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K-CONCIRCULAR VECTOR FIELDS AND HOLOMORPHICALLY PROJECTIVE MAPPINGS ON KÄHLERIAN SPACES

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ABSTRACT. In the paper K-concircular vector fields on Kählerian and hyperbolically Kählerian spaces are studied. Metric tensors of these spaces are found in explicit form. Metrics admitting K-concircular vector fields which are in holomorphically projective correspondence are found.

1. Introduction. S. Yamaguchi [14] investigated Kählerian torsion-forming vector fields which we call further K-concircular vector fields. K.R. Esenov [2], [3] deals with special cases of the above mentioned vector fields which we call further K-concircular vector fields.

This type of vector fields develops K. Yano's concircular vector fields [15] for the theory of Kählerian spaces (we understand by that both classic Kählerian spaces and hyperbolically Kählerian spaces).

In the paper we find metrics of Kählerian spaces in which K-concircular vector fields exist and we investigate holomorphically projective mappings of the spaces.

In this paper the concept of Kählerian spaces means a wider class of spaces in accordance with the following definition.

A (pseudo-)Riemannian space K_n is called a *Kählerian space* if it contains, along with the metric tensor $g_{ij}(x)$, an affine structure $F_i^h(x)$ satisfying the following relations

$$F_\alpha^h F_i^\alpha = e\delta_i^h\,, \quad F_i^\alpha g_{j\alpha} + F_j^\alpha g_{i\alpha} = 0\,, \quad F_{i,j}^h = 0\,. \tag{1}$$

where comma denotes the covariant derivative in K_n, δ_i^h is Kronecker symbol and $e = \pm 1$.

If $e = -1$ then K_n is an (*elliptically*) *Kählerian space* K_n^-, if $e = 1$ then K_n is a *hyperbolically Kählerian space* K_n^+.

The spaces K_n^- were introduced by P.A. Shirokov [13], the spaces K_n^+ by P.A. Rashevsky [11]. In their works these spaces were called *A-spaces*. Independently of P.A. Shirokov the spaces K_n^- were studied by E. Kähler [4]. In the available literature these spaces are mostly called *Kählerian*.

A vector field λ^h in K_n is called *Kählerian torso-forming* if the following condition

$$\lambda^h_{,i} = a\,\delta_i^h + b\,F_i^h + \varphi_i\,\lambda^h + e\,\varphi_\alpha F_i^\alpha\,\lambda^\beta F_\beta^h, \tag{2}$$

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holds, where a, b are functions, φ_i is a covector (for K_n^- see [14]).

If the covector λ_i ($\equiv \lambda^\alpha g_{\alpha i}$) is a gradient, then for $n > 4$ condition **(2)** can be written in the form

$$\lambda_{i,j} = a\, g_{ij} + c\left(\lambda_i \lambda_j - e\overline{\lambda}_i \overline{\lambda}_j\right), \tag{3}$$

where $\overline{\lambda}_i \equiv \lambda_\alpha F_i^\alpha$, c is a function. These vector fields λ_i we called *K-concircular.*

In [2] formula **(3)** is proved for λ^h being gradient and nonisotropic. If $a \neq 0$ then λ^h is nonisotropic. When we investigate the conditions of integrability of **(3)** we can learn that a and c are functions of parameter λ which generates the gradient $\lambda_i = \partial_i \lambda$, $\partial_i \equiv \partial / \partial x^i$.

Metrics of all Kählerian spaces which admit covariantly nonconstant convergent vector fields, that is K_n, in which a vector λ_i satisfying $\lambda_{i,j} = a\, g_{ij} \neq 0$ (a – const) exists, were shown [6], [7], [9]. These spaces admit nonaffine geodesic and nonaffine holomorphically projective mapping.

