Thomas Schmitt

Symplectic solution supermanifolds in field theory

In: Jan Slovák (ed.): Proceedings of the 16th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1997. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 46. pp. [153]--162.

Persistent URL: http://dml.cz/dmlcz/701605

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 46 (1997), pp. 153-162

SYMPLECTIC SOLUTION SUPERMANIFOLDS IN FIELD THEORY

T. SCHMITT

ABSTRACT. For a large class of classical field models used for realistic quantum field theoretic models, an infinite-dimensional supermanifold of classical solutions in Minkowski space can be constructed. This solution supermanifold carries a natural symplectic structure; the resulting Poisson brackets between the field strengthes are the classical prototypes of the canonical (anti-) commutation relations. Moreover, we discuss symmetries and the Noether theorem in this context.

THE Φ^4 TOY MODEL

We start with the usual toy model of every physicist working on quantum field theory, namely the Φ^4 theory on Minkowski $\mathbb{R}^{1+3}$, given by the Lagrangian

$$L[\Phi] = \frac{1}{2}\Big((\partial_0\Phi)^2 - \sum_{a=1}^{3}(\partial_a\Phi)^2 - m^2\Phi^2\Big) - q\Phi^4$$

with $m, q \geq 0$, which leads to the equation of motion

$$\frac{\delta L}{\delta \Phi} \equiv \Box\Phi - m^2\Phi - 4q\Phi^3 = 0. \tag{1}$$

It is well-known (cf. e. g. [5]) that for given Cauchy data $(\varphi^{\mathrm{Cau}}, \dot\varphi^{\mathrm{Cau}}) \in H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3)$ (here H_k is the standard Sobolev space with order $k > 1$, in order to ensure the algebra property of H_{k+1} under pointwise multiplication) there exists a unique solution $\varphi \in C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)) \subseteq C(\mathbb{R}^4)$ of the Cauchy problem

$$\frac{\delta L}{\delta \Phi}[\varphi] = 0, \quad \varphi(0) = \varphi^{\mathrm{Cau}}, \quad \partial_0\varphi(0) = \dot\varphi^{\mathrm{Cau}},$$

and that the arising nonlinear map

$$\Phi^{\mathrm{sol}} : H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3) \to C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)), \quad (\varphi^{\mathrm{Cau}}, \dot\varphi^{\mathrm{Cau}}) \mapsto \varphi \tag{2}$$

is continuous.

For our purposes, it turns out to be reasonable to use only smooth Cauchy data with compact support, i. e. of test function quality: The map (2) restricts to a map

$$\Phi^{\mathrm{sol}} : C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \to C_c^\infty(\mathbb{R}^4), \tag{3}$$

where $C_c^\infty(\mathbb{R}^{d+1})$ is the space of all $f \in C^\infty(\mathbb{R}^{d+1})$ such that there exists $R > 0$ with $f(t,x) = 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ with $|x| \geq |t| + R$. (The virtue of this space is that it is Poincaré invariant.)

Clearly, the image of this map, denoted M^{sol}, is the set of all smooth solutions of (1) which on some, and hence every, time slice are compact supported. (Of course, M^{sol} might miss to contain some interesting classical solutions; but, at any rate, it comes locally arbitrarily close to them.)

In fact, it tuns out that the map (2) (as well as (3)) is real-analytic. and its image M^{sol} is a submanifold of the infinite-dimensional manifold $C_c^\infty(\mathbb{R}^4)$, which we call the *manifold of classical solutions* of the Φ^4 model.

(I have to apologize that here and in the following, the lack of space forces me to use rather vague, "hand-waving" formulations. Exact formulations are given in [7, 8, 9]. Note also that the locally convex space $C_c^\infty(\mathbb{R}^4)$ is not even Frechet; in fact, it seems impossible to construct a convincing, Lorentz invariant theory using only Banach manifolds.)

Now let $\Sigma := (\partial L/\partial(\partial_0\Phi)) = \partial_0\Phi$ be the canonical momentum. In analogy with classical mechanics' $\sum dp_i dq_i$, we introduce the two form

$$\omega = \int_{\mathbb{R}^3} dx \delta\Xi_i(0,x)\delta\Sigma_i(0,x)$$

on M^{sol}; here δ is the exterior derivative for forms (cf. [6] for a detailed theory), and the product under the integral is the exterior product of one forms. Thanks to the use of spatially compactly supported solutions, the well-definedness of the integral is no problem.

