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CONNECTIONS FOR NON-HOLONOMIC 3-WEBS

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ABSTRACT. A $\{P, B\}$-structure determines on a manifold a four-tuple of distributions which may be non-integrable in general, and can be regarded as a generalization of a web. On a manifold endowed with a $\{P, B\}$-structure, we will find all connections which parallelize distributions of the structure. We will re-prove a formula for the so called canonical connection which is important especially in the integrable case. We also investigate integrability of the structure, and deduce some conditions for the torsion tensor of the canonical connection equivalent to integrability.

All manifolds, bundles, vector and tensor fields under consideration are supposed to be smooth (of the class C^∞). M will denote a manifold, TM its tangent bundle, $\mathfrak{X}(D)$ denote the set of all vector fields of the distribution D (i. e. on the subbundle $D \to M$ of $TM \to M$).

1. NON-HOLONOMIC THREE-WEBS

1.1. $\{P, B\}$-structures. A d-web of dimension (codimension) n on a manifold M consists of d foliations on M of the same dimension (codimension) n which are in general position (in the case of different dimensions it is difficult to give a nice theory). The most important case arise when $d = n + 1$ and dimension of the manifold M is a multiple of n. Here we will restrict ourselves to the case $n = 2$. For the purpose of tensor theory of webs it is more convenient to consider a web as a family of integrable distributions (in general position) tangent to given foliations. More generally, we can introduce a non-holonomic d-web as a family of d distributions in general position which are not necessarily integrable. We can give an alternative tensor definition for non-holonomic 3-webs.

Definition 1.1. *A $\{P, B\}$-structure* [Ng 3, Ng 1], or *a non-holonomic three-web* on a smooth manifold M is a triple (P, B, M) where P and B are smooth $(1,1)$-tensor fields on M such that P is a projector, $(P - I)P = 0$, B is involutory, i.e. $(B - I)(B + I) = 0$, and $PB + BP = B$.

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A $\{P, B\}$-structure determines on M a quadruple of n-dimensional distributions arising as invariant subspaces corresponding to characteristic roots of the endomorphisms P and B

$$(1) \quad D_1 = \ker(I - P), \quad D_2 = \ker P, \quad D_3 = \ker(B - I), \quad D_4 = \ker(B + I),$$

and a $\{P, B\}$-manifold endowed with this $\{P, B\}$-structure has even dimension $2n$. It can be verified that the above distributions (and also each triple of them) are in general position. So they define a non-holonomic 4-web[1] (D_1, D_2, D_3, D_4) of dimension n, and each triple of them forms a non-holonomic 3-subweb.

As *morphisms* of $\{P, B\}$-manifolds, we take local diffeomorphisms $f : M \to M'$ of base manifolds which commute with both tensor fields[2], $Tf \circ P = P' \circ Tf$, $Tf \circ B = B' \circ Tf$. $\{P, B\}$-manifolds together with their morphisms form a category. Each fibre T_xM, $x \in M$ is equipped with a structure of a $\{P, B\}$-vector space [Ng 3], and morphisms induce linear $\{P, B\}$-maps between fibres.

A frame is *adapted* with respect to a $\{P, B\}$-structure if it is of the form

$$\{X_1, \dots, X_n, BX_1, \dots, BX_n\}.$$

The set of all adapted frames forms a $GL(n, \mathbb{R})$-structure on M. On the other hand, it can be verified that the existence of a G-structure on a $2n$-dimensional manifold with $G = GL(n, \mathbb{R})$ is equivalent to the existence of a $\{P, B\}$-structure on M.

Definition 1.2. We say that a $\{P, B\}$-structure is *integrable* if the distributions D_1, D_2, D_3 are integrable.

Example 1.1. Any ordered 3-web (i. e. an ordered triple of smooth foliations in general position) on a manifold determines a $\{P, B\}$-structure the distributions D_1, D_2, D_3 of which are integrable [Ng 1, V 2]. Vice versa, any integrable $\{P, B\}$-structure defines an ordered 3-web formed by integral foliations of the corresponding distributions D_1, D_2, D_3 (some authors prefer D_4 to D_3). The remaining distribution D_4 (or D_3, respectively) is not necessarily integrable.

Since we will be interested at most in 3-webs we will pay more attention to the first triple (D_1, D_2, D_3) of distributions of the structure.

