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ON ALMOST GEODESIC MAPPINGS OF THE TYPE π_1 OF RIEMANNIAN SPACES PRESERVING A SYSTEM N-ORTHOGONAL HYPERSURFACES

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Abstract

We study almost geodesic mapping of the type π_1 preserving a system of n-orthogonal hypersurfaces. We find metrics of Riemannian spaces for which these mappings exist.

1 Introduction

N.S. Sinyukov defined almost geodesic mappings [5], [6]. He found their basic equations and he divided them into three types π_1, π_2 and π_3. We proved in our papers [1], [2], [4] that this classification is complete.

Mappings of the type π_1 are not studied very much because their equations are too difficult. In this paper we study mappings of the type π_1 which preserve system of n-orthogonal hypersurfaces. Under some additional assumptions, we find Riemannian metrics for which these mappings exist.

2 Basic definitions of almost geodesic mapping

We recall basic definitions for almost geodesic mappings of affine connected spaces without torsion [5]:

Definition 1 A curve of a space with affine connection A_n is called an *almost geodesic line* if its tangential vector $\lambda^h \equiv dx^h/dt$ satisfies the equations

$$\underset{2}{\lambda}{}^h = a(t)\lambda^h + b(t)\underset{1}{\lambda}{}^h$$

where $\underset{1}{\lambda}{}^h \equiv \lambda^h_{,\alpha}\lambda^\alpha$, $\underset{2}{\lambda}{}^h \equiv \underset{1}{\lambda}{}^h_{,\alpha}\lambda^\alpha$, the comma denotes the covariant derivative with respect to the connection A_n, a and b are functions of a parameter t.

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Definition 2 A diffeomorphism f of the space with affine connection A_n onto a space with affine connection $\bar{A}_n$ is called an *almost geodesic mapping* if any geodesic line of the space A_n turns into the almost geodesic line of the space $\bar{A}_n$.

Definition 3 The mapping $f: A_n \to \bar{A}_n$ is called *almost geodesic of the type* π_1 if the following conditions hold in general, with respect to the f-related coordinate systems:

$$A^h_{(ijk)} = a_{(ij}\delta^h_{k)} + b_{(i}P^h_{jk)}, \tag{1}$$

where b_i is a covector and a_{ij} is a symmetric tensor, δ^h_i is the Kronecker symbol, round brackets denote cycling of indices,

$$A^h_{ijk} \equiv P^h_{ij,k} + P^h_{\alpha i}P^\alpha_{jk}, \tag{2}$$

$$P^h_{ij}(x) \equiv \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{3}$$

is the tensor of deformation, Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are the affine connections of A_n and $\bar{A}_n$.

3 Mapping of Riemannian spaces preserving system n-orthogonal hypersurfaces

Suppose that the Riemannian spaces V, $\overline{V}$ have fixed systems of n-orthogonal hypersurfaces. Let the mapping f: $V_n \to \overline{V}_n$ preserves these systems of hypersurfaces. We take the coordinate system $x \equiv (x^1, x^2, \dots, x^n)$ so as the system of n-orthogonal hypersurfaces was the system of coordinate surfaces [3].

We denote $g_{ij}(x)$ and $\overline{g}_{ij}(x)$ components of metric tensors of spaces V_n and $\overline{V}_n$ in the coordinate system x. Elements of their inverse matrices are denoted by $g^{ij}(x)$ and $\overline{g}^{ij}(x)$. With respect to the system of n-orthogonal hypersurfaces in V_n and $\overline{V}_n$ we get

$$g_{ij} = \overline{g}_{ij} = g^{ij} = \overline{g}^{ij} = 0 \qquad (i, j = \overline{1, n};\ i \neq j). \tag{4}$$

Then the following equations hold

$$g^{ii} = \frac{1}{g_{ii}}, \qquad \overline{g}^{ii} = \frac{1}{\overline{g}_{ii}}. \tag{5}$$

Christoffel symbols Γ^h_{ij} of V_n have the following form:

$$\begin{array}{ll} \Gamma^i_{ii} = \frac{1}{2}\partial_i \ln g_{ii}, & \Gamma^i_{ij} = \frac{1}{2}\partial_j \ln g_{ii},\ (i \neq j), \\ \Gamma^h_{ij} = 0\ (h, i, j \neq), & \Gamma^j_{ii} = -\frac{1}{2}\partial_j g_{ii}/g_{jj}\ \ (i \neq j). \end{array} \tag{6}$$

Similar relationships hold for Christoffels symbols $\overline{\Gamma}^h_{ij}$ of $\overline{V}_n$. The following formulae are true for components of deformation tensor $P^h_{ij}(x) \equiv \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ for the mapping f: $V_n \to \overline{V}_n$ preserving system of n-orthogonal hypersurfaces:

$$\begin{array}{c} P^h_{ij} = 0 \quad (i, j, k \neq),\ P^i_{ii} = \dfrac{1}{2}\partial_i \ln\left(\dfrac{\overline{g}_{ii}}{g_{ii}}\right),\ P^i_{ij} = \dfrac{1}{2}\partial_j \ln\left(\dfrac{\overline{g}_{ii}}{g_{ii}}\right) \quad (i \neq j), \\ P^j_{ii} = \dfrac{1}{2}\left(\dfrac{\partial_j g_{ii}}{g_{jj}} - \dfrac{\partial_j \overline{g}_{ii}}{\overline{g}_{jj}}\right) \quad (i \neq j). \end{array} \tag{7}$$