It turns out that this two-form is Poincaré invariant (this is not obvious even for time shifts and makes use of the field equations). Thus, it equips the manifold M^{sol} with a (weakly) symplectic structure. In particular, we can define Poisson brackets, and we could also do prequantization (the easy part of quantizing a field theory). The detailed consequences of the symplectic structure are discussed below, in a more general context.

Realistic models: The setup

A quantum field model in Minkowski $\mathbb{R}^{1+d}$ is in general given by a Lagrangian $L[\Xi] = L[\Phi|\Psi]$ which is mathematically well-defined at least as differential polynomial which depends on N^Φ commuting variables $\Phi_1, \ldots, \Phi_{N^\Phi}$ and N^Ψ anticommuting variables $\Psi_1, \ldots, \Psi_{N^\Psi}$ and their space-time derivatives. While the Φ_i describe "ordinary", bosonic fields (for mathematicians: fields with integer spin), and their equations of motions are of second order, the Ψ_i describe fermionic fields, and their equations of motions are of first order. (A notable exception are the Faddeev-Popov ghost fields arising in the quantization of Yang-Mills theory: They are anticommuting but of second order. For notational simplicity, we will pretend here in the formulas that no ghosts are present.)

Of course, we have to constrain the Lagrangian: its general form is $L[\Xi] = L_{\text{kin}}[\Xi] + \Delta[\Xi]$ where the *kinetic Lagrangian* $L_{\text{kin}}[\Xi]$ is of second degree while the *interaction term* $\Delta[\Xi]$ is of lower degree ≥ 3.

$L_{\text{kin}}[\Xi]$ has the form

$$L_{\text{kin}}[\Xi] = \frac{1}{2}\sum_{i,j=1}^{N^\Phi}\left(\sum_{a,b=0}^{d}\partial_a\Phi_i g_{ij}^{ab}\partial_b\Phi_j + \Phi_i m_{ij}^\Phi\Phi_j\right) + \sum_{i,j=1}^{N^\Psi}\left(\sum_{a=0}^{d}\Psi_i\Gamma_{ij}^a\partial_a\Psi_j + \Psi_i m_{ij}^\Psi\Psi_j\right)$$

with rather technical requirements onto the numbers g_{ij}^{ab}, Γ_{ij}^a, m_{ij}^Φ, m_{ij}^Ψ; here it is sufficient to know that these requirements are satisfied for most "usual" field types, e. g.

- real and complex scalar fields,
- Yang-Mills fields with temporal gauge $A_0 = 0$,
- Yang-Mills fields with gauge-breaking term,

$$L_{\rm kin} = -\frac{1}{4}\sum \partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} + \frac{\zeta}{2}(\sum \partial_\mu A^\mu)^2,$$

  provided that we have the diagonal gauge $\zeta = 1$, so that we effectively have scalar fields,
- Faddeev-Popov ghost fields,
- Dirac, Weyl, Majorana, Majorana-Weyl spinor fields; presumably also the Rarita-Schwinger spin 3/2 field.

Also, we have to constrain the interaction term: We require that derivative couplings occur only linearly by first derivatives of second order fields; thus, $\Delta[\Xi]$ has the form

$$\Delta[\Xi] = \Delta'(\Phi, \Psi) + \sum_{i=1}^{N^\Phi}\sum_{a=0}^{d}\Delta^{a,i}(\Phi, \Psi)\partial_a\Phi_i$$

where $\Delta^{a,i}(\Phi, \Psi)$ and $\Delta'(\Phi, \Psi)$ are ordinary (non-differential) even polynomials of lower degree ≥ 2 and ≥ 3, respectively. (Actually, we may allow entire power series instead of polynomials; thus, there is no need to exclude e. g. sinus-Gordon and Liouville interactions.)

This restriction is necessary for the treatment of analytical questions. Fortunately, it seems to be satisfied for all physically reasonable local models, including in particular Yang-Mills theory.