2. Kählerian spaces with K-concircular vector fields.

Theorem 1. *Let a Riemannian space have a metric defined by the relations*

$$g_{ab} = g_{a+m\, b+m} = \partial_{ab} G + \partial_{a+m\, b+m} G; \quad g_{a\, b+m} = \partial_{a\, b+m} G - \partial_{a+m\, b} G, \tag{4}$$

where $G = G\,(x^1 + s(x^2, x^3, \dots, x^m, x^{m+2}, x^{m+3}, \dots, x^n))$; $G' \cdot G'' \neq 0$, $G, s \in C^3$ *are functions of the given arguments,* $a, b = 1, 2, \dots, m$; $m = n/2$, $|g_{ij}| \neq 0$.

Then this space is the Kählerian space K_n^- *which admits a* K*-concircular vector fields.*

Proof. In coordinates x, in which conditions **(4)** are valid, we define the affinor $F_i^h(x)$:

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a, \quad F_b^a = F_{b+m}^{a+m} = 0. \tag{5}$$

From **(1)** we get directly that $F_i^h(x)$ is the structure affinor K_n^- and that the vector $\lambda^h = \delta_1^h$ satisfies condition **(3)**, where

$$a = \frac{1}{2}\left(\ln G'\right)', \qquad c = \frac{1}{2}(\ln a)'/G''. \tag{6}$$

It is obvious that always $a \neq 0$.

Theorem 2. *Suppose a Kählerian spaces* K_n^- $(n > 4)$ *admitting* K*-concircular vector field for* $a \neq 0$. *Then in* K_n^- *a coordinate system exists such that its metric has the given form* **(4)**.

Proof. Since K-concircular vector field λ^h in K_n^- is analytic, i.e. the condition $\lambda^\alpha_{,\beta} F_\alpha^h F_i^\beta = \lambda^h_{,i}$ holds, then on the basis of [6], [7] an adapt coordinate system x, in which the structure F_i^h is of the form **(5)**, exists in K_n^- and $\lambda^h = \delta_1^h$. Then by an analysis of formulas **(1)** and **(3)** we get that the metric tensor K_n^- is of the form **(4)**.

Theorem 3. *Let a Riemannian space have a metric defined by the relations*

$$g_{a\, b+m} = \partial_{a\, b+m} G; \quad g_{ab} = g_{a+m\, b+m} = 0, \tag{7}$$

where $G = G\,(x^1 + x^{1+m} + s(x^2 + x^{2+m}, \dots, x^m + x^n))$, $G' \cdot G'' \neq 0$, $G, s \in C^3$ *are function of the given arguments,* $a, b = 1, 2, \dots, m$; $m = n/2$, $|g_{ij}| \neq 0$.

Then this spaces is the hyperbolically Kählerian space K_n^+ which admits a K-concircular vector field.

Proof. In the coordinates x, in which condition **(7)** holds, we define the affinor $F_i^h(x)$:

$$F_b^a = -F_{b+m}^{a+m} = \delta_b^a; \quad F_b^{a+m} = F_{b+m}^a = 0. \tag{8}$$

Analogically from **(1)** we get directly that $F_i^h(x)$ is a structure affinor of the hyperbolically Kählerian space K_n^+ and the vector $\lambda^h = \delta_1^h + \delta_{1+m}^h$ satisfies condition **(3)**, where functions a and c are given by **(6)**.

3. Holomorphically projective mappings of Kählerian spaces with K-concircular vector fields. An *analytically planar curve* of the Kählerian space K_n is a curve, defined by the equations $x^h = x^h(t)$, whose tangent vector $\lambda^h = dx^h/dt$, being parallely transfered, remains in the plane formed by the tangent vector λ^h and its conjugate $\overline{\lambda}^h \equiv \lambda^\alpha F_\alpha^h$, i.e., the condition

$$\nabla_t \lambda^h \equiv d\lambda^h/dt + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \rho_1(t)\lambda^h + \rho_2(t)\overline{\lambda}^h,$$

where ρ_1, ρ_2 are functions of the argument t, Γ_{ij}^h is the Christoffel symbols of K_n, fulfilled [10], [12].