1.2. Projectors of a $\{P, B\}$-structure. For simplicity, let us denote the complementary projector $I - P$ by $\tilde{P}$. It can be verified that

$$(2) \quad \begin{aligned} &P\tilde{P} = \tilde{P}P = PBP = \tilde{P}B\tilde{P} = 0, \\ &PB = B\tilde{P}, \quad BP = \tilde{P}B, \quad P = B\tilde{P}B, \quad \tilde{P} = BPB. \end{aligned}$$

[1]Note that this 4-web is of a very special type since D_4 is not arbitrary but an invariant subspace of B.

[2]Here T denotes the tangent functor.

As in (1), let D_i, $i = 1,2,3,4$ denote distributions of the given structure and let P_i^j be the corresponding projectors with kernel D_i and image D_j. Then $P_1^2 = P$, $P_2^1 = \tilde{P}$,

$$P_1^3 = P(I - B), \qquad P_3^1 = (I + B)\tilde{P},$$
$$P_2^3 = \tilde{P}(I - B), \qquad P_3^2 = P(I + B),$$
$$P_4^3 = -\frac{1}{2}(B - I), \qquad P_3^4 = \frac{1}{2}(B + I),$$

$B = B_{12}^3 = P_3^1 - P_2^3 = P_3^2 - P_1^3$, $B_{ij}^k = P_k^i - P_j^k$, $k = 3,4$, $i,j = 1,2$ etc. Since $B|D_3 = I$ and $B|D_4 = -I$ the distributions (constituted by invariant subspaces of B) are characterized by

$$\begin{aligned} D_3 &= \{BPX + PX\,;\, X \in \mathfrak{X}(TM)\} = \{B\tilde{P}X + \tilde{P}X\,;\, X \in \mathfrak{X}(TM)\}, \\ D_4 &= \{BPX - PX\,;\, X \in \mathfrak{X}(TM)\} = \{B\tilde{P}X - \tilde{P}X\,;\, X \in \mathfrak{X}(TM)\}. \end{aligned} \tag{3}$$

decomposition with

2. Parallelizing connections for $\{P, B\}$-structures

2.1. Connections preserving distributions of the structure.

Definice 2.1. Under a *parallelizing connection* for a given $\{P, B\}$-structure on M we understand an affine connection ∇ (the corresponding covariant derivation will be denoted by the same symbol) on M with respect to which the distribution D_i, $i = 1,2,3$ of the structure are parallel, i.e. for $i = 1,2,3$

$$\nabla_X Y \in \mathfrak{X}(D_i) \quad \text{for all } X \in \mathfrak{X}(TM), \quad Y \in \mathfrak{X}(D_i). \tag{4}$$

It can be verified the following:

Proposition 2.1. *Let ∇ be a linear connexion on a $\{P, B\}$-manifold. Then*

$$D_1 \text{ and } D_2 \text{ are parallel to } \nabla \Longleftrightarrow \nabla P = 0 \Longleftrightarrow \nabla(I - P) = 0,$$
$$D_3 \text{ is parallel} \Longleftrightarrow \nabla B = 0 \Longleftrightarrow D_4 \text{ is parallel.}$$

Corollary. *A connection ∇ paralelizes a $\{P, B\}$- structure if and only if*

$$\nabla P = \nabla B = 0.$$

Proposition 2.2. *A linear connection ∇ paralelizes a $\{P, B\}$-structure on M if and only if for all $X, Y \in \mathfrak{X}(TM)$*

$$\begin{aligned} \nabla_X Y &= P\nabla_X PY + BP\nabla_X(B\tilde{P}Y), \\ \nabla_X Y &= B\tilde{P}\nabla_X(BPY) + \tilde{P}\nabla_X\tilde{P}Y. \end{aligned} \tag{5}$$

Proof. Let a connection ∇ on M satisfy the formulas (5). An evaluation shows that $\nabla_X PY = P\nabla_X PY = P\nabla_X Y$, $\nabla_X BPY = BP\nabla_X PY = BP\nabla_X$ and $\nabla_X B\tilde{P}Y = P\nabla_X B\tilde{P}Y = B\tilde{P}\nabla_X Y$. We obtain $\nabla P = \nabla B = 0$. On the other hand, let ∇ be a parallelizing connection. Then

$$\nabla_X Y = \nabla_X(PY + \tilde{P}Y) = P\nabla_X PY + \tilde{P}B\nabla_X B\tilde{P}Y.$$

Similarly for the second formula in (5).

