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ON THE THEORY OF THE 4-QUASIPLANAR MAPPINGS OF ALMOST QUATERNIONIC SPACES *

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Abstract

4-quasiplanar mappings of almost quaternionic spaces with affine connection without torsion are investigated. Geometrically motivated definitions of these mappings are presented. Based on these definitions fundamental form of these mappings are found, which are equivalent to the forms of 4-quasiplanar mappings introduced a priori by I. Kurbatova.
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J. Mikesh and N.S. Sinyukov proposed a quite wide generalization of geodesic and holomorphically projective mappings of the spaces with affine connection without torsion having an affine structure – quasiplanar mappings [5].

I.N. Kurbatova analogically defined 4-quasiplanar mappings of spaces with affine connection without torsion with almost quaternionic structure [4]. However, in our opinion, these concepts are not sufficiently well-founded. The paper is devoted to this problem.

We remark that similar questions are also addressed in the setting of complex manifolds in [2].

1. A well-known definition says that an almost quaternionic space is a differentiable manifold M_n with almost complex structures $\overset{1}{F}$ and $\overset{2}{F}$ defined on it which satisfy

$$\text{a)}\quad \overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = -\delta^h_i;\quad \text{b)}\quad \overset{2}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i = -\delta^h_i;\quad \text{c)}\quad \overset{1}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i + \overset{2}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0, \tag{1}$$

where δ^h_i is the Kronecker symbol, see e.g. [3].

Tensor $\overset{3}{F}{}^h_i \equiv \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha$ defines almost complex structure, too. The relations among the tensors $\overset{1}{F}$, $\overset{2}{F}$, $\overset{3}{F}$ are the following

$$\overset{1}{F}{}^h_i = \overset{2}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha = -\overset{3}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha;\ \ \overset{2}{F}{}^h_i = \overset{3}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha = -\overset{1}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha;\ \ \overset{3}{F}{}^h_i = \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha = -\overset{2}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha. \tag{2}$$

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Any two of the three structures $\overset{1}{F}$, $\overset{2}{F}$, $\overset{3}{F}$ define an initial almost quaternionic structure.

In the sequel, whenever the affinors

$$\overset{1}{{}^*F}{}^h_i = \sum_{s=0}^{3} \alpha_s \overset{s}{F}{}^h_i \quad \text{and} \quad \overset{2}{{}^*F}{}^h_i = \sum_{s=0}^{3} \beta_s \overset{s}{F}{}^h_i \tag{3}$$

where α_s, β_s $(s = \overline{0,3})$ are functions and $\overset{0}{F}{}^h_i \equiv \delta^h_i$, satisfy the conditions **(1)** and **(2)**, we shall suppose that $\overset{1}{{}^*F}$ and $\overset{2}{{}^*F}$ define the same almost quaternionic structure as $\overset{1}{F}$ and $\overset{2}{F}$.

Necessary and sufficient conditions for creation of the same almost quaternionic structure are the following formulae

$$\alpha_0 = \beta_0 = 0; \quad \sum_{s=1}^{3} \alpha_s = \sum_{s=1}^{3} \beta_s = 1; \quad \sum_{s=1}^{3} \alpha_s \beta_s = 0. \tag{4}$$

Assume $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ is a space with affine connection Γ without torsion with almost quaternionic structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$.

Definition 1 A curve ℓ: $x^h = x^h(t)$ is called *4-planar* if the tangent vector $\lambda^h = dx^h/dt$ being parallely transported along this curve, remains in the linear 4-dimensional space generated by the tangent vector λ^h and the vectors $\overset{1}{F}{}^h_\alpha \lambda^\alpha$, $\overset{2}{F}{}^h_\alpha \lambda^\alpha$ and $\overset{3}{F}{}^h_\alpha \lambda^\alpha$, which are conjugated to it.

A curve is 4-planar if and only if the equations

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta} \lambda^\alpha \lambda^\beta = \sum_{s=0}^{3} \underset{s}{\rho} \; \overset{s}{F}{}^h_\alpha \lambda^\alpha, \tag{5}$$

where $\underset{s}{\rho}$ $(s = \overline{0,3})$ denote functions of the parameter t, hold.

