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NATURAL OPERATIONS OF HAMILTONIAN TYPE ON THE COTANGENT BUNDLE

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ABSTRACT. We study some geometrical constructions of Hamiltonian type on the cotangent bundle T^*M. In particular, we shall deal with constructions of vector fields by means of functions, of functions by means of pairs of functions and of 1–forms by means of pairs of 1–forms. All geometrical constructions are considered as natural operators and there is given the full classification in some particular cases.

1. INTRODUCTION

The cotangent bundle T^*M of a smooth manifold M plays an important role in analytical mechanics. It is the simplest example of a symplectic manifold. In [2] we have studied geometrical constructions of tensor fields on T^*M by means of tensor fields on M. The aim of this paper is to study geometrical constructions of tensor fields on T^*M by means of tensor fields on the same bundle. In particular, we shall pay an attention to geometrical constructions of Hamiltonian and Poisson type. We shall essentially use the canonical symplectic structure of T^*M, which naturally identifies the tangent and cotangent bundle of T^*M.

Denoting by TM the tangent bundle of a smooth manifold M, sections of the vector bundle $T^{(r,s)}M = \overset{r}{\otimes} TM \otimes \overset{s}{\otimes} T^*M$ are called tensor fields of type (r,s). According to the general theory, [3], geometrical constructions are natural differential operators of a specific type. Using such a point of view, geometrical constructions of (p,q)–tensor fields on T^*M by means of (r,s)–tensor fields on T^*M can be written as natural operators $T^{(r,s)}T^* \rightsquigarrow T^{(p,q)}T^*$, cf. the usual notation from [3]. In some particular cases it is also possible to determine all natural operators, i.e. to give the full list of all possible geometrical constructions in question. For example, in [2] we have showed that the canonical symplectic structure of T^*M is the only natural one. Similar problems were studied by Mikulski in [5] and [6]. We remark that the geometry of the cotangent bundle has been studied by many authors, see e.g. [1], [4] and [7].

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2. Constructions of vector fields and 1–forms by means of functions

Let (x^i) be the canonical coordinates on a smooth manifold M. Then the induced coordinates on the cotangent bundle $q_M : T^*M \to M$ will be denoted by (x^i, p_i). The cotangent bundle carries a canonical symplectic structure via the symplectic 2–form

$$\Lambda_M = d\lambda_M = dp_i \wedge dx^i, \tag{1}$$

where $\lambda_M = p_i dx^i$ is the classical Liouville 1–form on T^*M. Further, Λ_M induces a vector bundles isomorphism

$$S_M : T^{(1,0)}T^*M \to T^{(0,1)}T^*M. \tag{2}$$

It is well known that if $f : T^*M \to \mathbb{R}$ is an arbitrary function on T^*M, then there is a unique vector field X_f on T^*M such that $i_{X_f}\Lambda_M = df$. Vector field X_f is called *Hamiltonian vector field* corresponding to the function f. The coordinate expression of X_f is

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial p_i}. \tag{3}$$

If d is the exterior derivative operator, then the composition

$$\mathcal{H} := S_M^{-1} \circ d : T^{(0,0)}T^*M \to T^{(1,0)}T^*M \tag{4}$$

is called *Hamiltonian mapping.* In this way we can also define X_f by $X_f = \mathcal{H}(f)$. Clearly, $\mathcal{H}$ is an example of the first order natural differential operator $T^{(0,0)}T^* \rightsquigarrow T^{(1,0)}T^*$ which associates a vector field to each function on T^*M. The aim of this section is to determine all first order natural operators of this type.

Let L_{T^*M} be the classical Liouville vector field on T^*M generated by the homotheties of the vector bundle T^*M, in coordinates

$$L_{T^*M} = p_i \frac{\partial}{\partial p_i}. \tag{5}$$

If $f \in C^\infty(T^*M)$ is a function on T^*M, then the exterior differential df is a 1–form on T^*M. Evaluating the action of a vector on a covector, we have another function on T^*M,

$$I := \langle df, L_{T^*M}\rangle . \tag{6}$$

The coordinate expression of I is $I = \frac{\partial f}{\partial p_i} p_i$. We have

Proposition 1. *All first order natural operators $T^{(0,0)}T^* \rightsquigarrow T^{(1,0)}T^*$ transforming functions on T^*M into vector fields on T^*M are of the form*

$$f \mapsto \varphi(I, f) \cdot X_f + \psi(I, f) \cdot L_{T^*M} \tag{7}$$

where $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ *are arbitrary smooth functions of two variables.*