□

By a standard evaluation, we can prove the following.

Proposition 2.3. *Let Γ be an arbitrary linear connection on a $\{P, B\}$- manifold M. Then each of the formulas*

$$\begin{aligned} \nabla_X Y &= P\Gamma_X PY + BP\Gamma_X(B\tilde{P}Y), \\ \nabla_X Y &= B\tilde{P}\Gamma_X(BPY) + \tilde{P}\Gamma_X\tilde{P}Y \end{aligned} \tag{6}$$

defines a linear connection on M which is parallelizing for the given structure.

Using equalities (5) we can re-prove the well-known fact that for any $\{P, B\}$-structure there exists a unique linear connection ∇ [Ng 1, Va 3] which satisfies the following conditions:

$$\nabla P = \nabla B = 0, \qquad \mathcal{T}(PX, \tilde{P}Y) = 0$$

where $\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ (P and B are covariantly constant with respect to ∇, and each couple of "homogeneous" vectors $X \in \ker \tilde{P}$, $Y \in \ker P$ at the same point is conjugated with respect to the torsion tensor). It is called the *canonical connection* of a $\{P, B\}$-structure. The canonical connection of the integrable $\{P, B\}$-structure is called the *Chern connection* of the corresponding ordered 3-web.

Let there exist a conneciton ∇ satisfying the above conditions. Then the equalities $P\mathcal{T}(\tilde{P}X, PY) = 0$ and $\tilde{P}\mathcal{T}(PX, \tilde{P}Y) = 0$ yield the formulas

$$P\nabla_{\tilde{P}X}Y = P[\tilde{P}X, PY], \qquad \tilde{P}\nabla_{PX}Y = BP\nabla_{PX}B\tilde{P}Y = \tilde{P}[PX, \tilde{P}Y].$$

We substutite BY instead of Y and apply B on both sides of the equalities to obtain

$$\tilde{P}\nabla_{\tilde{P}X}Y = BP[\tilde{P}X, B\tilde{P}Y], \qquad P\nabla_{PX}Y = B\tilde{P}[PX, BPY].$$

Together, we obtain the formula

$$\nabla_X Y = B\tilde{P}[PX, BPY] + BP[\tilde{P}X, B\tilde{P}Y] + P[\tilde{P}X, PY] + \tilde{P}[PX, \tilde{P}Y]. \tag{7}$$

On the other hand, given (1,1)-tensor fields P, $\tilde{P}$ and B on M, we can ask when the formula (7) defines a linear connection on M. It is evident that the linearity conditions are satisfied. The formula $\nabla_{fX}Y = f\nabla_X Y$ holds if and only if $P\tilde{P} = \tilde{P}P = 0$. The formula $\nabla_X fY = f\nabla_X Y + (Xf)Y$ is satisfied if and only if the following conditions hold:

$$P + \tilde{P} = I, \quad B\tilde{P}BP + \tilde{P}^2 = P^2 + BPB\tilde{P} = P + \tilde{P}.$$

In the case of $\{P, B\}$-structures the above conditions are satisfied which proves that (7) is a linear connection.

By the above results we obtain immediately

Proposition 2.4. *The distributions D_i, $i = 1,2,3,4$ of a $\{P, B\}$-structure are parallel with respect to the canonical connection. Each distribution D_i is parallel to any D_j and autoparallel (for $i = j$).*

Corollary. *For an integrable $\{P, B\}$-structure, the Chern connection is reducible to the leaves of integral foliations of D_1, D_2 and D_3.*

Given a connection ∇ on a manifold M, any connection on M is of the form $\Gamma = \nabla + S$ where S is a $(1,2)$-tensor field. As in the integrable case [Va 1] we can prove

Proposition 2.5. *Let distributions D_1 and D_2 are parallel with respect to a linear connection ∇ on a $\{P, B\}$-manifold M, and let S be a $(1,2)$-tensor field given by*

$$S = B\nabla B(aP + b\tilde{P}), \qquad a + b = 1, \quad a, b \text{ non-negative reals.}$$

Then all three distributions of the structure are parallel with respect to a connection $\nabla + S$. Similarly for D_i, D_j if we substitute $P = P_i^j$, $\tilde{P} = P_j^i$, $B = B_{ij}^k$ for distinct $i, j, k \in \{1,2,3,4\}$.