Evidently, a curve 4-planar with respect to the structure $(\overset{1}{F} ,\overset{2}{F})$ is 4-planar with respect to the structure $(\overset{1}{{}^*F} , \overset{2}{{}^*F})$ defined by the formulas **(3)** and **(4)**, too. A 4-planar curve is a geodesic curve if $\rho_1 \equiv \rho_2 \equiv \rho_3 \equiv 0$.

2. Consider two spaces with affine connection without torsion A_n and $\bar{A}_n$ with objects of connection Γ and $\overline{\Gamma}$, respectively. Let an almost quaternionic structure $(\overset{1}{\overline{F}}$, $\overset{2}{\overline{F}}$) be defined on $\bar{A}_n$.

Definition 2 A diffeomorphism f: $A_n(\Gamma) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\overline{F}} ,\overset{2}{\overline{F}})$ is called *4-quasiplanar mappings*, if it maps any geodesic in A_n to a 4-planar curve in $\bar{A}_n$.

We prove the following

Theorem 1 *A mapping from A_n onto $\bar{A}_n$ is 4-quasiplanar if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{a}_{(i} \overset{s}{\overline{F}}{}^h_{j)} \tag{6}$$

hold, where Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are objects of the affine connections in A_n and $\bar{A}_n$, $\underset{s}{a}_i(x)$, $s = \overline{0,3}$, are covectors, $\overset{0}{\overline{F}}{}^h_i \equiv \delta^h_i$, $\overset{3}{\overline{F}}{}^h_i \equiv \overset{1}{\overline{F}}{}^\alpha_i \overset{2}{\overline{F}}{}^h_\alpha$, $(i\,j)$ denotes a symmetrization without division.

Proof. Let f: $A_n \to \bar{A}_n$ be 4-quasiplanar. Then any geodesic γ: $x^h = x^h(t)$ is characterized by the conditions

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho(t)\lambda^h, \tag{7}$$

where $\lambda^h \equiv dx^h/dt$ and ρ is some function of the parametr t. By the definition of a 4-quasiplanar mapping, in the common coordinate system x with respect to the mapping, f maps any geodesic to a 4-planar curve of the space $\bar{A}_n$, which is characterized by the equation

$$\frac{d\lambda^h}{dt} + \overline{\Gamma}^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{\rho}\,(t)\, \overset{s}{\overline{F}}{}^h_\alpha \lambda^\alpha \tag{8}$$

By subtraction of (7) and (8), we obtain the relation

$$P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{a}\ \overset{s}{\overline{F}}{}^h_\alpha \lambda^\alpha \tag{9}$$

where $\underset{0}{a} = \bar{\rho}_0 - \rho$; $\underset{s}{a} = \bar{\rho}_s$, $s = 1, 2, 3$ and

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{10}$$

where $P^h_{ij}(x)$ is the *tensor of deformation* of the affine connection.

For 4-quasiplanar mappings the relations (9) are satisfied identically in any point $x \in A_n$ for any vector field λ^h. Note that the functions $\underset{s}{a}$, $s = \overline{0,3}$ depend not only on the variable $x \in A_n$ but also on the vector field λ^h.

We multiply the formula (9) by $\lambda^i\ \overset{1}{\overline{F}}{}^j_\alpha \lambda^\alpha\ \overset{2}{\overline{F}}{}^k_\beta \lambda^\beta\ \overset{3}{\overline{F}}{}^l_\gamma \lambda^\gamma$ and then we alternate by the indexes h, i, j, k, l. As a result we get the relation

$$\left(P^{[h}_{\alpha\beta}\ \delta^i_\gamma\ \overset{1}{\overline{F}}{}^j_\delta\ \overset{2}{\overline{F}}{}^k_\varepsilon\ \overset{3}{\overline{F}}{}^{l]}_\eta \right) \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\lambda^\varepsilon\lambda^\eta = 0, \tag{11}$$

where $[h \ldots l]$ denotes an alternation.