CONFIGURATION SUPERMANIFOLDS

The very first question arising for such a model is what a configuration should be: simply taking the Ξ_i to be functions on $\mathbb{R}^{1+d}$ is incompatible with the required anticommutativity of the Ψ_j. Taking them to be functions with values in a Grassmann algebra resolves that point, but for the price of other conceptual difficulties. As we have argued in [8], the only convincing way is to view the configuration space of such a model not as a set consisting of individual elements, but as an infinite-dimensional supermanifold having even coordinates $\Phi_i(x)$ and odd coordinates $\Psi_j(x)$ where $x \in \mathbb{R}^{1+d}$.

A suitable calculus of real-analytic infinite-dimensional supermanifolds (smf's) has been constructed by the present author in [6, 7]. Here we remark that it assigns to every real $\mathbb{Z}_2$-graded locally convex space ($\mathbb{Z}_2$-lcs) $E = E_{\bar 0} \oplus E_{\bar 1}$ a *linear supermanifold* $\mathrm{L}(E)$ which is essentially a ringed space with underlying topological space $E_{\bar 0}$ while the structure sheaf $\mathcal{O}$ might be thought very roughly of as a kind of completion of $\mathcal{A}(\cdot) \otimes \Lambda E_{\bar 1}^*$; here $\mathcal{A}(\cdot)$ is the sheaf of real-analytic functions on $E_{\bar 0}$ while $\Lambda E_{\bar 1}^*$ is the exterior algebra over the dual of $E_{\bar 1}$. (The actual definition of the structure sheaf treats even and odd sector much more on equal footing.)

Given a second real $\mathbb{Z}_2$-lcs F, one constructs the sheaf $\mathcal{O}^F$ of F-valued superfunctions on $\mathrm{L}(E)$; very roughly, one may think of it as a completion of $\mathcal{O} \otimes_{\mathbb{R}} F$. Both sheafs coincide if F is finite-dimensional; in particular, the structure sheaf is just the sheaf of real-valued superfunctions: $\mathcal{O} = \mathcal{O}^{\mathbb{R}}$.

Actually, in considering more general smf's that superdomains, one has to enhance the structure of a ringed space slightly, in order to avoid "fake morphisms" (not every morphism of ringed spaces is a morphism of supermanifolds). What matters here is that the enhancement is done in such a way that the following holds:

Given F as above and an arbitrary smf Z, the set of morphisms $Z \to \mathrm{L}(F)$ is in natural 1-1-correspondence with the set $\mathcal{O}^F(Z)_{0,\mathbb{R}}$.

(Here the subscript stands for the real, even part.) This is the infinite-dimensional version of the fact that if $F = \mathbb{R}^{m|n}$ is the standard $m|n$-dimensional $\mathbb{Z}_2$-graded vector space then a morphism $Z \to \mathrm{L}(\mathbb{R}^{m|n})$ is known by knowing the pullbacks of the coordinate superfunctions, and these can be prescribed arbitrarily as long as parity and reality are OK.

If E, F are spaces of functions on $\mathbb{R}^d$ which contain the test functions as dense subspace then the Schwartz kernel theorem tells us that the multilinear forms $u_{k|l}$ are given by their integral kernels, which are generalized functions. Thus one can apply rather suggestive integral writings (cf. [6]) quite analogous to that of (2).

Returning to the description of the configuration space of our model, and guided by our experience in the Φ^4 model above, we use C_c^∞ as our basic functional-quality; thus, the configuration smf is the linear smf $M = \mathrm{L}(E)$ with model space

$$E := C_c^\infty(\mathbb{R}^{d+1}) \otimes \mathbb{R}^{N^\Phi|N^\Psi}.$$

with functional coordinates Φ_i, Ψ_j. The question for an action principle is a rather tricky one; this has nothing to do with the anticommuting degrees of freedom but with the fact that, even for Φ^4 theory, the action over the whole space-time is ill-defined; only the action over (say) compact space-time regions Ω is defined (in our context, it becomes a superfunction $\int_\Omega d^{d+1}x L[\Xi](x) \in \mathcal{O}(M)$). We will not dwell into that but simply take the variational derivatives of $L[\Xi]$,

$$L_i[\Xi] := \frac{\delta}{\delta \Xi_i} L[\Xi],$$

which are well-defined differential polynomials, as the field equations. Thus, we need the smf of Cauchy data,

$$M^{\mathrm{Cau}} := \mathrm{L}(E^{\mathrm{Cau}}), \quad E^{\mathrm{Cau}} := C_0^\infty(\mathbb{R}^d) \otimes \mathbb{R}^{2N^\Phi|N^\Psi}.$$

Recall that while for the second order equations governing the Φ_i, one needs both the initial position Φ_i^{Cau} and the initial velocity $\dot{\Phi}_i^{\mathrm{Cau}}$ as Cauchy data, one needs only the initial position Ψ_j^{Cau} for the first order fields Ψ_j; thus, the functional coordinates on M^{Cau} are $\Phi_i^{\mathrm{Cau}}, \dot{\Phi}_i^{\mathrm{Cau}}, \Psi_j^{\mathrm{Cau}}$.