The following is straightforward.

Proposition 2.6. *Let ∇ be a parallelizing connection on a $\{P, B\}$-manifold. A connection $\Gamma = \nabla + S$ is parallelizing if and only if the $(1,2)$-tensor field S satisfies*

$$X \in \mathfrak{X}(TM), \quad Y \in \mathfrak{X}(D_i) \Rightarrow S(X,Y) \in \mathfrak{X}(D_i), \qquad i = 1,2,3,4. \tag{8}$$

There are many parallelizing connections for a $\{P, B\}$-manifold, and a family of all such connections is described as follows:

Proposition 2.7. *Let ∇ be a parallelizing connection for all distributions of a $\{P, B\}$-structure. Let*[3]

$$\varphi_1 : M \to \operatorname{Hom}(TM, \operatorname{End} D_1)$$

be a differentiable map, and let us introduce its extension

$$\varphi : M \to \operatorname{Hom}(TM, \operatorname{End} TM)$$

by

$$(\varphi X)Y = (\varphi_1 X)(P_1^2 Y) + B \circ (\varphi_1 X) \circ B(P_2^1 Y), \qquad X, Y \in \mathfrak{X}(TM). \tag{9}$$

Let us define S by $S(X,Y) = (\varphi X)Y$. Then $\nabla + S$ is a parallelizing connection for $\{P, B\}$. Moreover, any parallelizing connection Γ can be given by

$$\Gamma(X,Y) = \nabla(X,Y) + (\varphi X)Y$$

[3] Analogously we can choose $\varphi_i : M \to \operatorname{Hom}(TM, \operatorname{End} D_i)$, $i = 2,3,4$ and modify correspondingly the formula for φ.

for some differentiable φ of the form (9). *Consequently, all parallelizing connections constitue a $2n^3$-dimensional vector space.*

By the above construction it follows that φX commutes with projectors which is crucial for the verification of (8).

2.2. Torsion of the structure. Now let us consider the torsion tensor of the canonical connection of a $\{P, B\}$-structure which is given by

$$\begin{aligned}\mathcal{T}(X,Y) = B\tilde{P}\Big([PX, BPY] + [BPX, PY]\Big) + BP\Big([\tilde{P}X, B\tilde{P}Y] \\ + [B\tilde{P}X, \tilde{P}Y]\Big) + [\tilde{P}X, PY] + [PX, \tilde{P}Y] - [X,Y].\end{aligned} \tag{10}$$

Let us evaluate $\mathcal{T}$ on couples of homogeneous vector fields belonging to the first (second) distribution:

$$\begin{aligned}\mathcal{T}(PX,PY) &= B\tilde{P}[PX, BPY] - B\tilde{P}[PY, BPX] - [PX, PY],\\ \tilde{P}\mathcal{T}(PX,PY) &= -\tilde{P}[PX,PY],\\ \mathcal{T}(\tilde{P}X,\tilde{P}Y) &= BP[\tilde{P}X, B\tilde{P}Y] - BP[\tilde{P}Y, B\tilde{P}X] - [\tilde{P}X,\tilde{P}Y],\\ P\mathcal{T}(\tilde{P}X,\tilde{P}Y) &= -P[\tilde{P}X,\tilde{P}Y].\end{aligned} \tag{11}$$

The properties of the torsion tensor anables us to give alternative conditions for integrability of distributions belonging to the structure. We will use the following identities:

$$\begin{aligned}P\mathcal{T}(PX,PY) &= B\tilde{P}[PX, BPY] + B\tilde{P}[BPX, PY] - P[PX,PY],\\ \mathcal{T}(BPX,BPY) &= BP[BPX,PY] + BP[PX,BPY] - [BPX,BPY],\\ B\tilde{P}\mathcal{T}(BPX,BY) &= P[BPX,PY] + P[PX,BPY] - B\tilde{P}[BPX,BPY].\end{aligned} \tag{12}$$

3. Integrable $\{P, B\}$-structures

3.1. Integrability conditions. It can be easily checked the following[4]:

(a) for $i = 1, 2$, D_i is integrable if and only if $[P,P](X,Y) = 0$ for X, $Y \in \mathfrak{X}(D_i)$;
(b) both D_1 and D_2 are integrable if and only if $[P,P] = 0$;
(c) for $j = 3, 4$, D_j is integrable if and only if $[B,B](X,Y) = 0$ for X, $Y \in \mathfrak{X}(D_j)$.

