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TORSIONS OF CONNECTIONS ON TANGENT BUNDLES OF HIGHER ORDER

MIROSLAV KUREŠ

ABSTRACT. General torsions of a connection on a natural bundle FM are defined as the Frölicher-Nijenhuis brackets of the associated horizontal projection and natural affinors on this bundle. All general torsions on T^rM are described. Further, special (i.e. r-linear) connections and second order case are studied in detail.
KEYWORDS. Connection, torsion, tangent bundle of higher order

0. INTRODUCTION

There are two classical approaches to torsion of a classical linear connection on a manifold M. If we consider Γ as a linear connection on TM, we can define the torsion as the covariant exterior differential, in the sense of Koszul, of the identity tensor on M. Secondly, if we interpret Γ as a principal connection on the frame bundle P^1M, we can introduce the torsion as the standard covariant exterior differential of the canonical $\mathbb{R}^m$-valued form on P^1M. This second approach was generalized by Yuen for r-th order frame bundle P^rM, [15]. Further, if we take a general connection introduced by Libermann, [13], as a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$ of an arbitrary fibered manifold, we can use the concept of a general torsion defined by Kolář and Modugno in [11] as the Frölicher-Nijenhuis bracket of Γ and an arbitrary natural affinor on Y. These torsions are completely described for bundles TM, T^1_kM, P^1M, T^2M and T^*M in [11].

The r-th order tangent bundle T^rM is the fundamental structure of higher order mechanics. For example, the papers [1], [4] refer to connections on T^rM. In the present paper we study general torsions of connections on T^rM and Proposition 2 gives their coordinate expression. Then we discuss torsions of the simplest class of connections and we interpret them geometrically. Our approach to torsions is based on the theory of natural operators, [6], [9]. We also compare our results with Yuen's approach.

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1. Higher order tangent bundle

Let M be an m–dimensional manifold. The *tangent bundle of order* r of M (which is also called the *bundle of velocities of order* r on M) is the $(r+1)m$–dimensional manifold T^rM of r–jets at $0 \in \mathbb{R}$ of differentiable mappings $\mu : \mathbb{R} \to M$. We denote by $\pi^r_0 : T^rM \to M$ the canonical projection defined by $\pi^r_0(j^r_0\mu) = \mu(0)$. Then $T^rM = J^r_0(\mathbb{R}, M)$ has a bundle structure over M. If $r = 1$, then $T^1M = TM$ is the tangent bundle of M. However, if $r > 1$, then T^rM is not a vector bundle. T^rM is also fibered over T^sM, $0 < s < r$. A projection $\pi^r_s : T^rM \to T^sM$ is defined by $\pi^r_s(j^r_0\mu) = j^s_0\mu$. It holds $\pi^r_q = \pi^s_q \circ \pi^r_s$ for any s, q, $0 \le q < s < r$. Given some local coordinates x^i on M, the r–th order Taylor expansion of a curve $x^i(t)$

$$x^i(t) + \frac{dx^i(t)}{dt} + \frac{1}{2}\frac{d^2x^i(t)}{dt^2} + \cdots + \frac{1}{r!}\frac{d^rx^i(t)}{dt^r}$$

determines the induced coordinates

$$x^i, y^{i,1} = \frac{dx^i}{dt}, y^{i,2} = \frac{1}{2}\frac{d^2x^i}{dt^2}, \ldots, y^{i,r} = \frac{1}{r!}\frac{d^rx^i}{dt^r}$$

on T^rM.

Every smooth map $f : M \to N$ induces a fiber bundle morphism $T^rf : T^rM \to T^rN$ (called the *tangent morphism of order* r) defined by the jet composition, i.e. $T^rf(j^r_0\mu) = j^r_0(f \circ \mu)$. Let $x^i, y^{i,1}, \ldots, y^{i,r}$ and $\bar{x}^p, \bar{y}^{p,1}, \ldots, \bar{y}^{p,r}$ are some local coordinates on T^rM and T^rN, respectively, and let $\bar{x}^p = f^p(x^i)$ be the coordinate expression of a smooth map $f : M \to N$. We express the tangent morphism of order r in local coordinates. For the map $T^r_xf : T^r_xM \to T^r_{f(x)}N$ we evaluate