COMPLETENESS AND THE SUPERMANIFOLD OF CLASSICAL SOLUTIONS

It is well-known that all-time solvability of non-linear evolution equations depends rather sensitively on the model, not only on the "right" signs of the coupling constants: the formally "same" Lagrangian may yield all-time solutions at lower dimensions and cease to do so for higher ones. Thus, the investigation of all-time solvability is a task for hard analysis; we simply state:

Lemma 1. *For a given model, the following conditions are equivalent:*

(i) For every smooth solution $\varphi \in C^\infty((a,b) \times \mathbb{R}^d) \otimes \mathbb{R}^{N^\Phi|0}$ of the underlying bosonic field equations on an open time interval such that $\operatorname{supp} \varphi(t)$ *is compact for all $t \in (a,b)$*

there exists a Sobolev index $k > d/2$ such that

$$\sup_{t\in(a,b)} \|\partial_t\varphi_i(t)\|_{H_{k+1}} < \infty. \tag{4}$$

(ii) The underlying bosonic equations are all-time solvable: Given bosonic Cauchy data $(\varphi^{\mathrm{Cau}}, \dot\varphi^{\mathrm{Cau}}) \in (\mathcal{H}_k^{\mathrm{Cau}})_0$ there exists a function $\varphi \in C(\mathbb{R}, H_{k+1}(\mathbb{R}^d) \otimes \mathbb{R}^{N^\Phi|0})$ with these Cauchy data which solves the field equations.

(iii) There exists a (necessarily unique) superfunction $\Xi^{\mathrm{sol}} = (\Phi^{\mathrm{sol}}, \Psi^{\mathrm{sol}}) \in \mathcal{O}^E(M^{\mathrm{Cau}})_0$ which solves the "universal" Cauchy problem

$$L_i[\Xi^{\mathrm{sol}}] = 0, \quad \Xi^{\mathrm{sol}}|_{t=0} = \Xi^{\mathrm{Cau}}, \quad \partial_t\Phi^{\mathrm{sol}} = \Phi^{\mathrm{Cau}}.$$

If these conditions are satisfied we call the model complete. *In that case, the image of the morphism $\Xi^{\mathrm{sol}} : M^{\mathrm{Cau}} \to M$ determined by the superfunction Ξ^{sol} is a sub-supermanifold $M^{\mathrm{sol}} \subseteq M$ which we call the* solution supermanifold *of the model.*

Of course, for a complete model, in the situation of (i), (4) holds for all $k \in \mathbb{R}$.

The equivalence of the a-priori estimate (i) with all-time solvability (ii) is a standard idea in the theory non-linear evolution equations.

The term "complete" has been chosen in view of the fact that on the Banach manifold $(\mathcal{H}_k^{\mathrm{Cau}})_0$, the short-time existence result produces a local flow, i. e. a local one-parameter automorphism group, and the model is complete iff this local flow is so. Also, we get a local flow on the supermanifold $M_k^{\mathrm{Cau}} = \mathrm{L}(\mathcal{H}_k^{\mathrm{Cau}})$, and it turns out that this "superflow" is complete iff the model is so.

M^{sol} can be interpreted as a moduli space for solution families; cf. [9].

We note that all this can be done also without support restriction (there is no new completeness notion); one gets a morphism

$$M_{C^\infty}^{\mathrm{Cau}} := \mathrm{L}(C^\infty(\mathbb{R}^d) \otimes \mathbb{R}^{2N^\Phi|N^\Psi}) \xrightarrow{\Xi^{\mathrm{sol}}} \mathrm{L}(C^\infty(\mathbb{R}^{d+1}) \otimes \mathbb{R}^{N^\Phi|N^\Psi}) =: M_{C^\infty}.$$

However, the symplectic structure to be considered below would be ill-defined in this situation.