4 Special almost geodesic mapping of the type π_1

In order to find nontrivial examples of almost geodesic mappings of the type π_1: $V_n \to \overline{V}_n$ which preserve systems of n-orthogonal hypersurfaces, we will take some presumptions which simplify equations (1). Unfortunately, we have not solved these equations in the general case.

We will study mappings of the type π_1: $V_n \to \overline{V}_n$ for which the tensor of deformation P^h_{ij} is recurrent, i.e.

$$P^h_{ij,k} = \varphi_k P^h_{ij} \tag{8}$$

where comma denotes the covariant derivative of an affine connection in V_n and φ_k is a covector. The following algebraic equations follow from (1) and (8)

$$P^h_{\alpha(i} P^\alpha_{jk)} = \delta^h_{(i} a_{jk)}, \tag{9}$$

where a_{ij} is a symmetric tensor.

The inverse mapping $(\pi_1)^{-1}$ is also almost geodesic mapping of the type π_1. It follows from (9) and [5].

Conversely, if the mapping f: $V_n \to \overline{V}_n$ fullfils equations (8) and (9), then f is the mapping of the type π_1.

The assumption

$$\text{a)}\quad P^h_{hi} = P^j_{ji} \quad (h,i,j \neq), \qquad \text{b)}\quad P^h_{hi} = -P^i_{ii} \quad (h \neq i) \tag{10}$$

simplifies the solution of equations (8) and (9), there is no summation in latin indices, and $(h,i,j \neq)$ denotes that indices h,i,k are different to each other.

If we analyse algebraic condition (8) we get

$$\begin{array}{lll} \text{a)} & 3P^k_{kk} P^h_{ii} = P^k_{ii} P^h_{kk} & (h,i,k \neq), \\ \text{b)} & 9P^j_{jj} P^h_{hh} = P^j_{hh} P^h_{jj} & (h \neq j). \end{array} \tag{11}$$

It is easy to check that for conditions (10) and (11) the algebraic equations (8) are identical.

Using (7) we can write conditions (10a):

$$\partial_i \ln(\overline{g}_{hh}/g_{hh}) = \partial_i \ln(\overline{g}_{jj}/g_{jj}) \qquad (h,i,j \neq).$$

Consequently, we obtain the formulae

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{jj}/g_{jj}) \cdot F_{hj}(x^h, x^j) \quad (h \neq j). \tag{12}$$

where $F_{hj} (\neq 0)$ are function of parameters x^h and x^j. Similarly

$$\overline{g}_{ii}/g_{ii} = (\overline{g}_{jj}/g_{jj}) \cdot F_{ij}(x^i, x^j) \quad (i \neq j),$$

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{ii}/g_{ii}) \cdot F_{hi}(x^h, x^i) \quad (h \neq i).$$

Using the last three equations we get

$$F_{hi}(x^h, x^i) = \frac{F_{hj}(x^h, x^j)}{F_{ij}(x^i, x^j)} \qquad (h, i, j \neq).$$

and consequently $F_{hi}(x^h, x^i) = f_{hj}(x^h) h_{ij}(x^i)$ $(h, i, j \neq)$. When we analyse these relationships we obtain $F_{hi}(x^h, x^i) = C_{hi} f_h^2(x^h) h_i^2(x^i)$ $(h \neq i)$ where $C_{hi} = \text{const} \neq 0$, f_h and h_i are nonzero function of mentioned parameters.

The formulae **(12)** can be written in the form

$$\bar{g}_{hh}/g_{hh} = (\bar{g}_{jj}/g_{jj}) \cdot C_{hj} \cdot f_h^2(x^h) \cdot h_j^2(x^j) \quad (h \neq j).$$

Now it is easy to see that $h_j(x^j) = f_j^{-1}(x^j)$. Therefore we get

$$\bar{g}_{hh}/g_{hh} = (\bar{g}_{jj}/g_{jj}) \cdot C_{hj} \cdot f_h^2(x^h) \cdot f_j^{-2}(x^j) \quad (h \neq j). \tag{13}$$

Use conditions **(10b)** and **(7)** we obtain

$$\partial_i \ln \left(\frac{\bar{g}_{hh}}{g_{hh}} \frac{\bar{g}_{ii}}{g_{ii}} \right) = 0 \qquad (h \neq i). \tag{14}$$

According to **(13)** for $h \to i$ and $j \to h$ we similarly get

$$\partial_i \ln \left(f_i(x^i) \frac{\bar{g}_{hh}}{g_{hh}} \right) = 0 \qquad (h \neq i).$$

Now we will integrate these equations and we get

$$\frac{\bar{g}_{hh}}{g_{hh}} = \frac{F_h(x^h)}{Q} \tag{15}$$

where

$$Q \doteq \prod_{\alpha=\overline{1,n}} f_\alpha(x^\alpha) \tag{16}$$

and $F_h(x^h)$ are functions.