Clearly, the expression in the parentheses doesn't depend on the vector λ^h. The relation (11) is satisfied for any vector field λ^h. Using this fact, the relation (11) implies

$$P^{[h}_{(\alpha\beta}\ \delta^i_\gamma\ \overset{1}{\overline{F}}{}^j_\delta\ \overset{2}{\overline{F}}{}^k_\varepsilon\ \overset{3}{\overline{F}}{}^{l]}_{\eta)} = 0. \tag{12}$$

Since $n \geq 8$, in any point x there exist two vectors y^h and z^h such that the vectors

$$y^h,\quad \overset{1}{F}{}^h_\alpha y^\alpha,\quad \overset{2}{F}{}^h_\alpha y^\alpha,\quad \overset{3}{F}{}^h_\alpha y^\alpha,\quad z^h,\quad \overset{1}{F}{}^h_\alpha z^\alpha,\quad \overset{2}{F}{}^h_\alpha z^\alpha,\quad \overset{3}{F}{}^h_\alpha z^\alpha,$$

are lineary independent.

First we contracte (12) by $y^\alpha y^\beta y^\gamma y^\delta y^\varepsilon y^\eta$. We obtain

$$y^\alpha y^\beta P^{[h}_{\alpha\beta}\, y^i\, \overset{1}{y}{}^j\, \overset{2}{y}{}^k\, \overset{3}{y}{}^{l]} = 0, \tag{13}$$

where $\overset{s}{y}{}^h \equiv \overset{s}{F}{}^h_\alpha y^\alpha$, $s = \overline{0,3}$.

From the relation (13) follows

$$P^h_{\alpha\beta} y^\alpha y^\beta = \sum_{s=0}^{3} \underset{s}{a}\, \overset{s}{y}{}^h, \tag{14}$$

where $\underset{s}{a}$ are some functions.

Convolving (12) by $y^\beta y^\gamma y^\delta y^\varepsilon y^\eta$ we obtain, using (14),

$$P^h_{i\alpha} y^\alpha = \sum_{s=0}^{3} \underset{s}{b}\, \overset{s}{F}{}^h_i + \sum_{s=0}^{3} \underset{s}{c}{}_i\, \overset{s}{y}{}^h, \tag{15}$$

where $\underset{s}{b}$ are some functions and $\underset{s}{c}{}_i$ are covectors.

Finally, contracting (12) by $y^\gamma y^\delta y^\varepsilon y^\eta$ and substituting the relations (14) and (15), we see that

$$P^h_{ij} = \sum_{s=0}^{3} \underset{s}{\rho}{}_{(i}\, \overset{s}{F}{}^h_{j)} + \sum_{s=0}^{3} \underset{s}{d}{}_{ij}\, \overset{s}{y}{}^h, \tag{16}$$

where $\underset{s}{\rho}{}_i$, $\underset{s}{d}{}_{ij}$ are some tensors.

Substituting (16) to (12), we obtain

$$\sum_{s=0}^{3} \underset{s}{d}{}_{(\alpha\beta}\, \overset{s}{y}{}^{[h}\, \delta^i_\gamma\, \overset{1}{F}{}^j_\delta\, \overset{2}{F}{}^k_\varepsilon\, \overset{3}{F}{}^{l]}_{\eta)} = 0. \tag{17}$$

Analogically, contracting (17) by $z^\alpha z^\beta z^\gamma z^\delta z^\varepsilon z^\eta$, we get

$$\sum_{s=0}^{3} \underset{s}{d}{}_{\alpha\beta}\, z^\alpha z^\beta\, \overset{s}{y}{}^{[h}\, \overset{0}{z}{}^i\, \overset{1}{z}{}^j\, \overset{2}{z}{}^k\, \overset{3}{z}{}^{l]} = 0, \tag{18}$$

where $\overset{s}{z}{}^h \equiv \overset{s}{F}{}^h_\alpha z^\alpha$, $s = \overline{0,3}$.