Symplectic structure and Poisson bracket

For the rest of this paper, we suppose to be given a fixed complete model. Let $\Sigma_j := (\partial L/\partial(\partial_0\Xi_j))$ be the canonical momenta (here the Lagrangian is considered as differential power series, not as superfunction). As in the Φ^4 model, we introduce the two form

$$\omega = \sum_{i=1}^{N^\Phi+N^\Psi} \int_{\mathbb{R}^d} dx\delta\Sigma_i(0,x)\delta\Xi_i(0,x)$$

on M^{sol}, turning it into a symplectic supermanifold. (In the terminology of [2], this would be called "weakly symplectic". Note, however, that our model space is not Banach, and that all our vector fields to be considered later are defined everywhere, not only on a dense domain.)

In fact, this symplectic structure is an intrinsic one, and does not depend on the accidental choice of the Cauchy hyperplane:

Theorem 2. *(Cf. also [10, 3]) ω is invariant under the Poincaré group as well as under change of the Lagrangian by a total derivative.*

Let

$$\mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}}) := \left\{ F \in \mathcal{O}(M^{\mathrm{sol}}) : \quad \exists\, \xi_F \in \mathcal{X}(M^{\mathrm{sol}}) : \ \xi_F \neg \omega = \delta F \right\}$$

(here $\neg$ is the interior derivative) be the set of all superfunctions which possess a *Hamiltonian vector field* ξ_F. Due to infinite-dimensionality, this is somewhat smaller than $\mathcal{O}(M^{\mathrm{sol}})$. As a rule of thumb, the coefficient functions of F should be smooth in spatial direction. For instance, for any testfunction $f \in C_0^\infty(\mathbb{R}^d)$ and fixed $i \in \{1, \ldots, N^\Phi\}$ we have

$$F := \int_{\mathbb{R}^d} dx\, f(x) \Phi_i(0,x) \in \mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}}), \quad \xi_F = \int_{\mathbb{R}^d} dx\, f(x) \frac{\delta}{\delta \Sigma_i(0,x)}.$$

(Note that the $\Phi_i(0,\cdot)|\Phi_j(0,\cdot)$ together with the bosonic momenta $\Sigma_1(0,\cdot), \ldots, \Sigma_{N^\Phi}(0,\cdot)$ form functional coordinates on M^{sol}, so that this makes sense.) However, the field strength $\Phi_i(0,x)$ at a point does not possess a Hamilton field since the unsmeared functional derivative $\frac{\delta}{\delta \Sigma_i(0,x)}$ is not a well-defined vector field on M^{sol}.

For $F \in \mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}})$, $G \in \mathcal{O}(M^{\mathrm{sol}})$ we define the *Poisson bracket* by

$$\{F, G\} := \xi_F(G).$$

This equips $\mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}})$ with the structure of a Lie superalgebra; since $\mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}})$ is also closed under product, it becomes a $\mathbb{Z}_2$-graded Poisson algebra. Explicitly,

$$\{F, G\} = \sum_{i=1}^{N^\Phi} \left(\int dx\, \frac{\delta}{\delta \Phi_i(0,x)} F \cdot \frac{\delta}{\delta \Sigma_i(0,x)} G - \int dx\, \frac{\delta}{\delta \Sigma_i(0,x)} F \cdot \frac{\delta}{\delta \Phi_i(0,x)} G \right)$$

$$- \frac{(-1)^{|F|}}{2} \sum_{i,j=1}^{N^\Psi} (\Gamma^0)^{-1}{}_{ij} \int dx\, \frac{\delta}{\delta \Psi_i(0,x)} F \cdot \frac{\delta}{\delta \Psi_j(0,x)} G$$

(all integrals over $\mathbb{R}^d$). Here, of course, Σ_j denotes only the bosonic momenta, the fermionic ones are not independent but linear combinations of the fermionic field strengthes. The Poisson brackets between the field strengthes and momenta yield the classical (=non-quantized) prototype of the *canonical (anti-) commutation relations*, the canonical brackets

$$\{\Phi_i(0,x), \Sigma_j(0,y)\} = \delta_{ij} \delta(x-y), \qquad \{\Psi_i(0,x), \Psi_j(0,y)\} = -\frac{1}{2} (\Gamma^0)^{-1}{}_{ij} \delta(x-y)$$