Proof. The proof is based on the classical equivalence between natural operators and equivariant maps of the corresponding standard fibres. Let G^r_m be the group of all invertible r-jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target zero. The canonical coordinates on G^r_m will be denoted by $(a^i_j, a^i_{jk}, \dots, a^i_{j_1,\dots,j_r})$ and the coordinates of the inverse element will be denoted by a tilde. By [3], the first order natural operators $T^{(0,0)}T^* \rightsquigarrow T^{(1,0)}T^*$ are in a canonical bijection with G^2_m-equivariant maps $(J^1T^{(0,0)}T^*)_0\mathbb{R}^m \oplus T^*_0\mathbb{R}^m \to (T^{(1,0)}T^*)_0\mathbb{R}^m$ of the corresponding standard fibres. If $f : T^*M \to \mathbb{R}$ is a function, then the partial derivatives $f_i = \frac{\partial f}{\partial x^i}$ and $f^i = \frac{\partial f}{\partial p_i}$ define the additional coordinates f_i and f^i on the standard fibre $(J^1T^{(0,0)}T^*)_0\mathbb{R}^m$. Moreover, the coordinate expression $\xi = \xi^i \frac{\partial}{\partial x^i} + \eta_i \frac{\partial}{\partial p_i}$ of a vector field ξ on T^*M induces the coordinates ξ^i and η_i on $(T^{(1,0)}T^*)_0\mathbb{R}^m$. Hence we have to find all G^2_m-equivariant maps of the form

$$\xi^i = \xi^i(p_i, f^i, f_i, f) \quad \text{and} \quad \eta_i = \eta_i(p_i, f^i, f_i, f).$$

One evaluates easily the equations of the action of G^2_m on the standard fibres

$$\begin{aligned} \overline{p}_i &= \widetilde{a}^j_i p_j, \\ \overline{f}^i &= a^i_j f^j, \\ \overline{f}_i &= \widetilde{a}^j_i f_j + a^k_{j\ell}\widetilde{a}^\ell_i p_k f^j, \\ \overline{\xi}^i &= a^i_j \xi^j, \\ \overline{\eta}_i &= \widetilde{a}^j_i \eta_j + \widetilde{a}^k_{i\ell} a^\ell_j p_k \xi^j. \end{aligned} \tag{8}$$

Consider first $\xi^i(p_i, f^i, f_i, f)$. Using equivariance on the kernel of the jet projection $G^2_m \to G^1_m$ we obtain

$$\xi^i(p_i, f^i, f_i, f) = \xi^i(p_i, f^i, f_i + a^k_{ji} p_k f^j, f)$$

which yields that ξ^i are independent of f_i. Then the homotheties $a^i_j = k\delta^i_j$ give

$$k\xi^i(p_i, f^i, f) = \xi^i(\frac{1}{k}p_i, kf^i, f).$$

By the tensor evaluation theorem from [3] we have $\xi^i = \varphi(p_i f^i, f) f^i$. Further, we may assume that η_i are of the form $\eta_i = \alpha f_i + \beta_i(p_i, f^i, f_i, f)$. Applying again equivariance on the kernel of the jet projection $G^2_m \to G^1_m$ we obtain $\alpha = -\varphi$. Moreover, using the full equivariance we find that β_i have the same transformation law as p_i. Finally, applying a similar procedure to β_i as that for ξ^i we deduce that $\beta_i = \psi(p_i f^i, f) p_i$. Hence we have proved

$$\xi^i = \varphi(p_i f^i, f) f^i, \quad \eta_i = -\varphi(p_i f^i, f) f_i + \psi(p_i f^i, f) p_i$$

which is nothing else but the coordinate form of (7). □

Taking into account the isomorphism S_M from (2) we have

Corollary 1. *All first order natural operators $T^{(0,0)}T^* \rightsquigarrow T^{(0,1)}T^*$ transforming functions on T^*M into 1-forms on T^*M are of the form*

$$f \mapsto \varphi(I, f) \cdot df + \psi(I, f) \cdot \lambda_M \tag{9}$$

where $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are arbitrary smooth functions of two variables.

An operator $A : T^{(0,0)}T^* \rightsquigarrow T^{(p,q)}T^*$ is called *absolute*, if $Af = AO_{T^*M}$ for all functions $f : T^*M \to \mathbb{R}$, where O_{T^*M} means the zero function on T^*M. For example, $\overset{p}{\otimes} L_{T^*M} \otimes \overset{q}{\otimes} \lambda_M$ is an absolute tensor field of type (p, q) on T^*M. On the other hand, we have

Corollary 2. *For all p and q there is a nonabsolute natural operator $T^{(0,0)}T^* \rightsquigarrow T^{(p,q)}T^*$.*

The following assertion can be proved in the same way as Proposition 1.