From linear independence follows that

$$\underset{s}{d}{}_{\alpha\beta} z^\alpha z^\beta = 0, \quad s = \overline{0,3}. \tag{19}$$

Convolving (17) by $z^\beta z^\gamma z^\delta z^\varepsilon z^\eta$, and using (19), we have

$$\underset{s}{d}{}_{i\alpha} z^\alpha = 0, \quad s = \overline{0,3},$$

and than by contracting (17) by $z^\gamma z^\delta z^\varepsilon z^\eta$, and using the preceding facts, we find

$$\underset{s}{d}_{ij} = 0, \quad s = \overline{0,3}.$$

Than the formulas (16) can be written in the form (6). Clearly, the formula (6) is sufficient, too. The theorem is proved.

3. Further, consider two spaces of affine connection without torsion A_n and $\bar{A}_n$ with objects of connections Γ_n and $\overline{\Gamma}_n$ and almost quaternionic structures $(\overset{1}{F}, \overset{2}{F})$ and $(\overset{1}{\overline{F}}, \overset{2}{\overline{F}})$, respectively. We introduce the following.

Definition 3 A diffeomorphism f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\overline{F}}, \overset{2}{\overline{F}})$ is called a *4-quasiplanar mapping of spaces with almost quaternionic structures*, if any 4-planar curve in A_n is transformed to a 4-planar curve in $\bar{A}_n$ by f.

The next theorem is valid.

Theorem 2 *A mapping f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\overline{F}}, \overset{2}{\overline{F}})$ is a 4-quasiplanar mapping in the common coordinate system x with respect to the mapping f if and only if the following conditions hold*

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{\psi}_{(i} \overset{s}{F}{}^h_{j)} \tag{20}$$

$$\overset{\tau}{\overline{F}}{}^h_i(x) = \sum_{s=1}^{3} \alpha^\tau_s \overset{s}{F}{}^h_i, \quad \tau = 1, 2, \tag{21}$$

where $\underset{s}{\psi}_i(x)$, $s = \overline{0,3}$ are some vectors and $\alpha^\tau_s(x)$, $s = 1, 2, 3$ are functions satisfying the conditions

$$\sum_{s=1}^{3} (\alpha^\tau_s)^2 = 1, \quad \tau = 1, 2; \qquad \sum_{s=1}^{3} \left(\alpha^1_s \alpha^2_s\right) = 0. \tag{22}$$

Remark. The formulae (21) and (22) express the fact that $(\overset{1}{F}, \overset{2}{F})$ and $(\overset{1}{\overline{F}}, \overset{2}{\overline{F}})$ define on A_n and $\bar{A}_n$ the same almost quaternionic structures.

Proof. Let f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\overline{F}}, \overset{2}{\overline{F}})$ be a 4-quasiplanar mapping since any geodesic is a special type of a 4-planar curve, the mapping f maps any geodesic to a 4-planar curve.

Then every geodesic in A_n is mapped to a 4-planar curve in $\bar{A}_n$ by this mapping. It follows by Theorem 1 that the formula (6) are satisfied.

Analogically, a special case of 4-planar curves in A_n are the curves ℓ: $x^h = x^h(t)$, for which the tangent vector $\lambda^h = dx^h/dt$ satisfies the equations

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \overset{\tau}{F}{}^h_\alpha\lambda^\alpha, \tag{23}$$

where τ is equal to 1 or 2.

By elimination of $\Gamma^h_{\alpha\beta}$ from formulas (23) using (6), we obtain

$$\frac{d\lambda^h}{dt}+\bar{\Gamma}^h_{\alpha\beta}\lambda^\alpha\lambda^\beta=\overset{\tau}{F}{}^h_\alpha\lambda^\alpha+\sum_{s=0}^{3}\underset{s}{\psi}{}_\alpha\lambda^\alpha\ \overset{s}{\bar{F}}{}^h_\beta\lambda^\beta. \tag{24}$$

Since this curve in $\bar{A}_n$ must be 4-planar, the condition

$$\overset{\tau}{F}{}^h_\alpha\lambda^\alpha=\sum_{s=0}^{3}\underset{s}{a}\ \overset{s}{\bar{F}}{}^h_\alpha\lambda^\alpha,\quad \tau=1,2, \tag{25}$$

where $\underset{s}{a}$, $s=\overline{0,3}$, are some functions of arguments $x\in A_n$ and also of vectors λ^h, must be satisfied as well.