(all other brackets vanish). However, these formulas are highly symbolical, and are well-defined only in "smeared" form: E. g., for $f \in C_0^\infty(\mathbb{R}^d)$,

$$\left\{ \int dx\, f(x) \Phi_i(0,x), \ \Sigma_j(0,y) \right\} = f(y).$$

GREEN FUNCTIONS AND POISSON BRACKET

We look at the linearized field equations:

$$L_{ij} := \frac{\delta}{\delta \Xi_j(x)} L_i \in \mathcal{O}^{\mathcal{D}'(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})}(M).$$

Apart from the Ξ-dependence, this is the Schwartz kernel of a second or first order differential operator. Explicitly, splitting into kinetic and interaction part, $L_i(\Xi) = \sum_j K_{ij}(\partial)\Xi_j + \Delta_i(\Xi, \partial\Phi)$, we get

$$L_{ij}[\Xi](x,y) = \left(K_{ij}(\partial) + \frac{\partial}{\partial \Xi_j}\Delta_i(\Xi,\partial\Phi)(x) + \sum_a \frac{\partial}{\partial(\partial_a \Xi_j)}\Delta_i(\Xi,\partial\Phi)(x)\partial_a\right)\delta(y-x).$$

We call an element

$$\mathcal{G} = (\mathcal{G}_{ij}) \in \mathcal{O}^{\mathcal{D}'(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})\otimes\mathbb{R}^{N^\Phi|N^\Psi}\otimes\mathbb{R}^{N^\Phi|N^\Psi}}(M^{\mathrm{sol}})_0$$

a *Green superfunction* if we have the usual Green function property

$$\sum_{j=1}^{N^\Phi+N^\Psi} \int dy L_{ij}[\Xi](x,y)\mathcal{G}_{jk}[\Xi](y,z) = \delta_{ik}\delta(x-z)$$

for $i,k = 1,\dots,N^\Phi + N^\Psi$.

Proposition 3. *There exist unique Green superfunctions $\mathcal{G}^+, \mathcal{G}^-$, called the* advanced *and* retarded *Green superfunctions, respectively, such that*

$$\mathcal{G}^+(x,y) = 0 \quad \textit{except for } x<y, \qquad \mathcal{G}^-(x,y) = 0 \quad \textit{except for } y<x.$$

Of course, for a free model, $\mathcal{G}^\pm$ are constant superfunctions the values of which are the usual free Green functions. We set

$$\widetilde{\mathcal{G}} := \mathcal{G}^+ - \mathcal{G}^-.$$

Theorem 4. *Let $A, B \in \mathcal{O}(M)$, and suppose that the coefficient functions of A, B are "smooth enough". Then the Poisson bracket of the on-shell restrictions of A, B is given by*

$$\{A,B\} = \sum_{i,j=1}^{N^\Phi+N^\Psi} \int_{\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}} dxdy\widetilde{\mathcal{G}}_{ij}[\Xi](x,y)\left(\frac{\delta}{\delta\Xi_i(x)}A[\Xi]\frac{\delta}{\delta\Xi_j(y)}B[\Xi]\right)$$

(with the necessary restrictions onto M^{sol} not being indicated).

This formula is not new, cf. [4], but the exact treatment, with the inclusion of first-order, antimmuting fields, is so. Somewhat symbolically (since the l.h.s. is not well-defined), we can write

$$\{\Xi_i(x), \Xi_j(y)\} = \widetilde{\mathcal{G}}_{ij}[\Xi](x,y).$$

That is, $\widetilde{\mathcal{G}}_{ij}(x,y)$ is an interacting, classical analogon of the Pauli-Jordan exchange function (of course, for a free scalar field, it is just equal to this function).

Symmetries

We call a superfunction $P \in \mathcal{O}^{C_c^\infty(\mathbb{R}^{d+1})}(M)$ *local* iff $\operatorname{supp} P(x) \subseteq \{x\}$ for all $x \in \mathbb{R}^{d+1}$ (effectively, this means that P is given by a space-time dependent differential power series).

