Konstantin Igorevich Beidar; Y. Fong; Alexander Stolin

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SYMMETRIC ALGEBRAS AND YANG-BAXTER EQUATION*

K. Beidar, Y. Fong, A. Stolin

Abstract. In 1982 Belavin and Drinfeld listed all elliptic and trigonometric solutions of CYBE for simple complex Lie algebras. Later the third author classified rational solutions of CYBE for the same Lie algebras. In this paper we consider rational solutions of CYBE and QYBE for algebras with a non-degenerate symmetric invariant bilinear form. Such algebras (both Lie and associative) are called symmetric. In the present paper the theory of rational solutions of YBE for these algebras is developed. This leads to new examples of rational solutions in both cases - classical and quantum. In particular in this paper we will find generalizations of the famous Yang rational solution of QYBE in $gl(n)$-case for all finite-dimensional associative symmetric algebras.

1. Introduction.

The first known solution of QYBE was found in [Y] and is of the form $1+\frac{P}{u} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, where P is the twist map $P(x\otimes y) = y \otimes x,\ x, y \in \mathbb{C}^n$.

The same element P considered as an element of $gl(n,\mathbb{C}) \otimes gl(n,\mathbb{C})$ provides the following rational solution of CYBE $r_0(u,v) = \frac{P}{u-v}$.

In other words $r_0(u, v)$ satisfies the following system:

$$[r_0^{12}(u_1,u_2), r_0^{13}(u_1,u_3)] + [r_0^{12}(u_1,u_2), r_0^{23}(u_2,u_3)] + [r_0^{13}(u_1,u_3), r_0^{23}(u_2,u_3)] = 0$$
$$r_0^{12}(u,v) = -r_0^{21}(v,u) \quad (1)$$

Recently in some papers (for instance [D1]) the following statement was tacitly used to provide a consistency of the Kniznik-Zamolodchikov equation.

Statement 1. Let L be a Lie algebra. We say that $t \in L \otimes L$ is invariant if

$$(*) \qquad [t, a\otimes 1 + 1 \otimes a] = 0 \quad for\ any \quad a \in L.$$

Consider the function $r(u,v) = \frac{t}{u-v}$ depending on the formal variables u, v. Then $r(u,v)$ satisfies CYBE.

The proof is straightforward from $(*)$. It is well-known that $P \in gl(n) \otimes gl(n)$ satisfies $(*)$.

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* This paper is in final form and no version of it will be submitted elsewhere

The relation $(*)$ provides that another statement holds for any Lie algebra L.

Statement 2. Let $r_1 \in L \otimes L$. Then $r(u,v) = \frac{t}{u-v} + r_1$ is a solution of (1) if and only if r_1 is a solution of CYBE.

These two statements are valid for any Lie algebra L but if we wish to establish some criteria for a function $r_2(u,v) = \frac{t}{u-v} + p(u,v)$ with a polynomial part $p(u,v)$ to be a solution of (1), we need some assumptions for the given Lie algebra L.

Definition 1.1. A Lie algebra L is called symmetric if L posseses an invariant symmetric non-degenerate bilinear form (a,b). Here invariancy means that $([a,b],c) = (a,[b,c])$.

Here we expose some examples of the symmetric Lie algebras.

Example 1.2. The semisimple Lie algebras with respect to the Killing form.

Example 1.3. The Manin triples (see [D2] for definition).

Example 1.4. The Drinfeld double of a finite dimensional Hopf algebra considered as a Lie algebra.

Lemma 1.5. Let $\{e_i\}$ be a basis of a symmetric Lie algebra L and $\{f^i\}$ be the dual basis. Then the element $\sum_i e_i \otimes f^i = t \in A \otimes A$ is an invariant element, which does not depend on a choice of basis.

Proof. Clearly t does not depend on a choice of basis. Therefore it remains to prove that $[t, e_j \otimes 1 + 1 \otimes e_j] = 0$ for all j.

Let $[e_j, e_j] = c_{ij}^k e_k$. Then $c_{ij}^k = ([e_i, e_j], f^k) = (e_i, [e_j, f^k])$ and hence $[e_j, f^k] = c_{sj}^k f^s$.(We assume the summation is repeated indices).

We have

$$[t, e_j \otimes 1 + 1 \otimes e_j] = [e_i \otimes f^i, e_j \otimes 1 + 1 \otimes e_j] = [e_i, e_j] \otimes f^i$$

$$+ e_i \otimes [f^i, e_j] = c_{ij}^k e_k \otimes f^i - c_{sj}^i e_i \otimes f^s = 0 \; .$$

In what follows we will be looking for solutions of system (1) in the form

$$r(u,v) = \frac{t}{u-v} + p(u,v) \; .$$

We will call $r(u,v)$ a rational solution of CYBE for the symmetric Lie algebra L.