The diffeomorphism of K_n onto $\overline{K}_n$ is a holomorphically projective mapping (*HPM*) if it transforms all analytically planar curves of K_n into anlytically planar curves of $\overline{K}_n$.

Under HPM the structure of the spaces K_n and $\overline{K}_n$ is preserved, i.e., in the coordinate system x, generally with respect to the mapping, the conditions $\overline{F}_i^h(x) \equiv F_i^h(x)$ are satisfied. To be more precise $\overline{F}_i^h(x) = \pm F_i^h(x)$ for K_n.

The necessary and sufficient conditions for the holomorphically projective mappings of K_n onto $\overline{K}_n$ are the fulfillment of the following condition in a common coordinate system with respect to the mapping:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}\delta_{i)}^h - \overline{\psi}_{(i}F_{i)}^h$$

where $\overline{\Gamma}_{ij}^h$ is Christoffel symbol of $\overline{K}_n$, $(i\,j)$ denotes a symmetrization without division, ψ_i is the covariant vector and $\overline{\psi}_i \equiv \psi_\alpha F_i^\alpha$. This relations are equivalent to the equation (see [16], [12], [10]):

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k} - e\,\overline{\psi}_{(i}\overline{F}_{j)k}, \tag{9}$$

where $\overline{F}_{ij} \equiv \overline{g}_{i\alpha}F_j^\alpha$, $\overline{g}_{ij}$ is the metric tensor of $\overline{K}_n$.

V.V. Domashev and J. Mikeš found for K_n^- [1], [12] and I.N. Kurbatova for K_n^+ [5] that the Kählerian space K_n admits of a nontrivial holomorphically projective mapping if only if the system of equations

$$a_{ij,k} = \xi_{(i}g_{j)k} - e\,\overline{\xi}_{(i}F_{j)k}\,, \tag{10}$$

has a nontrivial solution for the unknown tensors a_{ij} $(= a_{ji} = -e\,a_{\alpha\beta}F_i^\alpha F_j^\beta,\ |a_{ij}| \neq 0)$ and $\xi_i \neq 0$, where $F_{jk} \equiv g_{j\alpha}F_k^\alpha$, $\overline{\xi}_i \equiv \xi_\alpha F_i^\alpha$. The solutions of **(9)** and **(10)** are connected by the relations

$$a_{ij} = \exp(2\psi)\,\overline{g}^{\alpha\beta}g_{\alpha i}\,g_{\beta j}\,, \quad \xi_i = -\exp(2\psi)\,\overline{g}^{\alpha\beta}g_{\alpha i}\,\psi_\beta\,, \tag{11}$$

where ψ is a function generated by the gradient $\psi_i = \psi_{,i}$, $\|\overline{g}^{ij}\| = \|\overline{g}_{ij}\|^{-1}$.

Let K_n be the Kählerian space shown in the Theorem 1 and Theorem 3. In these spaces K-concircular vector field λ^h exists, which satisfies **(3)**, where $a \neq 0$.

Let

$$a_{ij} \equiv \alpha\, g_{ij} - \frac{\beta}{a}\left(\lambda_i\lambda_j - e\overline{\lambda}_i\overline{\lambda}_j\right), \tag{12}$$

where α, β are nonzero constants such that $\det\|a_{ij}\| \neq 0$.

The constructed tensor a_{ij} satisfies the fundamental equations **(10)** from the theory of holomorphically projective mappings.

From here we get

Theorem 4. *The Kählerian space K_n with K-concircular vector field λ^h (where $a \neq 0$) admits nontrivial holomorphically projective mapping.*

For holomorphically projective mapping K_n with K-concircular vector field maps itself into $\overline{K}_n$ with K-concircular vector field as well [2].

