k, \end{aligned} \tag{28}$$

where γ^0_x denotes the constant curve $\gamma^0_x(t) = x$, $\forall t \in \mathbb{R}$. The pullback with s^k_0 forgets the fibre dependence of the functions on T^kM, so s^k_0 is smooth.
Now it is easy to follow our intention and to state:

Proposition: 1 *The map $\rho : \mathcal{D}^k_{\mathbb{R}} \longrightarrow End_{\mathbb{R}}(\mathcal{C}_{T^kM})$ defined by*

$$d_i \mapsto \rho(d_i) := \pi^{k*}_i \circ m_i \circ s^{k*}_0 \tag{29}$$

is a representation of the coalgebra of derivations $\mathcal{D}^k_{\mathbb{R}}$.

The implementation of the Leibniz-rule is taking over the Leibniz-rule for the operator $\frac{1}{i!}\frac{d^i}{dt^i}$ by the microlocalization map.

4 Conclusion

"Representation" of a purely algebraic construction by geometric objects and their morphisms can be a strong method in the study of geometric spaces [1]; roughly speaking it reverses the main idea of algebraic topology. The microlocalization as representation of the derivation coalgebra studies the microlocal structure of a space. Microlocal is a refinement of the term local; the latter relates to the uniform structure given by the neighborhoods of points in a topological space. The way to the local point of view is managed by the procedure of localization which assigns to a point $x \in M$ the algebra $\mathcal{C}_{M,x}$ of germs of smooth functions. Algebraically $\mathcal{C}_{M,x}$ is the localization [Ei] of the algebra $\mathcal{C}_M$ at the maximal ideal $m_x = \{f \in \mathcal{C}_M | f(x) = 0\}$. The topological tool for this is sheaf theory; it produces the localization as the inverse image $i_x^{-1}(\mathcal{C}_M)$ of the sheaf associated to $\mathcal{C}_M$ by the inclusion map $i_x : x \to M$. Instead of the neighborhoods the microlocalization procedure is based on the infinitesimal neighborhoods. For that we take the fibre

$$T_x^k M := (\pi_0^k)^{-1}(x) = \{[\gamma]_k | \gamma(0) = x\} \tag{30}$$

over x and install on it the smooth structure which makes the natural inclusion

$$t_x^k : T_x^k M \to T^k M \tag{31}$$

a smooth map. Infinitesimal neighborhoods of order k of x are the neighborhoods of $[\gamma_x^0]$ in $T_x^k M$ and the microlocalization procedure arises by the inverse image with the inclusion map i_x replaced by

$$i_x^k := s_0^k \circ i_x : x \longrightarrow T_x^k M \tag{32}$$

With these things at hand we define a special contact of smooth spaces:

Definition: 6 *Let M_1, M_2, M be smooth spaces and $e_1 : M \to M_1$, $e_2 : M \to M_2$ smooth maps. We say that M_1 and M_2 have M-contact of order k if*

$$T_{e_1(x)}^k M_1 \cong T_{e_2(x)}^k M_2 \,, \; \forall x \in M. \tag{33}$$

An important fact is that in this sense the spaces $T^k M$ infinitesimally approximate the product space $M \times M$ by $e_1 = s_0^k$ and $e_2 = \Delta$.
As an concluding outlook we want to mention an extension of the theme treated here. With an obvious change in the microlocalization map the Froelicher-Kriegl method allows also to take over to smooth spaces the pseudodifferential structure, which exists on $\mathbb{R}$. The space for the degree of order changes from ($k \in \mathbb{N}$) to ($k \in \mathbb{R}$) and a representation of the more refined coalgebra of pseudoderivations with a continuous version of the Leibniz-rule [F2] is in order.

[1] See M. Karoubis lecture in this winterschool.

References

[Be] Robert Berger, Differentiale hoeherer Ordnung und Koerpererweiterungen bei Primzahlcharakteristik, Sitzungsberichte der Heidelberger Akademie der Wissenschaften 1966

[Ei] David Eisenbud, Commutative Algebra, Springer 1994

[FK] Alfred Froehlicher, Andreas Kriegl, Linear Spaces and Differentiation Theory, John Wiley & Sons, New York 1988

[F1] Gerald Fischer, Differential Geometry over the Structure Sheaf: A Way to Quantum Physics, Proceedings Srni 1997

[F2] Gerald Fischer, Work in progress.

[Ha] Rudolf Haag, Local Quantum Physics, Springer 1993

[La] Serge Lang, Algebra, Addison-Wesley, New York 1984

Institut für Theoretische Physik
Universität Regensburg
Regensburg, Germany