Formulas (25) hold in any point $x\in A_n$ and for all vectors λ^h. Multiplying (24) by $\lambda^i\ \overset{1}{\bar{F}}{}^j_\beta\lambda^\beta\ \overset{2}{\bar{F}}{}^k_\gamma\lambda^\gamma\ \overset{3}{\bar{F}}{}^l_\varepsilon\lambda^\varepsilon$ and than alternating over all indexes we obtain

$$\left(\overset{\tau}{F}{}^{[h}_\alpha\,\delta^i_\beta\ \overset{1}{\bar{F}}{}^j_\gamma\ \overset{2}{\bar{F}}{}^k_\delta\ \overset{3}{\bar{F}}{}^{l]}_\varepsilon\right)\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\lambda^\varepsilon=0,\qquad \tau=1,2.$$

It follows from here that

$$\overset{\tau}{F}{}^{[h}_{(\alpha}\,\delta^i_\beta\ \overset{1}{\bar{F}}{}^j_\gamma\ \overset{2}{\bar{F}}{}^k_\delta\ \overset{3}{\bar{F}}{}^{l]}_{\varepsilon)}=0,\qquad \tau=1,2. \tag{26}$$

Analysing (26) similarly as in the proof of the preceding theorem, we can prove that

$$\overset{\tau}{F}{}^h_i=\sum_{s=0}^{3}\alpha^\tau_s\ \overset{s}{\bar{F}}{}^h_i,\quad \tau=1,2, \tag{27}$$

where α^τ_s are some functions.

As $(\overset{1}{F},\overset{2}{F})$ and $(\overset{1}{\bar{F}},\overset{2}{\bar{F}})$ are quaternionic structures, the α^τ_s satisfy conditions analogous to (4). This fact is expressed by the formulas (21) and (22).

On the other hand, the conditions (20), (21) and (22) are evidently sufficient for f to be a 4-quasiplanar mapping.

4. Theorem 2 formulates the fact that a 4-quasiplanar mapping preserves almost quaternionic structures.

Therefore, a priori studying such mappings, we shall further suppose that the almost quaternionic structure is preserved, which is in fact equivalent to the condition

$$\overset{s}{\bar{F}}{}^h_i(x)=\overset{s}{F}{}^h_i(x),\qquad s=1,2,3. \tag{28}$$

On the other hand, Theorem 2 implies that under the conditions (28), 4-quasiplanar mappings $f\colon A_n(\Gamma,\overset{1}{F},\overset{2}{F})\to\bar{A}_n(\bar{\Gamma},\overset{1}{\bar{F}},\overset{2}{\bar{F}})$ are characterized by conditions (20).

By these conditions I.N. Kurbatova [4] defined 4-quasiplanar mappings preserving almost-quaternionic structure.

Finally we will suppose that the space $A_n(\Gamma,\overset{1}{F},\overset{2}{F})$ is mapped onto Riemannian space $\bar{V}_n(\bar{g},\overset{1}{\bar{F}},\overset{2}{\bar{F}})$

Theorem 3 *A mapping* $f\colon A_n(\Gamma,\overset{1}{F},\overset{2}{F}) \to \bar V_n(\bar g,\overset{1}{\bar F},\overset{2}{\bar F})$ *is 4-quasiplanar if and only if the metric tensor* $\bar g_{ij}(x)$ *satisfied the following equations:*

$$\bar g_{ij,k} = \sum_{s=0}^{3} \left(\underset{s}{\psi}_{k}\, \bar g_{\alpha(i}\, \overset{s}{F}{}^{\alpha}_{j)} + \underset{s}{\psi}_{(i}\, \bar g_{j)\alpha}\, \overset{s}{F}{}^{\alpha}_{k} \right) \tag{29}$$

where comma is covariant derivative in A_n *and the almost quaternionic structure is preserved.*

Proof follows from the fact that formulas (20) and (29) are equivalent.

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