Lemma 1.6. Let $L((u^{-1}))$ be the Lie algebra of Loran series in u^{-1} with coefficients from L. Then $L((u^{-1}))$ has an invariant symmetric inner product defined as follows:

$$(au^k, bu^n) = (a,b)\delta_{k+n,-1} \; .$$

The proof is straightforward.

Remark. Throughout this paper we denote both forms in L and $L((u^{-1}))$ by $(\ ,\ )$ what should not be misleading.

The aim of this paper is to develop a theory of rational solutions of CYBE and QYBE for symmetric algebras including non-trivial examples. In particular, we will generalize the Yang solution in different ways.

2. Rational solutions of CYBE for symmetric algebras

This section is devoted to a theory of the rational solutions of CYBE for arbitrary symmetric Lie algebras developed for simple Lie algebras in [S1, S2]

Following [S1] we prove the basic result.

Theorem 2.1. Let L be a symmetric Lie algebra. There is a natural one-to-one correspondence between the rational solutions of CYBE (1) of the form $\frac{t}{u-v}+p(u,v)$ and the subspaces $W \subset L((u^{-1}))$ such that

(i) W is a subalgebra in $L((u^{-1}))$

(ii) $W \oplus L[u] = L((u^{-1}))$

(iii) $W = W^{\perp}$ with respect to the inner product $(\ ,\ )$ for $L((u^{-1}))$ wich was defined in Lemma 1.2. We will call such subspaces Lagrangian.

(iv) $W \supseteq u^{-N}L((u^{-1}))$ for some $N > 0$.

Proof. For any $p(u,v) \in L[u] \otimes L[v] = (L \otimes L)[u,v]$ we define the subspace $W_p \subset L((u^{-1}))$ as follows:

Let $\zeta_p : u^{-1}L[[u^{-1}]] \to L[u]$ be the following linear map $\zeta_p(a) = (p^{(2)}(u), a)p^{(1)}(u)$ where we write formally $p(u,v) = p^{(1)}(u) \otimes p^{(2)}(v)$. Then $W_p = \{a + \zeta_p(a) : a \in u^{-1}L[[u^{-1}]]\} \subset L((u^{-1}))$. Clearly $W_p \oplus L[u] = L((u^{-1}))$ (note that $L[u] \cap W_p = (0)$!), and $W_p \supset u^{-N}L[[u^{-1}]]$ for some $N > 0$ because p is polynomial. Conversely, having W satisfying (ii) and (iv) we can restore $p(u,v)$.

Consider the following decomposition of

$$L((u^{-1})) = L[u] \oplus u^{-1}L[[u^{-1}]] \, .$$

Then the projection $j : W_p \to u^{-1}L[[u^{-1}]]$ along $L[u]$ is an isomorphism due to (ii). Let i be the projection of W to $L[u]$ along $u^{-1}L[[u^{-1}]]$. We can define $\rho : u^{-1}L[[u^{-1}]] \to L[u]$ as $i \cdot j^{-1}$.

If $\{e_1, \cdots, e_n\}$ and $\{f^1, \cdots, f^n\}$ are dual bases of L, then

$$p(u,v) = \sum_{m,k=0} \rho(e_m u^{-k-1}) \otimes f^m \cdot v^k \, .$$

Then (iv) ensures us that $p(u,v)$ is polynomial, because $Ker\ \rho \supset u^{-N}L[[u^{-1}]]$. Therefore (ii) and (iv) are proved.

The next step is to prove that (iii) is equivalent to the skew-symmetry of $p(u,v)$, i.e. $p(u,v) = -p^{21}(v,u)$. To do this we note that the skew-symmetry of $p(u,v)$ is equivalent to the following property of ρ_p : $0 = (a, \rho_p(b) + (b, \rho_p(a))$ (1) for any $a, b \in u^{-1}L[[u^{-1}]]$.

Rewriting (1) as $(a+\rho_p(a), b+\rho_p(b)) = 0$ we conclude that (1) implies $W^{\perp} \supseteq W$. We have to prove that if $(a, W) = 0$ then $a \in W$. Using (ii) we have $a = (x+\rho_p(x))+s$, where $x \in u^{-1}L[[u^{-1}]]$, $s \in L[u]$. Therefore $(s, W) = 0$ and $(s, u^{-1}L[[u^{-1}]]) = 0$. Hence $s = 0$ and implies that (1) $\Longrightarrow$ (iii).