$$\begin{aligned}
\bar{y}^{p,1} &= a^p_{i_1}y^{i_1,1} \\
\bar{y}^{p,2} &= a^p_{i_1i_2}y^{i_1,1}y^{i_2,1} + a^p_{i_1}y^{i_1,2} \\
&\cdots \\
\bar{y}^{p,r} &= \sum_{q=1}^{r}\sum_{\pi\in\mathbb{P}(r,q)} k_\pi a^p_{i_1\ldots i_q}y^{i_1,r_1}\ldots y^{i_q,r_q},
\end{aligned} \tag{1}$$

where $a^p_{i_1} = \frac{\partial f^p(x)}{\partial x^{i_1}}, a^p_{i_1i_2} = \frac{1}{2}\frac{\partial^2 f^p(x)}{\partial x^{i_1}\partial x^{i_2}}, \ldots, a^p_{i_1\ldots i_q} = \frac{1}{q!}\frac{\partial^q f^p(x)}{\partial x^{i_1}\ldots\partial x^{i_q}}$, $\mathbb{P}(r,q)$ is the set of decompositions of number r to q additive terms $r_1 \le \cdots \le r_q \in \mathbb{N}$, $r_1 + \cdots + r_q = r$ (i.e. we sum as to all such decompositions $\pi \in \mathbb{P}(r,q)$) and k_π is the number of permutations of the set of components of a decompositions π.

Remark 1. It is useful to take into account the identity $\sum_{\pi\in\mathbb{P}(r,q)} k_\pi = \binom{r-1}{q-1}$.

Remark 2. The algebraic properties of T^rM are studied in [10]. The tangent morphism of order r represents an r–graded linear morphism. Besides, on every fiber T^r_xM is defined a structure of linear r–tower and the graded linear map (1) is a morphism of linear r–towers. Moreover, the higher order tangent bundle is a Weil bundle, [6], [14].

2. All natural affinors

Let $V^{\pi^r_s}T^rM \subset TT^rM$ denote the vertical bundle with respect to the tangent projection $T\pi^r_s$, $0 \le s < r$. We have r exact sequences of vector bundles over T^rM:

$$0 \longrightarrow V^{\pi^r_0}T^rM \xrightarrow{i_1} TT^rM \xrightarrow{s_1} T^rM \times_M TM \longrightarrow 0$$
$$0 \longrightarrow V^{\pi^r_1}T^rM \xrightarrow{i_2} TT^rM \xrightarrow{s_2} T^rM \times_{TM} TTM \longrightarrow 0$$
$$\dots$$
$$0 \longrightarrow V^{\pi^r_{r-1}}T^rM \xrightarrow{i_r} TT^rM \xrightarrow{s_r} T^rM \times_{T^{r-1}M} TT^{r-1}M \longrightarrow 0$$

There exist r canonical isomorphisms of vector bundles

$$\begin{aligned} h_1 :\ & T^rM \times_M TM \to V^{\pi^r_{r-1}}T^rM, \\ h_2 :\ & T^rM \times_{TM} TTM \to V^{\pi^r_{r-2}}T^rM, \\ & \dots, \\ h_r :\ & T^rM \times_{T^{r-1}M} TT^{r-1}M \to V^{\pi^r_0}T^rM. \end{aligned}$$

Thus, r canonical (1,1)–tensor fields are defined by

$$A_j := i_j \circ h_{r-j+1} \circ s_{r-j+1},$$

$j = 1, \dots, r$. In coordinates,

$$A_j : (dx^i, dy^{i,1}, \dots, dy^{i,r}) \mapsto (\underbrace{0, \dots, 0}_{j-times}, dx^i, dy^{i,1}, \dots, dy^{i,r-j}).$$

Further, let us denote A_0 the identical (1,1)–tensor field.

A *natural affinor* on a natural bundle F over m–dimensional manifolds is a system of (1,1)–tensor fields $A_M : TFM \to TFM$ for every m–dimensional manifold M satisfying $TFf \circ A_M = A_N \circ TFf$ for every local diffeomorphism $f : M \to N$.

