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EQUIVARIANT COHOMOLOGY OF THE SKYRMION BUNDLE

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1. THE UNGAUGED SKYRME MODEL

The SKYRME model[7] in theoretical nuclear physics is an effective field theory related to quantum chromodynamics (QCD) by its underlying symmetry. It models the low energy limit of QCD and covers interactions between mesons and baryons, but not interactions between these particles and electromagnetic fields. For this reason, it is also called *ungauged* SKYRME model. In this ungauged SKYRME model, the meson fields π^a on space-time M generate differentiable functions $U: M \to \mathrm{SU}_{N_F}$ defined by ($N_F = 2, 3, \ldots, 6$ denotes the number of flavors in QCD)

$$U = \exp(i \sum_{a=1}^{N_F^2-1} \pi^a \lambda_a) \qquad \text{with} \quad \lambda_a = (\lambda_a)^\dagger \in \mathbb{C}^{N_F \times N_F},\ \mathrm{Tr}(\lambda_a) = 0.$$

The vacuum is represented by the unit matrix $\mathbb{1} \in \mathrm{SU}_{N_F}$. Requiring $\pi^a(r) \to 0$ and thus $U(r) \to \mathbb{1}$ for $r \to \infty$ one can compactify euclidian space $\mathbb{R}^3$, resp., space-time $\mathbb{R}^4$, so that the meson fields constitute functions $U: \mathbb{R}_{(t)} \times \mathbb{S}^3 \to \mathrm{SU}_{N_F}$, resp., $U: \mathbb{S}^4 \to \mathrm{SU}_{N_F}$, cf. A. SCHMITT's article in this volume,[8] where $N_F = 2$.

The cohomology of the unitary groups plays an important role for the ungauged SKYRME model. For convenience, we need some basic definitions: Let $L := U^{-1}\, dU = U^\dagger\, dU$ and $R := (dU)U^{-1} = (dU)U^\dagger \in \mathcal{A}_1(\mathrm{U}_m, \mathbb{C}^{m\times m})$ denote the left, resp., right invariant currents: $\mathbb{C}^{m\times m}$-valued 1-forms that are invariant under multiplication with constant elements of U_m from the left, resp., from the right and obey $L(\mathcal{X})(\mathbb{1}) = R(\mathcal{X})(\mathbb{1}) = X$ for all vector fields $\mathcal{X} \in \mathcal{D}^1(\mathrm{U}_m)$ with $\mathcal{X}(\mathbb{1}) = X \in \mathfrak{u}_m = L(\mathrm{U}_m)$. For any constant $Q \in \mathbb{C}^{m\times m}$, we define λ_k^Q and $\rho_k^Q \in \mathcal{A}_k(\mathrm{U}_m, \mathbb{C})$ by

$$\lambda_k^Q := \mathrm{Tr}(QL^k) := \mathrm{Tr}(Q\underbrace{L \wedge \cdots \wedge L}_{k}), \qquad \rho_k^Q := \mathrm{Tr}(QR^k) := \mathrm{Tr}(Q\underbrace{R \wedge \cdots \wedge R}_{k}).$$

These are left, resp., right invariant complex-valued k-forms on U_m; for $Q = \mathbb{1}$ we have

$$\omega_k := \lambda_k^{\mathbb{1}} = \rho_k^{\mathbb{1}} = \mathrm{Tr}(L^k) = \mathrm{Tr}(R^k) \in \mathcal{A}_k(\mathrm{U}_m, \mathbb{C}),$$

which are invariant under all multiplications. Obviously $\omega_{2l} = 0$. The forms ω_{2l+1} are closed since the MAURER-CARTAN identities $dL = -L \wedge L$, $dR = R \wedge R$ yield

$$\begin{aligned} dL^{2l+1} &= -L^{2l+2}, \quad dR^{2l+1} = R^{2l+2}, \qquad dL^{2l+2} = dR^{2l+2} = 0, && (1)\\ d(UL^{2l}) &= UL^{2l+1}, \quad d(L^{2l}U^\dagger) = -L^{2l+1}U^\dagger, \quad d(UL^{2l+1}) = d(L^{2l+1}U^\dagger) = 0. && (2) \end{aligned}$$

Moreover, ω_{2l+1} generate the DE-RHAM cohomology $H^*(\mathrm{SU}_m, \mathbb{C})$, resp., $H^*(\mathrm{U}_m, \mathbb{C})$ as an exterior algebra.

In the SKYRME model, baryons appear as topological soliton solutions of the meson fields U. Their number B can be computed by an integration of the pullback $U^*\omega_3$ over the space manifold:[5, 8] $B(U) = \int_{\mathbb{S}^3} -\frac{1}{24\pi^2} U^*\omega_3$. In a similar fashion, ω_5 is connected with the so-called WESS-ZUMINO term in the lagrangian, which describes the anomalous processes of QCD.[5, 9] Note that $\omega_5 = 0$ on SU_2, which is why the WESS-ZUMINO term only appears if $N_F > 2$.