Lemma 3.1. *The following conditions are equivalent:*

(i) $B[B,B](X,Y) = [B,B](X,Y)$ *for* X, $Y \in \mathfrak{X}(D_1)$,
(ii) $B[B,B](X,Y) = [B,B](X,Y)$ *for* X, $Y \in \mathfrak{X}(D_2)$,
(iii) D_3 *is integrable.*

[4] [,] denotes the Nijenhuis bracket.

Proof. An evaluation shows that for $X, Y \in \mathfrak{X}(TM)$,

$$(B-I)[PX+BPX, PY+BPY] = -[B,B](PX,PY) + B[B,B](PX,PY).$$

Therefore D_3 is integrable if and only if $(B[B,B]-[B,B])(PX,PY)=0$. Similarly for the couple $\tilde{P}X$, $\tilde{P}Y$.

□

An analogous statement (up to a change of sign in the formulas) is true for D_4.

Remark. A polynomial structure B satisfying $B^2 = I$ is called *integrable* if there exist local coordinates such that with respect to the corresponding holonomic frame the matrix representation of B is $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. It is well known by the general theory of polynomial structures with simple roots that B is integrable if and only if all couples of projectors vanish. In our case, the conditions $[P_4^3, P_4^3] = [P_4^3, P_3^4] = [P_3^4, P_3^4] = 0$ are equivalent with a single condition $[B,B]=0$.

We obtain a characterization of structures for which both D_3 and D_4 are integrable.

Proposition 3.1. *The following conditions are equivalent:*

(i) *B is integrable,*
(ii) $[B,B]=0$,
(iii) *an almost product structure $[D_3, D_4]$ is integrable, in other words, D_3 and D_4 are simultaneously integrable,*
(iv) *both D_3 and D_4 are integrable.*

The equivalence between (iii) and (iv) is due to the fact that an almost product structure $[D, D']$ is integrable if and only if each of D, D' is integrable (which is not the case for an almost product structure $[D_1, \ldots, D_s]$ with $s > 2$).

3.2. Integrability through torsion. As we have seen above the integrability conditions for a $\{P, B\}$-structure $X, Y \in \mathfrak{X}(D_i) \Longrightarrow [X,Y] \in \mathfrak{X}(D_i)$, $i = 1,2,3$ can be expressed via Nijenhuis brackets, $[P,P]=0$, $[B,B](X,Y)=0$ for X, $Y \in \mathfrak{X}(D_3)$. Another possibility for formulation of integrability conditions yields the torsion tensor $\mathcal{T}$. By (11) it follows immediately:

$$D_1 \text{ is integrable if and only if } \tilde{P}\mathcal{T}(PX,PY)=0, \tag{13}$$

$$D_2 \text{ is integrable if and only if } P\mathcal{T}(\tilde{P}X,\tilde{P}Y)=0. \tag{14}$$

Denote

$$F(X,Y) = B[B,B](X,Y) - [B,B](X,Y). \tag{15}$$

The integrability of D_3 is equivalent with either of the conditions

$$F(PX,PY)=0 \qquad \text{or} \qquad F(\tilde{P}X,\tilde{P}Y)=0.$$

An evaluation shows that

$$\tilde{P}F(PX,PY) = BPF(PX,PY).$$

Since B is an isomorphism it follows

$$F(PX,PY) = 0 \iff PF(PX,PY) = 0.$$

Let us suppose that both D_1 and D_2 are integrable. Then

$$PB[B,B](PX,PY) = -B\tilde{P}\mathcal{T}(BPX,BPY),$$
$$-P[B,B](PX,PY) = P\mathcal{T}(PX,PY).$$

Together,

$$PF(PX,PY) = 0 \iff -B\tilde{P}\mathcal{T}(BPX,BPY) + P\mathcal{T}(PX,PY) = 0.$$

Proposition 3.2. *A $\{P,B\}$-structure is integrable if and only if the following conditions are satisfied*

$$\tilde{P}\mathcal{T}(PX,PY) = 0, \qquad P\mathcal{T}(\tilde{P}X,\tilde{P}Y) = 0,$$
$$\tilde{P}\mathcal{T}(BPX,BPY) = BP\mathcal{T}(PX,PY).$$

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