Proposition 1. *All natural affinors on T^rM constitute an $(r+1)$–parameter family linearly generated by A_j, $j = 0, 1, \dots, r$.*

Proof. Kolář and Modugno proved in [11] that all natural affinors on an arbitrary Weil bundle correspond to the multiplication by the elements of the relevant Weil algebra. The Weil algebra associated with the functor T^r is $\mathbb{A} := \mathbb{R}[t]/\langle t^{r+1}\rangle$, where $\langle t^{r+1}\rangle$ denotes the ideal generated by t^{r+1}. The elements of $\mathbb{A}$ have the form $a_0 + a_1t + \cdots + a_rt$ and that is why the elements $1, t, \dots, t^r$ determine $r+1$ natural affinors $A_0, A_1, \dots, A_r$. $\square$

3. General connections and their general torsions

We use the concept of general connection on an arbitrary fibered manifold, [13]. Consider a general connection $\Gamma : T^rM \to J^1T^rM$ with following equations of the corresponding horizontal lifting $\gamma : T^rM \times_M TM \to TT^rM$:

$$
\begin{aligned}
dy^{i,1} &= F_j^{i,1}(x,y)dx^j \\
dy^{i,2} &= F_j^{i,2}(x,y)dx^j \\
&\dots \\
dy^{i,r} &= F_j^{i,r}(x,y)dx^j
\end{aligned}
$$

The connection Γ can be identified with the associated horizontal projection, which is a special (1,1)–tensor field on T^rM with the coordinate expression

$$\delta_j^i \frac{\partial}{\partial x^i} \otimes dx^j + F_j^{i,1} \frac{\partial}{\partial y^{i,1}} \otimes dx^j + \dots + F_j^{i,r} \frac{\partial}{\partial y^{i,r}} \otimes dx^j.$$

The *general torsion* is defined as the Frölicher–Nijenhuis bracket $[\Gamma, A]$, where A is a natural affinor, [11]. We do not consider identical affinor A_0, because $[\Gamma, A_0] = 0$ for every connection Γ. According to Proposition 1 we can evaluate all general torsions of Γ as $\tau_n = [\Gamma, A_n]$, $n = 1, \dots, r$, and their linear combinations. The special cases of bundles TM and T^2M are discussed in detail in [11].

Proposition 2. *All general torsions of a general connection Γ on T^rM form a r–parameter family linearly generated by τ_n, $n = 1, \dots, r$, where τ_n has the coordinate expression*

$$
\begin{aligned}
(2) \qquad & \frac{\partial F_j^{k,\alpha}}{\partial y^{i,n}} \frac{\partial}{\partial y^{k,\alpha}} \otimes dx^i \wedge dx^j - \frac{\partial F_j^{k,\beta}}{\partial x^i} \frac{\partial}{\partial y^{k,\beta+n}} \otimes dx^i \wedge dx^j \\
& - \frac{\partial F_j^{k,\alpha}}{\partial y^{i,\beta+n}} \frac{\partial}{\partial y^{k,\alpha}} \otimes dx^i \wedge dy^{j,\beta} + \frac{\partial F_j^{k,\beta}}{\partial y^{i,\alpha}} \frac{\partial}{\partial y^{k,\beta+n}} \otimes dx^i \wedge dy^{j,\alpha},
\end{aligned}
$$

$n < r$, $\alpha = 1, \dots, r$, $\beta = 1, \dots, r - n$, and τ_r has the coordinate expression

$$(2') \qquad \frac{\partial F_j^{k,\alpha}}{\partial y^{i,r}} \frac{\partial}{\partial y^{k,\alpha}} \otimes dx^i \wedge dx^j,$$

$\alpha = 1, \dots, r$.

Proof. We obtain this formula by a direct evaluation of the Frölicher–Nijenhuis bracket in local coordinates. □

We call τ_n the *n–th general torsion* of Γ. The r–th general torsion τ_r is also called the *weak torsion*, [1].

4. r–LINEAR CONNECTIONS AND THEIR GENERAL TORSIONS

There are no linear connections on T^rM for $r > 1$ (because T^rM is not a vector bundle). We consider the simplest class of special connections on T^rM defined by the property that the flows of the corresponding horizontal lifts of all vector fields on

M are constituted by tangent morphisms of order r. Such a connection Γ is called *r-linear connection* and its coordinate expression is

$$(3)\qquad \begin{aligned} dy^{i,1} &= \Gamma^i_{i_1 j}(x) y^{i_1,1} dx^j \\ dy^{i,2} &= (\Gamma^i_{i_1 i_2 j}(x) y^{i_1,1} y^{i_2,1} + \Gamma^i_{i_1 j}(x) y^{i_1,2}) dx^j \\ &\dots \\ dy^{i,r} &= (\sum_{q=1}^{r} \sum_{\pi\in\mathbb{P}(r,q)} k_\pi \Gamma^i_{i_1\dots i_q j}(x) y^{i_1,r_1}\dots y^{i_q,r_q}) dx^j, \end{aligned}$$

where Γ's are symmetric in subscripts $i_1,\dots,i_q$ (cf. (1)).