We call a vector field $\chi \in \mathcal{X}(M)$ an *infinitesimal Lagrangian symmetry*, or *symmetry*, for shortness, iff

- $\chi(\Xi_i)$ is local, and
- χ transforms the Lagrangian into a total derivative, i. e. there exist local $P_a \in \mathcal{O}^{C_c^\infty(\mathbb{R}^{d+1})}(M)$ such that

$$\chi(L[\Xi]) = \sum_{a=0}^{d} \partial_a P_a[\Xi].$$

(We remark that it would be possible to allow an additional term which lies in the square of the differential ideal of the field equations.)

From the point of view that the classical theory should be determined not by the Lagrangian itself but by its equivalence class modulo total derivatives, this definition is quite logical: χ is a symmetry iff this equivalence class is infinitesimally invariant under χ.

The standard examples for symmmetries are space-time translations and rotations as well as BRST symmetry, supersymmetry, and several rigid interior symmetries. (Unfortunately, models with Yang-Mills symmetry do not fit into our model class unless the Yang-Mills symmetry is broken in order to make the Cauchy problem well-posed.)

Having fixed a symmetry χ, the usual quantum field theory textbook argument shows that the superfunction

$$Q_\chi(t) := \int_{\mathbb{R}^d} dx \left(\sum_{i=1}^{N^\Phi + N^\Psi} \chi(\Xi_i)\Sigma_i - P_0 \right)(t,x) \in \mathcal{O}(M^{\mathrm{sol}}),$$

called the *Noether charge* associated to χ, is independent of the time instant $t \in \mathbb{R}$ (in fact, it is also independent of the particular choice of the P_a, and hence well-defined).

Now if the $\chi(\Xi_i)$ have no explicit time dependence (which is equivalent to commutation of χ with infinitesimal time shift), then it follows that $Q_\chi := Q_\chi(0)$ is invariant under time evolution, i. e. a conserved quantity:

$$Q_\chi = U_t^*(Q_\chi)$$

for all $t \in \mathbb{R}$.

However, this cannot be the whole story. If we want χ to survive in the quantized theory, we have to guarantee that χ restricts to a Hamiltonian vector field on M^{sol} which is generated by Q_χ. Indeed, this is always the case, even if χ does not commute with time shift (a property which is not Poincaré invariant anyway), so that Q_χ is not conserved:

Theorem 5. *Fix a symmetry* χ.

(i) χ *is restrictable to a vector field* $\chi^{\mathrm{sol}} \in \mathcal{X}(M^{\mathrm{sol}})$.

(ii) χ^{sol} *is the Hamiltonian vector field on* M^{sol} *generated by* Q_χ, *i. e.*

$$\delta Q_\chi = \chi^{\mathrm{sol}} \neg \omega$$

where $\neg$ *is the interior derivative. Thus, for any* $F \in \mathcal{O}(M^{\mathrm{sol}})$ *we have*

$$\chi^{\mathrm{sol}}(F) = \{Q_\chi, F\}. \tag{5}$$

In particular, the symplectic structure is invariant under χ, i. e. its Lie derivative vanishes:

$$\mathcal{L}_{\chi^{\mathrm{sol}}}(\omega) = 0.$$

This result once again shows that the symplectic structure is a very intrinsic one.

Note that Q_χ may describe an observable only if χ is *even* (i. e. an ordinary symmetry, not a supersymmetry).

For Poincaré transformations as well as for strictly interior symmetries, the Theorem follows by a rather straightforward compuation. However, if no additional properties of χ are to be used, the proof of assertion (ii) is somewhat delicate.

Now let be given an *Lie superalgebra of infinitesimal symmetries* of the model, i. e. a Lie superalgebra (lsa) $\mathfrak{g}$ together with a homomorphism of it into the lsa of symmetries of the model. Now ass. (iii) of the Theorem yields an lsa homomorphism

$$\mathfrak{g} \to \left(\mathcal{O}^{\mathrm{Ham}}(M^{\mathrm{sol}}), \{\cdot,\cdot\}\right), \tag{6}$$

i. e. an infinitesimal Hamiltonian action of $\mathfrak{g}$ on M^{sol}. If we suppose $\mathfrak{g}$ finite-dimensional then (6) is the pullback of linear superfunctions of a unique smf morphism

$$J : M^{\mathrm{sol}} \to \mathrm{L}(\mathfrak{g}^*)$$

where $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$; it is natural to call J the *moment morphism.* Of course, if the model is bosonic, and $\mathfrak{g}$ is an ordinary Lie algebra then J is the usual moment map.