Substitute **(16)** in **(14)** in order to get $F_h = c_h f_h^2(x^h)$, where $c_h = \text{const} \neq 0$.

In the end we obtain the equations

$$\frac{\bar{g}_{hh}}{g_{hh}} = \frac{c_h f_h^2(x^h)}{Q}. \tag{17}$$

The components of deformation tensor P_{ij}^h after substitution **(17)** in **(7)** are of the form

$$P_{ii}^i = \tfrac{1}{2}\partial_i \ln f_i, \quad P_{ij}^i = -\tfrac{1}{2}\partial_j \ln f_j \quad (i \neq j), \quad P_{ij}^h = 0 \quad (i, j, k \neq),$$
$$P_{ii}^j = \frac{1}{2}\left(\frac{\partial_j g_{ii}}{g_{jj}} \left(\frac{c_i f_i^2}{c_j f_j^2} - 1 \right) - \frac{c_i f_i^2 g_{ii}}{c_j f_j^2 g_{jj}} \partial_j \ln f_j \right) \quad (i \neq j). \tag{18}$$

Now we will study equations (8). For $k = i$, $(h, i, j \neq)$ we get $P^h_{jj}\Gamma^j_{ii} + P^h_{ii}\Gamma^i_{ij} = 0$. From (6) for $\partial_j g_{ii} \neq 0$ we obtain

$$P^h_{jj}/g_{jj} = P^h_{ii}/g_{ii} \quad (h, i, j \neq). \tag{19}$$

Using (19) and dividing (11a) by $g_{ii}g_{kk}$, we obtain $P^k_{ii}/g_{ii} = 3P^k_{kk}/g_{kk}$, i.e.

$$P^h_{ii} = 3\frac{g_{ii}}{g_{hh}}P^h_{hh} \quad (h \neq i). \tag{20}$$

The formula (19) holds identically. For $h \neq i$, (18) implies

$$\partial_h \ln g_{ii} = \frac{3c_h f_h^2 - c_i f_i^2}{c_h f_h^2 - c_i f_i^2}\partial_h \ln f_h. \tag{21}$$

If we analyse equations (8) we find that $\varphi_k = -2\partial_k \ln f_k$ and

$$\partial_i \ln g_{ii} = \frac{2\partial_{ii}\ln f_i}{\partial_i \ln f_i} + 4\partial_i \ln f_i + \frac{g_{ii}}{\partial_i \ln f_i}\sum_{\alpha\neq i}\frac{(\partial_\alpha \ln f_\alpha)^2}{g_{\alpha\alpha}}\frac{3c_\alpha f_\alpha^2 - c_i f_i^2}{c_\alpha f_\alpha^2 - c_i f_i^2}. \tag{22}$$

According to the assumption, functions f_i are not constant. Hence we can find transformation of the coordinate system $x'^h = x'^h(x^h)$, $h = \overline{1,n}$ preserving the system of n-orthogonal coordinate hypersurfaces, such that

$$f_i(x^i) = e^{-x^i}$$

Formula (21) and (22) then will have the form

$$\begin{aligned}\partial_i \ln g_{ii} &= -4 - g_{ii}\sum_{\alpha\neq i}\frac{1}{g_{\alpha\alpha}}\frac{3c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}}{c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}},\\ \partial_h \ln g_{ii} &= \frac{c_i e^{-2x^i} - 3c_h e^{-2x^i}}{c_h e^{-2x^h} - c_i e^{-2x^i}} \qquad (h \neq i).\end{aligned} \tag{23}$$

It is clear that

$$g_{ii} = e^{-x^i}\prod_{\alpha\neq i}(c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i})e^{-x^\alpha}, \qquad c_i = \text{const} \neq 0, \tag{24}$$

is the partial solution of the system (23). This solution is obviously the general solution of this system because this system is of the Cauchy type.

Next theorem follows from this solution and from equations (8) and (9).

Theorem 1 *The Riemannian space V_n with the metric*

$$g_{ii} = e^{-x^i}\prod_{\alpha\neq i}(c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i})e^{-x^\alpha}, \quad g_{ij} = 0 \ (i \neq j), \quad c_i = const \neq 0,$$

admits almost geodesic mapping of the type π_1 preserving a system of n-orthogonal hypersurfaces onto Riemannian space $\overline{V}_n$ with metric

$$\overline{g}_{ii} = c_i e^{-2x^i}\prod_{\alpha\neq i}(c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}), \qquad \overline{g}_{ij} = 0 \ (i \neq j).$$

Remark. This mapping is not geodesic. It means that Riemannian spaces from the theorem are among non trivial examples of mappings of the type π_1 and a there are only very few of them known so far, [5].

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