2. COUPLING OF THE ELECTROMAGNETIC FIELD

As was said before, the ungauged SKYRME model only treats interactions between baryons and mesons, but not with electromagnetic fields. Now it is well known that the latter can conveniently be described by a MAXWELL connection Γ on a principal fiber bundle $P(M, G)$ over space-time M, where the group of the bundle is the electromagnetic (gauge) group $G \cong \mathrm{U}_1 \cong \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We will write G additively and — for compatibility with the results in literature — extract the electromagnetic charge e such that $eG = \mathbb{R}/\mathbb{Z}$, resp., $e\mathfrak{g} = \mathbb{R}$, where $\mathfrak{g}$ denotes the LIE algebra of G. Normally P is a trivial bundle, except in the case, when magnetic monopoles are present. Their locations must be excluded from M.

For any fiber bundle $B(M, F, G)$ with bundle manifold B, base manifold M, fiber F and LIE group G, let $\pi: B \to M$ denote the projection onto the base and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ a bundle atlas, where $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M and $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F: b \mapsto (\pi(b), \pi_\alpha(b))$ define local projections $\pi_\alpha: \pi^{-1}(U_\alpha) \to F$ onto the fiber. For the left effective LIE group action $L: G \times F \to F$, we write $L(g, f) = L_g(f) = \tau_f(g)$, where $L_g: F \to F$ and $\tau_f: G \to F$ are differentiable for all $g \in G$ and $f \in F$. It is common to write $B(M, F, G) = P \times_G F$ for a bundle B associated with a principal bundle P via any left action L. If R denotes the free right action of G on P such that the base M is the orbit space then we have a free right action $\tilde{R}$ on $P \times F$ given by $\tilde{R}_g(p, f) := (R(p, g), L(g^{-1}, f)$ such that $P \times_G F$ is the orbit space and $P \times F$ is a principal bundle $(P \times F)(P \times_G F, G)$.

Recall that a connection Γ on a principal bundle is uniquely defined by its connection 1-form $\omega^\Gamma \in \mathcal{A}_1(P, \mathfrak{g})$. If G is abelian, the curvature 2-form $\Omega^\Gamma \in \mathcal{A}_2(P, \mathfrak{g})$ simply reads $\Omega^\Gamma = d\omega^\Gamma$. Let $\sigma_{\alpha,0}: U_\alpha \to \pi^{-1}(U_\alpha)$ denote the local sections given by $\sigma_{\alpha,0}(x) := \psi_\alpha^{-1}(x, 0)$, where 0 is the neutral element in G. Then the (local) gauge potentials A^α and gauge fields F^α are given by

$$\mathrm{A}^\alpha = \sigma_{\alpha,0}^\star(\omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_1(U_\alpha, \mathfrak{g}), \qquad \mathrm{F}^\alpha = \sigma_{\alpha,0}^\star(\Omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_2(U_\alpha, \mathfrak{g}). \quad (3)$$

In contrast to the gauge potentials, the F^α constitute a *global* gauge field $\mathrm{F} \in \mathcal{A}_2(M, \mathfrak{g})$. Thus for the electromagnetic group G, the form $\tilde{\mathrm{F}} = e\,\mathrm{F} \in \mathcal{A}_2(M)$ is a closed real-valued 2-form on M; in fact, it is a representative for the first CHERN class of P. If P is trivial then also the local forms $\tilde{\mathrm{A}}^\alpha = e\,\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha)$ constitute a global $\tilde{\mathrm{A}}$ with $\tilde{\mathrm{F}} = d\tilde{\mathrm{A}}$ and thus the CHERN class is zero.

Every connection Γ on $P(M,G)$ induces horizontal and vertical projections h, v of differential forms not only on P but on every associated bundle $B(M,F,G) = P \times_G F$:

$$h, v \colon \mathcal{A}(B,V) \to \mathcal{A}(B,V), \qquad \omega \mapsto \omega h, \omega v,$$

for any finite dimensional vector space V such as $V = \mathbb{R}, \mathbb{C}, \mathfrak{g}$, etc.

Now every V-valued form $\chi \in \mathcal{A}(F,V)$ can locally be extended to a form $\chi^\alpha := \pi_\alpha^\star \chi \in \mathcal{A}(\pi^{-1}(U_\alpha), V)$. Normally these forms do not constitute a global form on B. Yet, as was shown previously,[5] if χ is invariant, i. e., $L_g^\star \chi = \chi$ for all $g \in G$, then the vertical projections $\chi^\alpha v$ constitute a global form "χv"$\in \mathcal{A}(B,V)$. So vertical projection $v \colon \mathcal{A}(B,V) \to \mathcal{A}(B,V)$ on the bundle not only maps global forms to global forms but also these *locally* embedded G-invariant forms on the fiber to *global* vertical forms on the bundle, i. e., we have a well-defined map $v \colon \mathcal{A}(F,V)_{\text{inv}} \to \mathcal{A}(B,V)$.