The implication (iii) $\Longrightarrow$ (1) is trivial because (iii) provides that $(a + \rho_p(a), b + \rho_p(b)) = 0$ for any $a, b \in u^{-1}L[[u^{-1}]]$.

Surely, the main part of Theorem 2.1, namely (i) remains unproved but now we can formulate it in a different way.

Statement. $\frac{t}{u-v}+p(u,v)$ satisfies

$$CYBE \Longleftrightarrow ([a+\rho_p(a), b+\rho_p(b)], c+\rho_p(c))=0$$

for any $a,\ b,\ c \in u^{-1}L[[u^{-1}]]$.

Taking into account that $\frac{t}{u-v}$ satisfies CYBE as well, we can rewrite LHS of CYBE as

$$[p^{12}(u_1,u_2),p^{13}(u_1,u_3)]+[p^{12}+p^{13},\frac{t^{23}}{u_2-u_3}]+\{cyclic\ perm.\ (1,2,3)\}=0.$$

Substituting

$$p(u,v)=\sum_{m,k}\rho(e_m u^{-k-1})\otimes f^m\cdot v^k$$

into the first summand we get:

$$[p^{12},p^{13}]=\sum_{i,j,m,n}[\rho(e_m u_1^{-i-1}),\rho(e_n u_1^{-j-1})]\otimes f^m u_2^i\otimes f^n u_3^j$$

and

$$T=[p^{12}+p^{13},\frac{t^{23}}{u_2-u_3}]=\sum_{s,k}[\rho(e_s u_1^{-k-1})\otimes[f^s u_2^k\otimes 1+1\otimes f^s u_3^k,\ \frac{t^{23}}{u_2-u_3}]$$

Since $t=e_n\otimes f^n$ is invariant, it is easily seen that

$$[f^s u_2^k\otimes 1+1\otimes f^s u_3^k,\ \frac{t^{23}}{u_2-u_3}]=[f^s,e_n]\otimes f^n\cdot\frac{u_2^k-u_3^k}{u_2-u_3}$$

Finally using the invariancy of the form (,) we obtain that

$$[f^s,e_n]=-\sum_m([e_m,e_n],f^s)f^m$$

and hence,

$$T=\sum_{p,m,i+j=k-1}(\rho([e_m u_1^{-i-1},\ e_n u_1^{-j-1}]),e_l u_1^{-p-1})f^l u_1^p\otimes f^m u_2^i\otimes f^n u_3^j$$

(note that we have no longer the summation in s !)

Applying the same considerations to the other terms of LHS of CYBE and denoting by $a=e_m u^{-i-1},\ b=e_n u^{-j-1},\ c=e_l u^{-p-1}$ we yield that

$$K=([\rho(a),\rho(b)]-\rho([a,b]),\ c)+\{cyclic\ perm.\ (a,b,c)\}=0$$

If we recall that $([a,b],c) = ([\rho(a),\rho(b)],\rho(c)) = 0$ and $(\rho(a),b) + (a,\rho(b)) = 0$, we get

$$K = ([a+\rho(a),\ b+\rho(b)],\ c+\rho(c)) = 0$$

This observation completes the proof of the **statement** and Theorem 2.1.

Corollary 2.2. Let $r(u,v) = \frac{t}{u-v} + p(u,v)$ be a rational solution of CYBE in L. Then $p(u,v) = const = r$ if and only if $W_p \subset L[[u^{-1}]]$. Moreover, this solution defines a subalgebra $L_0 \subset L$ and a skew-symmetric non-degenerate bilinear form B_0 on L_0 such that $B_0([x,y],z) + B_0([z,x],y) + B_0([y,z],x) = 0$, $r \in L_0 \otimes L_0$ and $r = B_0^{-1}$ as elements of $L_0 \otimes L_0$.

Proof. Clearly, if $p(u,v) = const$ then $W_p = \{a + \rho_p(a)\} \subset L[[u^{-1}]]$. Conversely, if $W \subset L[[u^{-1}]]$ satisfies all the conditions of Theorem 2.1, then $W = W^{\perp} \supset (L[[u^{-1}]])^{\perp} = u^{-2}L[[u^{-1}]]$. Further, ρ defined by such W satisfies $Ker(\rho) \supseteq u^{-2}L[[u^{-1}]]$. Therefore $p(u,v) = \sum_{k=0}^{\infty} \rho(eu^{-k-1}) \otimes fv^k$ reduces only to $k = 0$ and hence $p(u,v) = const$. Moreover, we have proved that ρ maps $u^{-1}L[[u^{-1}]]$ to $L \subset L[u]$. Since $W = \{a + \rho(a)\}$ is a subalgebra, it is easily seen that $Im(\rho) \subset L$ is a subalgebra. Let $L_0 = Im(\rho)$. Then $r \in L_0 \otimes L_0$ because $r \in L_0 \otimes L$ by the construction and r is skew-symmetric.