Proposition 3. *n-th general torsion τ_n of the r-linear connection Γ has the coordinate expression*

$$(4)\qquad \Gamma^k_{ij}\frac{\partial}{\partial y^{k,n}} \otimes dx^i \wedge dx^j + \sum_{q=1}^{\beta}\sum_{\pi\in\mathbb{P}(\beta,q)} (k_\pi \Gamma^k_{i_1\dots i_q i,j} + l_\pi \Gamma^k_{i_1\dots i_q ij}) y^{i_1,\beta_1}\dots y^{i_q,\beta_q}\frac{\partial}{\partial y^{k,\beta+n}} \otimes dx^i \wedge dx^j,$$

$n<r$, $\beta=1,\dots,r-n$, $\pi\in\mathbb{P}(\beta,q)$, $\pi\cup\{n\}=\hat\pi\in\mathbb{P}(\beta+n,q+1)$, $n_{\hat\pi}$ is the number of occurences of n in $\hat\pi$, $l_\pi=n_{\hat\pi}k_{\hat\pi}$, $\Gamma^k_{i_1\dots i_q i,j}=\frac{\partial\Gamma^k_{i_1\dots i_q i}}{\partial x^j}$, and τ_r has the coordinate expression

$$(4')\qquad \Gamma^k_{ij}\frac{\partial}{\partial y^{k,r}} \otimes dx^i \wedge dx^j$$

Proof. This is a direct application of (2), (2') for (3). □

Remark 3. Geometrically viewing, it is important that general torsions of r-linear connections do not depend on fiber components of vector fields on T^rM. In other words, they are projectable to $VT^rM \otimes \Lambda^2T^*M$.

Remark 4. A further important geometrical property is provided by the easy provable identity $\tau_n = A_{n-h}(\tau_h)$, $0<h\le n\le r$. In addition, if τ_n denotes general torsions on T^rM and $\bar\tau_n$ general torsions on T^sM, $0<s<r$, we can verify that $\bar\tau_n = T\pi^r_s(\tau_n)$ for $n=1,\dots,s$. Consequently, the torsion τ_1 of an r-linear connection (if r is sufficiently great) provides all information about all such torsions on higher order tangent bundles. Practically, it is useful for coordinate computations.

5. Principal connections on higher order frame bundles

Let $P^rM = invJ^r_0(\mathbb{R}^m, M)$ be the r-th frame bundle of M. The group $G^r_m = invJ^r_0(\mathbb{R}^m,\mathbb{R}^m)_0$ acts smoothly on P^rM on the right by the jet composition. The tangent bundle of order r is a fiber bundle associated with P^rM with standard fiber $L^r_{1,m} = J^r_0(\mathbb{R},\mathbb{R}^m)_0$. A principal connection Γ on P^rM is a G^r_m-invariant section $\Gamma: P^rM \to J^1P^rM$.

Proposition 4. *The principal connections on P^rM are in bijection with the r–linear connections on T^rM.*

Proof. P^rM can be locally identified with the trivial principal bundle $U\times G^r_m$, $U\subset M$. Let $a=(a^i), b=(b^i):\mathbb{R}^m\to\mathbb{R}^m$ be any maps satisfying $A=j^ra, B=j^rb\in G^r_m$. If $c=b\circ a$, then the group multiplication $C=j^rc=j^rb\circ j^ra$ in jet coordinates is

$$
\begin{aligned}
c^i_{j_1} &= \beta^i_{k_1}a^{k_1}_{j_1}\\
&\dots\\
c^i_{j_1\dots j_r} &= \sum_{q=1}^{r}\sum_{\pi\in\mathbb{P}(r,q)} k_\pi\beta^i_{k_1\dots k_q}(x)a^{k_1}_{j_1\dots j_{r_1}}\dots a^{k_q}_{j_1\dots j_{r_q}},
\end{aligned}
$$

where $a^i_{j_1},\dots,a^i_{j_1\dots j_r}$ are group coordinates and $\beta^i_{k_1}=\frac{\partial b^i}{\partial a^{k_1}},\dots,\beta^i_{k_1\dots k_r}=\frac{1}{r!}\frac{\partial^r b^i}{\partial a^{k_1}\dots\partial a^{k_r}}$.