In order to compute $\chi^\alpha v$ in our case, take $E \in \mathfrak{g}$ with $eE = 1$ and let $\mathcal{E} \in \mathcal{D}^1(F)$ denote the vector field on F induced by E via $\mathcal{E}(f) := [d\tau_f]_0(E) \in T_f(F)$. Also let $\imath$ denote the inner product between a vector field and a differential form, i. e., if $\chi \in \mathcal{A}_n(F,V)$ and $\mathcal{F}, \mathcal{F}^{(i)} \in \mathcal{D}^1(F)$, then $\imath_\mathcal{F}\chi \in \mathcal{A}_{n-1}(F,V)$ is given by

$$(\imath_\mathcal{F}\chi)(\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(n-1)}) := n \cdot \chi(\mathcal{F}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(n-1)}).$$

If for any G-invariant $\chi \in \mathcal{A}_n(F,V)$, we define $\nu \in \mathcal{A}_{n-1}(F,V)$ by $\nu := \imath_\mathcal{E}\chi$, then also ν is G-invariant and we obtain

$$\chi^\alpha v = \chi^\alpha + \tilde{\mathrm{A}}^\alpha \wedge \nu. \tag{4}$$

Since $(\imath_\mathcal{E})^2 = 0$, we have $\nu^\alpha v = \nu^\alpha$.

Recall that the LIE derivation $\mathrm{L}_\mathcal{X}$ of forms with respect to a vector field $\mathcal{X}$ is given by the homotopy identity $\mathrm{L}_\mathcal{X} = d \circ \imath_\mathcal{X} + \imath_\mathcal{X} \circ d$. Every G-invariant χ is also $\mathfrak{g}$-invariant, i. e., $\mathrm{L}_\mathcal{E}\chi = 0$. In fact, since G is connected, both statements are equivalent. This yields:

Lemma 2.1 *If $\chi \in \mathcal{A}(F)_{\text{inv}}$ is closed, then also $\nu = \imath_\mathcal{E}\chi \in \mathcal{A}(F)_{\text{inv}}$ is closed.*

Proof. $d\nu = d\imath_\mathcal{E}\chi = \mathrm{L}_\mathcal{E}\chi - \imath_\mathcal{E} d\chi = 0.$ □

For later purposes, we also note that invariant forms ω on a G-manifold F, that obey $\imath_\mathcal{X}\omega = 0$, are called *basic* forms, and that the exterior algebra of these forms, $\mathcal{A}(F)_{\text{bas}}$, is closed under d. For a principal bundle $P(M,G)$ with free right action R, the projection $\pi \colon P \to M$ induces an isomorphism $\pi^\star \colon \mathcal{A}(M) \to \mathcal{A}(P)_{\text{bas}}$.

3. THE SKYRMION BUNDLE

For the purpose of treating interactions between electromagnetic fields on the one hand and mesons, resp., baryons on the other hand, we now gauge the SKYRME model and introduce the skyrmion bundle as follows:[4] instead of considering maps $U \colon M \to \mathrm{SU}_{N_F}$ we now think of the meson fields as of global sections in a bundle $B(M, \mathrm{SU}_{N_F}, G) = P(M,G) \times_G \mathrm{SU}_{N_F}$. The left action of G on SU_{N_F} is given by the inner automorphisms

$$L_g(U) \;=\; \tau_U(g) \;=\; e^{-iegQ} U e^{+iegQ},$$

where Q is the hermitian $n \times n$-matrix containing the quark charges in units of e; for $N_F = 2$, resp., 3, we have

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}, \qquad \text{resp.,} \qquad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

Note that while P admits global sections only if P is trivial, the skyrmion bundle always has global sections, e. g., the vacuum $U \equiv \mathbb{1}$. Also note that all ω_{2l+1}, ρ_l^Q and λ_l^Q are G-invariant and that $\mathcal{E}(U) = [d\tau_U]_0(E) = -ie\,E\,[Q, U]$ for all $U \in \mathrm{SU}_{N_F}$.

For calculations within the skyrmion bundle, we need an analogue of the topological charge and the WESS-ZUMINO term. This requires generalizations of the invariant forms $\omega_{2l+1} \in \mathcal{A}_{2l+1}(\mathrm{SU}_{N_F}, \mathbb{C})$: especially ω_3 and ω_5 have to be extended to the bundle in order to get *closed* forms ω_3^B, resp., ω_5^B on B, whose pullbacks by the mesonic sections $U\colon M \to B$ can be integrated over space-time, resp., the space manifold only. The topological charge $B^A(U)$ for the skyrmion bundle is then defined analogously. From the mathematical point of view, we need closed differential forms $\omega_{2l+1}^B \in \mathcal{A}_{2l+1}(B, \mathbb{C})$ such that the restriction to the fibers reproduces ω_{2l+1}, i. e.,

$$(i_{\alpha,x})^\star \omega_{2l+1}^B = \omega_{2l+1}, \tag{5}$$

where $i_{\alpha,x} = (\pi_{\alpha,x})^{-1}\colon \mathrm{SU}_{N_F} \to \pi^{-1}(x) \subset B$ denote the local injections for $x \in U_\alpha$. Note that $\omega_3 v$ and $\omega_5 v$, although they obey (5), are not appropriate since they are not closed.

If P is trivial and hence $B \cong M \times F$, these forms can, of course, always be found: simply take $(\mathrm{pr}_F)^\star \omega_{2l+1}$, where $\mathrm{pr}_F\colon B \to F$ denotes the global projection onto the fiber that exists (only) for trivial bundles. Yet this does not solve the problem for the most popular application of the skyrmion bundle: the description of proton decay catalyzed by magnetic monopoles.[2, 3] Thus we are looking for ω_{2l+1}^B that exist independently of P, i. e., independently of the number and distribution of monopole singularities excluded from M and of the transition functions.