We will find metrics of two Kählerian spaces K_n and $\overline{K}_n$ with K-concircular vector fields, such that holomorphically projective mapping exists between them. By an analysis of **(11)** and **(12)** we can see that the metric tensor $\overline{g}_{ij}$ is of the form

$$\overline{g}_{ij} = \frac{1}{\alpha}\exp(2\psi)\left\{g_{ij} - \frac{\beta}{a + \beta\lambda_\alpha\lambda^\alpha}\left(\lambda_i\lambda_j - e\overline{\lambda}_i\overline{\lambda}_j\right)\right\}. \tag{13}$$

By the covariant differentiation of **(13)** we get according to **(3)** and **(9)** that

$$\partial_i\psi \equiv \psi_i = \frac{-\beta a}{a + \beta\,\lambda_\alpha\lambda^\alpha}\,\lambda_i\,. \tag{14}$$

In the corresponding coordinates **(3)** or **(7)** we integrate equations **(14)** and find the explicit form of the following objects: λ^h, a, ψ, λ_i, $\overline{\lambda}_i$, $\lambda^\alpha\lambda_\alpha$.

On the basis of Theorem 1

$$\lambda^h = \delta^h_1, \quad a = \frac{1}{2}(\ln G')', \quad \lambda_i = G''\tau_i; \quad \lambda_\alpha\lambda^\alpha = G''(\tau);$$

$$\overline{\lambda}_a = G''\tau_{a+m}, \quad \overline{\lambda}_{a+m} = -G''\tau_a, \quad \psi = -\frac{1}{2}\ln|1 + 2\beta G'| + \psi_0,$$

hold in K_n^-, where $G = G(\tau)$, $\tau = x^1 + s(x^2, \dots, x^m, x^{m+2}, \dots, x^n)$, $\tau_i \equiv \partial_i\tau$, ψ_0 is constant, $a, b = \overline{1, m}$, $m = n/2$.

Analogically on the basis of Theorem 3

$$\lambda^h = \delta^h_1 + \delta^h_{1+m}, \quad a = \frac{1}{2}\left(\ln G'\right)', \quad \lambda_i = G''\tau_i, \quad \lambda_\alpha\lambda^\alpha = 2G''(\tau),$$

$$\overline{\lambda}_a = G''\tau_a, \quad \overline{\lambda}_{a+m} = -G''\tau_{a+m}, \quad \psi = -\frac{1}{4}\ln|1 + 4\beta G'| + \psi_0,$$

hold in K_n^+, where $G = G(\tau)$, $\tau = x^1 + x^{1+m} + s(x^2 + x^{m+2}, \dots, x^m + x^n)$, $\tau_i \equiv \partial_i\tau$, ψ_0 is constant, $a, b = \overline{1, m}$, $m = n/2$.

4. Global aspects of the existence of K-concircular vector fields. Now we will study the existence of K-concircular vector fields on the compact Kählerian space K_n without a boundary. We suppose that a function $\lambda \in C^2$ is defined globally on K_n and determines the gradient K-concircular vector fields.

Theorem 5. *Compact Kählerian spaces K_n with nondefined signature of metrics do not admit K-concircular vector field with $a \neq 0$ (Remark: K_n^+ has always a nondefined signature).*

Proof. For any point $x_0 \in K_n$ a coordinate neighbourhood U_{x_0} can be find such that a positively defined form $A^{\alpha\beta}(x)y_\alpha y_\beta$, $A^{\alpha\beta}(x) \in C^0(U_{x_0})$, exists in it such that $g_{\alpha\beta}(x)A^{\alpha\beta}(x) = 0$.

After contracting **(3)** with A^{ij} we get

$$A^{\alpha\beta}\lambda_{,\alpha\beta} + B^\alpha\lambda_{,\alpha} = 0,$$

where $B^\alpha \in C^0(U_{x_0})$ are components which can depend on λ.

These formulas hold in all U_{x_0}, that is why only trivial solution $\lambda \equiv$const of **(3)** exists according to a modification of Hopf theorem [8]. It is a contradiction to $a \neq 0$.

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