We take two sections $s,\sigma:M\to P^rM$, $s:x\mapsto(x,B)$, $\sigma:x\mapsto(x,C)$. The condition of G^r_m–invariancy means $(j^1s)A=j^1\sigma$. If we denote $\psi=j^1\sigma$ and $\Gamma^i_{l_1k}(x)=\frac{\partial s^i_{l_1}}{\partial x^k},\dots,\Gamma^i_{l_1\dots l_rk}=\frac{\partial s^i_{l_1\dots l_r}}{\partial x^k}$ we obtain

$$
\begin{aligned}
\psi^i_{j_1k} &= \Gamma^i_{l_1k}a^{l_1}_{j_1}\\
&\dots\\
\psi^i_{j_1\dots j_rk} &= \sum_{q=1}^{r}\sum_{\pi\in\mathbb{P}(r,q)} k_\pi\Gamma^i_{l_1\dots l_qk}(x)a^{l_1}_{j_1\dots j_{r_1}}\dots a^{l_q}_{j_1\dots j_{r_q}}.
\end{aligned}
$$

But we identify $a^i_{j_1},\dots,a^i_{j_1\dots j_r}$ with fiber coordinates on P^rM which we denote $\phi^i_{j_1},\dots,\phi^i_{j_1\dots j_r}$ (we introduce these coordinates including the factorial numbers as well as on T^rM). For every $j^r_0f\in T^r_xM$ and $j^r_0\phi\in P^r_xM$ we have $j^r_0(\phi^{-1}\circ f)\in L^r_{1,m}$, and conversely, every $j^r_0g\in L^r_{1,m}$ and $j^r_0\phi\in P^r_xM$ determine $j^r_0(\phi\circ g)\in T^r_xM$. If we evaluate these properties in coordinates, we come directly to (3). □

Torsions of principal connections on higher order frame bundles were introduced by Yuen, [15]. These connection are investigated in [2], [3], [8]. Let $u=j^r_0f\in P^rM$, $A\in T_uP^rM$. There is a canonical $\mathbb{R}^m\oplus\mathfrak{g}^{r-1}_m$–valued form θ on P^rM defined by

$$\theta(A)=\tilde{u}^{-1}\circ T\pi^r_{r-1},$$

where $\tilde{u}:\mathbb{R}^m\oplus\mathfrak{g}^{r-1}_m\to T_{j^{r-1}_0f}P^{r-1}M$ is the linear isomorphism determined by u, π^r_{r-1} denotes the canonical projection $P^rM\to P^{r-1}M$. The exterior covariant differential $D\theta$ with respect to a principal connection Γ on P^rM is a 2–form Θ called *Yuen's torsion form.*

6. Geometry of the second order case

We are going to consider the bundle T^2M. Let us remind that connections on T^2M are studied in [12], for example. Let x^i, $y^i=y^{i,1}$, $z^i=y^{i,2}$ are local coordinates on T^2M. The coordinate expression of a 2–linear connection Γ on T^2M is

$$
\begin{aligned}
dy^i &= \Gamma^i_{kj}y^kdx^j\\
dz^i &= (\Gamma^i_{klj}y^ky^l+\Gamma^i_{kj}z^k)dx^j.
\end{aligned}
$$

Corollary 5. *All general torsions of* Γ *form a 2-parameter family linearly generated by* τ_1, τ_2*:*

$$\begin{aligned}
\tau_1: &\quad \Gamma^k_{ij}\frac{\partial}{\partial y^k}\otimes dx^i\wedge dx^j + (\Gamma^k_{li,j}+2\Gamma^k_{lij})y^l\frac{\partial}{\partial z^k}\otimes dx^i\wedge dx^j \\
\tau_2: &\quad \Gamma^k_{ij}\frac{\partial}{\partial z^k}\otimes dx^i\wedge dx^j
\end{aligned}$$

Proof. This is a direct corollary of Proposition 3. □

It is clear that Γ is projectable with respect to π^2_1, i.e. there exists a connection $\tilde{\Gamma}$ on TM, whose coordinate expression is

$$dy^i = \Gamma^i_{kj}y^k dx^j,$$

such that $(J^1\pi^2_1)\circ\Gamma = \tilde{\Gamma}\circ\pi^2_1$. Let us denote $\mathcal{T}$ the torsion of the linear connection $\tilde{\Gamma}$ defined by the classical way. We geometrize τ_2 as $\pi^{2*}_0(\mathcal{T})$, i.e. the second general torsion of 2–linear connection Γ on T^2M is just the pullback of $\mathcal{T}$ with respect to π^2_0.