The correct setting for this problem is the so-called *equivariant cohomology* of the G-manifold SU_{N_F}. Before discussing this cohomology in the next section, let us recall the previously[4, 5] obtained result for ω_3 and ω_5. We have G-invariant differential forms $\chi_{2l-2i+1}^i \in \mathcal{A}_{2l-2i+1}(\mathrm{SU}_{N_F}, \mathbb{C})$ such that for every skyrmion bundle the forms ω_3^B and ω_5^B may be defined by

$$\begin{aligned} \omega_3^B &:= \omega_3 v + \tilde{\mathrm{F}} \wedge \chi_1^1 v \;:=\; \omega_3 v + \tilde{\mathrm{F}} \wedge 3i(\rho_1^Q + \lambda_1^Q) v, & (6)\\ \omega_5^B &:= \omega_5 v + \tilde{\mathrm{F}} \wedge \chi_3^1 v + \tilde{\mathrm{F}} \wedge \tilde{\mathrm{F}} \wedge \chi_1^2 v \;:=\; \omega_5 v + \tilde{\mathrm{F}} \wedge 5i(\rho_3^Q + \lambda_3^Q) v \\ &\quad + \tilde{\mathrm{F}} \wedge \tilde{\mathrm{F}} \wedge -5[2(\rho_1^{Q^2} + \lambda_1^{Q^2}) + \mathrm{Tr}(Q\, dU\, QU^{-1} - QUQ\, dU^{-1})] v. & (7) \end{aligned}$$

ω_3^B and ω_5^B are closed and adapted to the given MAXWELL connection Γ — in the sense that their horizontal parts are merely exterior powers of $\tilde{\mathrm{F}}$. They generate cohomology groups $H^3(B, \mathbb{C}) \cong \mathbb{C}$, resp., $H^5(B, \mathbb{C}) \cong \mathbb{C}$ for $N_F > 2$ for all skyrmion bundles B, independently of P.

4. EQUIVARIANT DE RHAM COHOMOLOGY FOR $\mathbb{S}^1$-MANIFOLDS

The WEIL algebra $\mathrm{W}(\mathfrak{g})$ is by definition the tensor product of the exterior algebra and the symmetric algebra on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$:

$$\mathrm{W}(\mathfrak{g}) := \bigwedge \mathfrak{g}^* \otimes \mathrm{S}(\mathfrak{g}^*).$$

It is a graded algebra, where $\tilde{\omega} = E^* \in \mathfrak{g}^*$ is given degree 1 in the exterior algebra, but degree 2 in the symmetric algebra, where this element will be denoted by $\tilde{\Omega}$. $\mathrm{W}(\mathfrak{g})$ is freely generated as a commutative graded algebra — i. e., $\alpha_p \cdot \beta_q = (-1)^{pq}\beta_q \cdot \alpha_p$ for elements of degree p, resp., q — by the generators $\tilde{\omega}$ and $\tilde{\Omega}$. In our case, since $\mathfrak{g}^*$ is one-dimensional, $\bigwedge \mathfrak{g}^* = \mathbb{R} \oplus \mathfrak{g}^*$ and thus

$$\mathrm{W}(\mathfrak{g}) = \mathrm{S}(\mathfrak{g}^*) \oplus \mathfrak{g}^* \otimes \mathrm{S}(\mathfrak{g}^*).$$

We also have a right G-action on $\mathrm{W}(\mathfrak{g})$ which is induced by the co-adjoint action on $\mathfrak{g}^*$. Yet here this is simply the trivial action.

The WEIL algebra is equipped with a differential operator D and an inner product $\imath$, which on the generators are defined by

$$\begin{aligned} D\tilde{\omega} &= \tilde{\Omega}, \quad D\tilde{\Omega} = 0; \\ \imath_X\tilde{\omega} &= x, \quad \imath_X\tilde{\Omega} = 0 \quad \text{for all} \quad X = xE \in \mathfrak{g}. \end{aligned}$$

Obviously, $D^2 = 0$ and $(\imath_X)^2 = 0$ for every $X \in \mathfrak{g}$.

Note the relationship with connection 1-forms and curvature 2-forms. Indeed, the *universal* connection and curvature over the WEIL algebra are given by $\omega := \tilde{\omega}E \in \mathrm{W}_1(\mathfrak{g}) \otimes \mathfrak{g}$ and $\Omega := \tilde{\Omega}E \in \mathrm{W}_2(\mathfrak{g}) \otimes \mathfrak{g}$, independently of the choice of basis. Now every connection Γ on a principal bundle determines maps $\mathfrak{g}^* \to \mathcal{A}_1(P)$ and $\mathfrak{g}^* \to \mathcal{A}_2(P)$ via its connection 1-form and curvature 2-form. These maps, in turn, induce the so-called WEIL homomorphism of the graded algebra $\mathrm{W}(\mathfrak{g})$ into $\mathcal{A}(P)$, which carries the universal connection and curvature over $\mathrm{W}(\mathfrak{g})$ to the given connection and curvature forms on P and commutes with the differentials and the inner derivations $\imath_X$.[6]