Proposition 2. *All first order natural operators transforming functions on T^*M into functions on T^*M are of the form*

$$f \mapsto \varphi(I, f)$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary smooth function of two variables.

Remark 1. Corollary 1 describes all (first order) geometrical constructions of 1-forms by means of functions on the cotangent bundle. For comparison's sake, all natural operators transforming functions on M into 1-forms on M are of the form $g \mapsto \varphi(g)dg$ with an arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$. This is a consequence of a more general result of Kolář, cf. [3].

Remark 2. If (M, ω) is an arbitrary symplectic manifold (not necessarily the cotangent bundle), then we can pose a more general question: To find all (first order) natural operators transforming functions on M into vector fields on M. In this case other methods are to be used, because $\det(\omega_{ij}) \neq 0$ and we can not apply some methods which were used in this section (for instance, assumptions of the tensor evaluation theorem are not satisfied).

3. Constructions of functions by means of pairs of functions

Let X_f be the Hamiltonian vector field corresponding to a function f defined on T^*M. If f and g are two functions on T^*M, then the *Poisson bracket* $\{f, g\}$ of f and g is defined by $\{f, g\} := \Lambda_M(X_f, X_g)$. This is an example of the first order natural operator $T^{(0,0)}T^* \oplus T^{(0,0)}T^* \rightsquigarrow T^{(0,0)}T^*$. If $f_i = \frac{\partial f}{\partial x^i}$ and $f^i = \frac{\partial f}{\partial p_i}$, then the coordinate form of $\{f, g\}$ is

$$\{f, g\} = f^i g_i - f_i g^i. \tag{10}$$

We prove that the Poisson bracket is the only nontrivial construction of functions by means of pairs of functions on the cotangent bundle.

Proposition 3. *Let $dimM \geq 2$. Then all first order natural operators $T^{(0,0)}T^* \oplus T^{(0,0)}T^* \rightsquigarrow T^{(0,0)}T^*$ transforming pairs of functions on T^*M into functions on T^*M are of the form*

$$f,g \mapsto \varphi(\langle df, L_{T^*M}\rangle, \langle dg, L_{T^*M}\rangle, \{f,g\}, f, g) \tag{11}$$

where $\varphi : \mathbb{R}^5 \to \mathbb{R}$ is an arbitrary smooth function of five variables.

Proof. We have to find all G^2_m–equivariant maps of the form

$$h = h(p_i, f^i, f_i, g^i, g_i, f, g). \tag{12}$$

The transformation laws of g^i and g_i are the same as those of f^i and f_i in (8). Denote by $I_1 = \langle df, L_{T^*M}\rangle = f^i p_i$, $I_2 = \langle dg, L_{T^*M}\rangle = g^i p_i$, $I_3 = \{f,g\} = f^i g_i - f_i g^i$, $I_4 = f$ and $I_5 = g$. Using homotheties and then the tensor evaluation theorem from [3] we prove that

$$h = \varphi(I_1, I_2, J_1, J_2, I_4, I_5, f_i f^i, g_i g^i),$$

where $J_1 = g_i f^i$ and $J_2 = f_i g^i$. Replacing J_1 and J_2 by the new couple of independent variables $I_3 = J_1 - J_2$ and J_2 we have

$$h = \varphi(I_1, \dots, I_5, f_i g^i, f_i f^i, g_i g^i).$$

It suffices to deduce that φ depends on $I_1, \dots, I_5$ only. Evaluating equivariance on the kernel of the jet projection $G^2_m \to G^1_m$ we have

$$\varphi(I_1, \dots, I_5, f_i g^i, f_i f^i, g_i g^i) =$$
$$= \varphi(I_1, \dots, I_5, (f_i + a^k_{ij} p_k f^j) g^i, (f_i + a^k_{ij} p_k f^j) f^i, (g_i + a^k_{ij} p_k g^j) g^i).$$

Setting $f^i = (1, 0, \dots, 0)$ and $g^i = (0, 1, 0, \dots, 0)$ we get

$$\varphi(I_1, \dots, I_5, f_2, f_1, g_2) = \varphi(I_1, \dots, I_5, f_2 + a^k_{12} p_k, f_1 + a^k_{11} p_k, g_2 + a^k_{22} p_k).$$

If $a^k_{11} = 0 = a^k_{22}$, then

$$\varphi(I_1, \dots, I_5, f_2, f_1, g_2) = \varphi(I_1, \dots, I_5, f_2 + a^k_{12} p_k, f_1, g_2)$$

so that φ is independent of f_2. Quite analogously, using a^k_{11} and a^k_{22} we deduce that $\varphi = \varphi(I_1, \dots, I_5)$. This completes the proof. □

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