We geometrize τ_1 by another way than in [11]. For this purpose, we are going to illustrate relations with second order frame bundle P^2M. The r–th order frame bundle P^rM is an open dense subset in $T^r_mM = J^r_0(\mathbb{R}^k, M)$. The Weil algebra associated with the functor T^2_m is $\mathbb{A} = \mathbb{R}[t^1,\dots,t^m]/\langle(t^1,\dots,t^m)^3\rangle$. The elements of $\mathbb{A}$ have the form $a + b_it^i + c_{ij}t^it^j$ and the elements 1, t^i, t^it^j determine $\frac{m^2}{2}+\frac{3m}{2}+1$ natural affinors. The restrictions of them are natural affinors on P^2M, so that t^i and t^it^j determine these two types of affinors on P^2M:

$$\begin{aligned}
A_s: &\quad \delta^i_j\frac{\partial}{\partial\phi^i_s}\otimes dx^j + \delta^i_j\frac{\partial}{\partial\phi^i_{sk}}\otimes d\phi^j_k \\
A_{st}: &\quad \delta^i_j\frac{\partial}{\partial\phi^i_{st}}\otimes dx^j,
\end{aligned}$$

where ϕ^i_j, ϕ^i_{jk} are fiber coordinates on P^2M. There are interesting results concerning connections on P^2M in [5], [7].

The equations of a principal connection Δ on P^2M are

$$\begin{aligned}
d\phi^i_j &= \Gamma^i_{lk}\phi^l_j dx^k \\
d\phi^i_{jk} &= (\Gamma^i_{mnl}\phi^m_j\phi^n_k + \Gamma^i_{ml}\phi^m_{jk})dx^l.
\end{aligned}$$

Proposition 6. *All general torsions of* Δ *form a* $(m+m^2)$*-parameter family linearly generated by* τ_s, τ_{st}, $s,t=1,\dots,m$*:*

$$\begin{aligned}
\tau_s: &\quad \Gamma^k_{ij}\frac{\partial}{\partial\phi^k_s}\otimes dx^i\wedge dx^j + (\Gamma^k_{li,j}+2\Gamma^k_{lij})\phi^l_m\frac{\partial}{\partial\phi^k_{sm}}\otimes dx^i\wedge dx^j \\
\tau_{st}: &\quad \Gamma^k_{ij}\frac{\partial}{\partial\phi^k_{st}}\otimes dx^i\wedge dx^j
\end{aligned}$$

Proof. We obtain this formula by a direct evaluation of the Frölicher–Nijenhuis bracket in local coordinates. □

Let us suppose fixed indices s, t. Then A_s, A_{st} on P^2M are evidently in bijection with A_1, A_2 on T^2M. That is why the torsions on T^2M correspond to torsions on P^2M, how we see immediately in coordinate formulas. More or less, we expected this property after the formulation of Proposition 4. But there is yet the interesting question of Yuen's torsion (denoting by $\overset{\mathcal{Y}}{\mathcal{T}}$) on P^2M. We evaluate the coordinate expression of its form as

$$\tilde{\phi}^i_k \Gamma^k_{lm} dx^l \wedge dx^m + (\Gamma^i_{km}\Gamma^k_{jl} + 2\Gamma^i_{jlm}) dx^l \wedge dx^m$$

and we see that it gives a different information.

We take difference tensor $\tau_s - \overset{\mathcal{Y}}{\mathcal{T}}$, $s = i$. We denote by Z the corresponding $(1,2)$-tensor on T^2M.

Proposition 7. *The geometrical interpretation of τ_1 is given by the identity*

$$Z = A_1(C(\tilde{\Gamma})),$$

where $C(\tilde{\Gamma})$ denotes the curvature of the underlying connection $\tilde{\Gamma}$ on TM.

Proof. The coordinate expression of Z is

$$(\Gamma^k_{li,j} + \Gamma^k_{mi}\Gamma^m_{lj}) y^l \frac{\partial}{\partial z^k} \otimes dx^i \wedge dx^j.$$

A direct evaluation of the right hand side yields the same result. □

In other words, $\tau_s = \overset{\mathcal{Y}}{\mathcal{T}}$ if and only if $\tilde{\Gamma}$ is integrable. We recollect Remark 4 and we see that the interpretation of the first general torsion τ_1 on T^2M represents the exhaustive geometrical answer.

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