Analogously to the LIE derivative of forms, an operator L is introduced on the WEIL algebra by $\mathrm{L}_X = \imath_X \circ D + D \circ \imath_X$ for $X \in \mathfrak{g}$. Yet in our case, one quickly verifies that $\mathrm{L}_X = 0$ on $\mathrm{W}(\mathfrak{g})$, and thus the basic subcomplex of $\mathrm{W}(\mathfrak{g})$ is $\mathrm{S}(\mathfrak{g}^*)$, i. e., the ring of polynomials on $\mathfrak{g}$. Also, since $D(\mathrm{S}(\mathfrak{g}^*)) = \{0\}$, we obtain

$$H_D^*(\mathrm{W}(\mathfrak{g})_{\mathrm{bas}}) = \mathrm{W}(\mathfrak{g})_{\mathrm{bas}} = \mathrm{S}(\mathfrak{g}^*).$$

Now for any G-manifold F, the WEIL algebra is being tensored with $\mathcal{A}(F)$. Then the differentials, G-actions and inner products on $\mathrm{W}(\mathfrak{g})$ and $\mathcal{A}(F)$ carry over to this tensor product, which becomes a differential graded algebra with right G-action (trivial on the first factor), derivations $\imath_X$ and operators L_X. Note that we let G act from the right, whereas on F, it was supposed to act from the left. This requires a minus sign for the inner derivations on $\mathcal{A}(F)$: e. g., $\imath_E\chi = -\imath_{\mathcal{E}}\chi$ for $\chi \in \mathcal{A}(F)$, where, as before, $\mathcal{E} \in \mathcal{D}^1(F)$ denotes the vector field induced by E, and thus $\imath_E(\tilde{\omega} \otimes \chi) = \chi + \tilde{\omega} \otimes \imath_{\mathcal{E}}\chi$.

Now we can form the basic subalgebra of this complex $\mathrm{W}(\mathfrak{g}) \otimes \mathcal{A}(F)$:

$$\mathcal{A}(F)^G := [\mathrm{W}(\mathfrak{g}) \otimes \mathcal{A}(F)]_{\mathrm{bas}} = [\mathrm{S}(\mathfrak{g}^*) \otimes \mathcal{A}(F) \oplus \mathfrak{g}^* \otimes \mathrm{S}(\mathfrak{g}^*) \otimes \mathcal{A}(F)]_{\mathrm{bas}}.$$

Its elements are called *equivariant differential forms* on F and its cohomology $H^*(F)^G$ is named the *equivariant cohomology* of the G-manifold F. An element in $\mathcal{A}(F)^G$ is given by $\chi = \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes (\chi^i + \tilde{\omega} \otimes \nu^i)$ with invariant forms $\chi^i, \nu^i \in \mathcal{A}(F)_{\mathrm{inv}}$ (note that $\tilde{\Omega}^i$ means the i-th power of $\tilde{\Omega}$, while the i in χ^i and ν^i is just an index and not their degree), such that

$$\begin{aligned} 0 = \imath_E \chi &= \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes (-\imath_{\mathcal{E}} \chi^i + \nu^i + \tilde{\omega} \otimes \imath_{\mathcal{E}} \nu^i) \\ &= \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes (-\imath_{\mathcal{E}} \chi^i + \nu^i) + \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes \tilde{\omega} \otimes \imath_{\mathcal{E}} \nu^i. \end{aligned}$$

Thus $\nu^i = \imath_{\mathcal{E}} \chi^i$, from which $\imath_{\mathcal{E}} \nu^i = 0$ follows immediately. Hence

$$\mathcal{A}(F)^G = \mathrm{S}(\mathfrak{g}^*) \otimes j_0(\mathcal{A}(F)_{\mathrm{inv}}),$$

with injective algebra morphism $j_0 \colon \mathcal{A}(F)_{\mathrm{inv}} \to \mathcal{A}(F)^G$ defined by $j_0(\chi) := \chi + \tilde{\omega} \otimes \imath_{\mathcal{E}} \chi$. Note that under the WEIL homomorphism $W \colon \mathrm{W}(\mathfrak{g}) \otimes \mathcal{A}(F) \to \mathcal{A}(P \times F)$, the basic form $W(j_0(\chi))$ corresponds to χv given by (4).

Analogously, we compute $D\chi$ for $\chi = \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes (\chi^i + \tilde{\omega} \otimes \nu^i)$:

$$D\chi = d\chi^0 + \sum_{i=1}^{\infty} \tilde{\Omega}^i \otimes (d\chi^i + \nu^{i-1}) - \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes \tilde{\omega} \otimes d\nu^i. \tag{8}$$

Lemma 4.1 *Let $j \in \mathbb{N}_0$. If $d\chi^0 = 0$ and $d\chi^{i+1} = -\imath_{\mathcal{E}} \chi^i$ holds for all $i < j$, then all $\imath_{\mathcal{E}} \chi^i$ for $i \leq j$ are closed.*

Proof. For $j = 0$ this follows immediately from Lemma 2.1. For $j > 0$, we have $d\imath_{\mathcal{E}} \chi^j = \mathrm{L}_{\mathcal{E}} \chi^j - \imath_{\mathcal{E}} d\chi^j = -\imath_{\mathcal{E}} d\chi^j$ because χ^j is invariant. Now $d\chi^j = -\imath_{\mathcal{E}} \chi^{j-1}$ and thus $d\imath_{\mathcal{E}} \chi^j = (\imath_{\mathcal{E}})^2 \chi^{j-1} = 0$. □

Thus χ in (8) is closed iff

$$d\chi^0 = 0 \qquad \text{and} \qquad d\chi^{i+1} = -\imath_{\mathcal{E}} \chi^i = -\nu^i \qquad \text{for all} \quad i \in \mathbb{N}_0. \tag{9}$$

Those $\phi \in \mathcal{A}(F)_{\mathrm{inv}}$, for which a series (9) with $\chi^0 := \phi$ exists, will be called *G-transgressive* forms on the G-manifold F. Their set is denoted by $\mathcal{A}(F)_{G-\mathrm{trans}}$.