Now let χ be the generator of infinitesimal time translation. Of course, the corresponding Noether charge

$$\mathcal{H} := \int_{\mathbb{R}^d} dx \left(\sum_{i=1}^{N^\Phi + N^\Psi} \partial_t \Xi_i \, \Sigma_i - L \right)(0,x)$$

is just the *Hamiltonian* of the model. (5) specializes to

$$\partial_t U_t(F)|_{t=0} = \{\mathcal{H}, F\}.$$

Specializing F further to the field strengthes and momenta, we get the *canonical form of the equations of motion*

$$\partial_t \Phi_i(t,x) = \frac{\delta}{\delta \Sigma_i(t,x)} \mathcal{H}(t,x), \quad \partial_t \Sigma_i(t,x) = -\frac{\delta}{\delta \Phi_i(t,x)} \mathcal{H}(t,x)$$

in the second order sector, and

$$\partial_t \Psi_i(t,x) = \frac{1}{2} \sum_j (\Gamma^0)^{-1}_{ij} \frac{\delta}{\delta \Psi_j(t,x)} \mathcal{H}(t,x)$$

in the first order sector.

OUTLOOK

The symplectic smf M^{sol} might be the starting point for a geometric quantization. Of course, while prequantization is no problem (and will be discussed elsewhere), it is a rather tricky question what the infinite-dimensional substitute for the symplectic volume needed for integration should be; we guess that it is some improved variant of Berezin's functional integral (cf. [1]). Note, however, that although the integration domain M^{sol} is isomorphic to the linear supermanifold M^{Cau}, this isomorphism is for a model with interaction highly Lorentz-non-invariant, and Berezin's functional integral makes use of that linear structure.

Also, in the interacting case, the usual problems of quantum field theory, in particular renormalization, will have to show up on this way, too, and the chances to construct a Wightman theory are almost vanishing. Nevertheless, it might be possible to catch some features of the physicist's computational methods (in particular, Feynman diagrams), overcoming the present mathematician's attitude of contempt and disgust to these methods, and giving them a mathematical description of Bourbakistic rigour.

What certainly can be done is a mathematical derivation of the rules which lead to the tree approximation S_{tree} of the scattering operator. S_{tree} should be at least a well-defined power series; of course, the wishful result is that S_{tree} is defined as an automorphism of the solution smf M^{free} of the free theory.

A final comment: The paper containing more precise formulation of the results and complete proofs is yet in preparation and will be published later.

References

[1] Berezin F A: *The Method of Second Quantization.* Pure & Appl. Phys. 24, Academic Press, New York – London 1966

[2] Chernoff P R, Marsden J E: *Properties of Infinite Dimensional Hamiltonian systems.* Lecture Notes in Math. No. 425, Springer-Verlag 1974, Berlin

[3] Crnkovic C, Witten E: *Covariant description of canonical formalism in geometrical theories.* In *Three Hundred Years of Gravitation*, Eds. S. W. Hawking and W. Israel (Cambridge 1987)

[4] DeWitt B: *Dynamical Theory of Groups and Fields.* Gordon and Breach, New York 1965

[5] Reed M, Simon B: *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness.* Academic Press, New York, 1975

[6] Schmitt T: *Infinitedimensional Supermanifolds.*
I. Report 08/88 des Karl-Weierstraß-Instituts für Mathematik, Berlin 1988.
II, III. Mathematica Gottingensis. Schriftenreihe des SFBs Geometrie und Analysis, Heft 33, 34 (1990). Göttingen 1990

[7] Schmitt T: *Functionals of classical fields in quantum field theory.* Reviews in Mathematical Physics, Vol. 7, No. 8 (1995), 1249-1301

[8] Schmitt T: *Supergeometry and Quantum Field Theory, or: What is a Classical Configuration?* Preprint No. 419/1995 der TU Berlin, hep-th/9607132

[9] Schmitt T: *The Cauchy Problem for Classical Field Equations with Ghost and Fermion Fields.* Preprint No. 420/1995 der TU Berlin, hep-th/9607133

[10] Segal I: *Symplectic Structures and the Quantization Problem for Wave Equations.* Symposia Math. 14 (1974), 99-117

Author's address: Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 135, D-10623 Berlin, Federal Republic Germany