Lemma 4.2 *$\mathcal{A}(F)_{G-\mathrm{trans}}$ is a subalgebra of $\mathcal{A}(F)_{\mathrm{inv}}$ in the kernel of d.*

Proof. For $\alpha, \beta \in \mathcal{A}(F)_{G-\mathrm{trans}}$, we may define

$$\chi^k_{(\alpha \wedge \beta)} := \sum_{i+j=k} \chi^i_{(\alpha)} \wedge \chi^j_{(\beta)}.$$

□

Obviously, if $\phi_n \in \mathcal{A}_n(F)_{G-\mathrm{trans}}$, then the series terminates after at most $[\frac{n}{2}]$ steps. Let $H^*(F)_{G-\mathrm{trans}} := \mathcal{A}(F)_{G-\mathrm{trans}}/d\mathcal{A}(F)$ and define $j\colon \mathcal{A}(F)_{G-\mathrm{trans}} \to \mathcal{A}(F)^G$ by

$$j(\phi_n) = \sum_{i=0}^{[\frac{n}{2}]} \underbrace{\tilde{\Omega}\cdot\cdots\cdot\tilde{\Omega}}_{i} \otimes (\chi^i + \tilde{\omega} \otimes \imath_{\mathcal{E}}\chi^i). \tag{10}$$

By (9), if the equivariant cohomology class of a G-transgressive form is zero, then already the G-transgressive form is exact. On the other hand, if $\phi \in \mathcal{A}_n(F)_{G-\mathrm{trans}}$ is exact, we find $\alpha \in \mathcal{A}(F)_{\mathrm{inv}}$ with $\phi = d\alpha$. We may take $\chi^1 := \imath_{\mathcal{E}}\alpha$ (thus $\nu^1 = 0$ and the series ends after one step) and then $j(\phi) = \phi + \tilde{\omega} \otimes \imath_{\mathcal{E}}\phi + \tilde{\Omega} \otimes \imath_{\mathcal{E}}\alpha = D(\alpha + \tilde{\omega} \otimes \imath_{\mathcal{E}}\alpha)$. Thus j is an injective algebra morphism which commutes with the differentials and induces an injective morphism $[j]\colon H^*(F)_{G-\mathrm{trans}} \to H^*(F)^G$.

Hence $\ker D|_{\mathcal{A}(F)^G} = \mathrm{S}(\mathfrak{g}^*) \otimes j(\mathcal{A}(F)_{G-\mathrm{trans}})$ and $H^*(F)^G$ is the quotient

$$H^*(F)^G = \ker D|_{\mathcal{A}(F)^G}/D\mathcal{A}(F)^G = \mathrm{S}(\mathfrak{g}^*) \otimes j(\mathcal{A}(F)_{G-\mathrm{trans}})/[\mathrm{S}(\mathfrak{g}^*) \otimes Dj_0(\mathcal{A}(F)_{\mathrm{inv}})].$$

Let us check, in what case $H^*(F)^G$ differs from $\mathrm{S}(\mathfrak{g}^*) \otimes [j](H^*(F)_{G-\mathrm{trans}})$. So take $\chi = \sum_{i=0}^{\infty} \tilde{\Omega}^i \otimes (\chi^i + \tilde{\omega} \otimes \imath_{\mathcal{E}}\chi^i) \in \mathcal{A}(F)^G$ and $\alpha = \sum_{i=k+1}^{\infty} \tilde{\Omega}^i \otimes (\alpha^i + \tilde{\omega} \otimes \imath_{\mathcal{E}}\alpha^i) \in \mathrm{S}(\mathfrak{g}^*) \otimes j(\mathcal{A}(F)_{G-\mathrm{trans}})$ with $0 \neq [\alpha^{k+1}] \in H^*(F)_{G-\mathrm{trans}}$. One easily checks that

$$\alpha = D\chi \iff 0 = d\chi^0 = d\chi^1 + \imath_{\mathcal{E}}\chi^0 = \cdots = d\chi^k + \imath_{\mathcal{E}}\chi^{k-1}, \quad \alpha^i = d\chi^i + \imath_{\mathcal{E}}\chi^{i-1}, \quad i > k.$$

But then $[\imath_{\mathcal{E}}\chi^k] = [\alpha^{k+1}] \neq 0 \in H^*(F)_{G-\mathrm{trans}}$. Thus the series for $\phi = \chi^0 \in \mathcal{A}(F)_{\mathrm{inv}}$ terminates after exactly k steps, i. e., ϕ is a closed invariant form on F, which is not G-transgressive. On the other hand, if the series (9) for a closed $\phi \in \mathcal{A}(F)_{\mathrm{inv}}$ terminates after k steps, then $\tilde{\Omega}^{k+1} \otimes \imath_{\mathcal{E}}\chi^k \in \mathcal{A}(F)^G$ represents a non-trivial element $\tilde{\Omega}^{k+1} \otimes [\imath_{\mathcal{E}}\chi^k] \in \mathrm{S}(\mathfrak{g}^*) \otimes H^*(F)_{G-\mathrm{trans}}$, but it is D-exact in $\mathcal{A}(F)^G$, namely $\tilde{\Omega}^{k+1} \otimes \imath_{\mathcal{E}}\chi^k = D\left(\sum_{i=0}^{k} \tilde{\Omega}^i \otimes (\chi^i + \tilde{\omega} \otimes \imath_{\mathcal{E}}\chi^i)\right)$. We thus have proven:

Theorem 4.3 *For $G \cong \mathbb{S}^1$ and any G-manifold F, the equivariant cohomology $H^*(F)^G$ contains a subgroup isomorphic to $H^*(F)_{G-\mathrm{trans}} \leq H^*(F)$. In addition, the following statements are equivalent:*

- $H^*(F) = H^*(F)_{G-\mathrm{trans}}$;
- $H^*(F)^G \cong \mathrm{S}(\mathfrak{g}^*) \otimes H^*(F)_{G-\mathrm{trans}}$;
- $H^*(F)^G \cong \mathrm{S}(\mathfrak{g}^*) \otimes H^*(F)$.

5. THE UNIVERSAL BUNDLE

Let $EG(BG, G)$ denote the universal principal bundle for the LIE group G, and let $EG \times_G F$ denote the universal bundle for the G-manifold F. Recall that $EG \cong \mathbb{S}^\infty$ and $BG \cong \mathbb{C}P^\infty$ for $G \cong \mathbb{S}^1$ and recall from Section 2 that $\mathcal{A}(EG \times_G F) \cong \mathcal{A}(EG \times F)_{\mathrm{bas}}$. Now for a compact connected LIE group, the WEIL homomorphism $\mathrm{W}(\mathfrak{g}) \to \mathcal{A}(EG)$ induces an isomorphism on the cohomology of the basic subcomplexes and we have:[1]

Theorem 5.1 *For $G \cong \mathbb{S}^1$ and any G-manifold F, there is a natural isomorphism:*

$$H^*(EG \times_G F) \cong H^*(F)^G.$$

Choosing F to be a single point, we immediately get the following well-known fact:

Corollary 5.2 *For $G \cong \mathbb{S}^1$, there is a natural isomorphism:*

$$H^*(BG) \cong \mathrm{W}(\mathfrak{g})_{\mathrm{bas}} = \mathrm{S}(\mathfrak{g}^*).$$

By the universal property of $EG \times_G F$, there exists a bundle map $b: B \to EG \times_G F$ for every bundle $B = P \times_G F$ that comes with the given left action on F. Hence we have homomorphisms $b^*: \mathcal{A}(EG \times_G F) \to \mathcal{A}(B)$ and $[b^*]: H^*(EG \times_G F) \to H^*(B)$ for all these bundles. As a consequence, in order to find those invariant closed forms on F, that extend to closed forms on every bundle in the sense of (5), we have to determine the invariant closed forms ω on F that generalize to a closed form ω^B on $EG \times_G F$ such that

$$(i_p)^*\omega^B = \omega \qquad \text{for all} \quad p \in EG, \tag{11}$$

where $i_p: F \to EG \times_G F$ is defined by $i_p(f) := (p, f)G$. To accomplish this, Corollary 5.2 is not very helpful, because $\pi \circ i_p = \pi(p): F \to BG$ is constant and thus $(i_p)^*(H^*(BG)) = \{0\}$. Instead, we should look at those closed elements $\chi \in \mathcal{A}(F)^G$, for which $\chi^0 \neq 0$. But we already saw that those are exactly given by the G-transgressive forms on F. We thus have proven:

Theorem 5.3 *For $G \cong \mathbb{S}^1$ and any G-manifold F, let $\phi_n \in \mathcal{A}_n(F)_{\mathrm{inv}}$ be closed. Then the following statements are equivalent:*

- *ϕ_n defines a cohomology class $[\phi_n^B] \in H^n(B)$ with representative ϕ_n^B such that $(i_{\alpha,x})^*\phi_n^B = \phi_n$ for all $x \in U_\alpha$ for any fiber bundle $B(M, F, G)$ that comes with the given G-action on F;*
- *ϕ_n defines a cohomology class $[\phi_n^B] \in H^n(B)$ with representative ϕ_n^B on the universal bundle $B = EG \times_G F$ such that $(i_p)^*\phi_n^B = \phi_n$ for all $p \in EG$;*
- *ϕ_n defines an equivariant cohomology class in $H^n(F)^G$;*
- *ϕ_n is G-transgressive, i. e., there exist $\chi^i \in \mathcal{A}_{n-2i}(F)_{\mathrm{inv}}$ for $0 \leq i \leq [\frac{n}{2}]$ such that with $\nu^i := \imath_E \chi^i \in \mathcal{A}_{n-2i-1}(F)_{\mathrm{inv}}$ we have*

$$\chi^0 = \phi_n, \qquad d\chi^1 = -\nu^0, \qquad d\chi^2 = -\nu^1, \quad \ldots \quad \nu^{[\frac{n}{2}]} = 0.$$

In that case, for any connection Γ on $P(M, G)$ with gauge field $\mathrm{F} = \tilde{\mathrm{F}}E \in \mathcal{A}_2(M, \mathfrak{g})$,

$$\phi_n^B := \sum_{i=0}^{[\frac{n}{2}]} \underbrace{\tilde{\mathrm{F}} \wedge \cdots \wedge \tilde{\mathrm{F}}}_{i} \wedge \chi^i v = \sum_{i=0}^{[\frac{n}{2}]} \chi^i v \wedge \underbrace{\tilde{\mathrm{F}} \wedge \cdots \wedge \tilde{\mathrm{F}}}_{i} \in [\phi_n^B] \in H^n(B) \tag{12}$$

is such a representative for the generated cohomology class $[\phi_n^B]$, which is adapted to the given connection Γ.

Recall that by (4), ϕ_n^B in (12) is indeed the pullback under $\tilde{\pi}: P \times F \to B$ of the basic form on $P \times F$, that is the image of $j(\phi_n)$ in (10) under the Weil homomorphism.

6. G-TRANSGRESSIVE FORMS FOR THE SKYRMION BUNDLE

Let us return to the skyrmion bundle now. According to Theorem 5.3, our task is to find the G-transgressive forms for the skyrmion bundle. We will prove that every invariant closed form on SU_{N_F} for $N_F \leq 6$ is G-transgressive, thus

$$H^*(\mathrm{SU}_{N_F})_{G-\mathrm{trans}} = H^*(\mathrm{SU}_{N_F}). \tag{13}$$

Due to Lemma 4.2, only the generators of $H^*(\mathrm{SU}_{N_F})$ need to be considered. Recall that we have already shown in (6) and (7) that ω_3 and ω_5 are G-transgressive. The constant map $1 \in C^\infty(\mathrm{SU}_{N_f})$ is obviously G-transgressive. Now direct computation for ω_{2l+1} shows that 2 steps of the series (9) are always possible, e. g., one may take

$$\begin{aligned}
\chi^1_{2l-1} &= (2l+1)(\rho^Q_{2l-1} + \lambda^Q_{2l-1}),\\
\chi^2_{2l-3} &= (2l+1)\Big[2(\rho^{Q^2}_{2l-3} + \lambda^{Q^2}_{2l-3}) + \sum\nolimits_{j=1}^{l-2} \mathrm{Tr}(QR^{2j-1}QR^{2l-2j-2} + QL^{2j-1}QL^{2l-2j-2})\\
&\qquad + \sum\nolimits_{j=1}^{l-1} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j-2}U^{-1} + QUL^{2j-2}QL^{2l-2j-1}U^{-1})\Big].
\end{aligned}$$

(For $l = 1$, resp., $l = 2$, we obtain χ^1_1 in (6), resp., χ^1_3 and χ^2_1 in (7).) Fortunately, this is sufficient for $N_F \leq 6$. In the worst case, starting with ω_{11}, we end up with $\nu^2_6 \in \mathcal{A}(\mathrm{SU}_{N_F}, \mathbb{C})_{\mathrm{inv}}$. By Lemma 4.1, ν^2_6 is closed, and since $H^6(\mathrm{SU}_m) = 0$, it is exact. Hence we find $\chi^3_5 \in \mathcal{A}(\mathrm{SU}_{N_F}, \mathbb{C})_{\mathrm{inv}}$ with $d\chi^3_5 = \nu^2_6$. Analogously, because $H^4(\mathrm{SU}_m) = H^2(\mathrm{SU}_m) = 0$, we always find $\chi^i_1 \in \mathcal{A}(\mathrm{SU}_{N_F}, \mathbb{C})_{\mathrm{inv}}$ with $d\chi^i_1 = \nu^{i-1}_2$.

It remains to prove that $\nu^i_0 = \imath_\mathcal{E}\chi^i_1 = 0$. As before, Lemma 4.1 yields that $d\nu^i_0 = 0$, whence ν^i_0 is a constant function and it suffices to prove $\nu^i_0(\mathbb{1}) = 0$. Now G acts on SU_{N_F} by conjugation, and thus $\mathcal{E}(\mathbb{1}) = 0$. This yields $\nu^i_0(\mathbb{1}) = \chi^i_1(\mathbb{1})[\mathcal{E}(\mathbb{1})] = 0$, which completes the proof of (13).

From Theorems 4.3 and 5.1 and Corollary 5.2 we obtain our final result:

Theorem 6.1 *For the skyrmion bundle with $N_F \leq 6$, we have*

$$H^*(EG \times_G \mathrm{SU}_{N_F}) \cong H^*(\mathrm{SU}_{N_F})^G \cong \mathrm{S}(\mathfrak{g}^*) \otimes H^*(\mathrm{SU}_{N_F}) \cong H^*(BG) \otimes H^*(\mathrm{SU}_{N_F}).$$

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