Let us define $B_0(\rho(a),\rho(b)) = (\rho(a),b)$. To show that B_0 is well-defined on L_0 it is sufficient to prove that $(Ker(\rho),b) = 0$, which is obvious because $b \in u^{-1}L[[u^{-1}]]$ and $Ker(\rho) \subset u^{-1}L[[u^{-1}]]$. Thus, B_0 is non-degenerate by its construction. Since $(\rho(a),b) + (a,\rho(b)) = 0$, B_0 is skew-symmetric. The fact that $r = B_0^{-1}$ follows now from the construction of r and B_0 and elementary linear algebra.

It remains to prove that B_0 is a 2-cocycle, which is equivalent to the following identity for any $a,b,c \in u^{-1}L[[u^{-1}]]$

$$(\rho[a,b],c) + (\rho[c,a],b) + (\rho[b,c],a) = 0\ .$$

The fact that $W = \{x + \rho(x)\} \subset L[[u^{-1}]]$ is a subalgebra immediately implies that

$$\rho([a,b] + [a,\rho(b)] + [\rho(a),b]) = [\rho(a),\rho(b)]$$

Hence, $(\rho[a,b],c) = ([\rho(a),\rho(b)],c) + ([a,\rho(b)],\rho(c)) + ([\rho(a),b],\rho(c))$ (we have used that $(\rho([a,\rho(b)]),c) = -([a,\rho(b)],\rho(c))$. On the other hand we have proved in Theorem 2.1 that $([a+\rho(a),b+\rho(b)],c+\rho(c)) = 0$ for any $a,b,c \in u^{-1}L[[u^{-1}]]$. In our case $([a,b],\rho(c)) = ([a,b],c) = ([\rho(a),\rho(b)],\rho(c)) = 0$. Therefore we have $([\rho(a),b],\rho(c)) + ([\rho(a),b],\rho(c)) + ([\rho(a),\rho(b)],c) = 0$. This observation completes the proof.

Remark. Here we note that Theorem 2.1 and Corollary 2.2 are valid for symmetric Lie algebras over arbitrary fields, in particular over real numbers.

Proposition 2.3. Let L be a symmetric Lie algebra over field F and A be an associative symmetric commutative algebra over the same field, which means that there exists $f \in A^*$ such that $f(xy)$ is a non-degenerate symmetric bilinear form. Then $L \otimes_F A$ is a symmetric Lie algebra with respect to the operation $[\ell_1 \otimes a_1, \ell_2 \otimes a_2] = [\ell_1,\ell_2] \otimes a_1a_2$.

If B is the invariant symmetric non-degenerate bilinear form on L then

$$(B\otimes f)(\ell_1\otimes a_1,\ell_2\otimes a_2)=B(\ell_1,\ell_2)\cdot f(a_1a_2)$$

is the form which defines the structure of the symmetric Lie algebra on $L\otimes_F A$.

Proof. Clearly $L\otimes_F A$ is a Lie algebra with respect to the indicated operation and $B\otimes f$ is the symmetric non-degenerate form.

The invariancy of $B\otimes f$ is also clear because

$$\begin{aligned}&(B\otimes f)([\ell_1\otimes a_1,\ell_2\otimes a_2],\ell_3\otimes a_3)=\\ =&B([\ell_1,\ell_2],\ell_3)f(a_1a_2a_3)=B(\ell_1,[\ell_2,\ell_3]\cdot f(a_1a_2a_3)=\\ =&(B\otimes f)(\ell_1\otimes a_1,\ [\ell_2\otimes a_2,\ell_3\otimes a_3])\ .\end{aligned}$$

Corollary 2.4. Let L be a symmetric Lie algebra and $W\subset L((u^{-1}))$ satisfying the conditions of Theorem 2.1. If A is a symmetric associative algebra, the $W\otimes A\subset (L\otimes A)((u^{-1}))$ also satisfies the conditions of Theorem 2.1 and therefore defines a rational solution of CYBE in $L\otimes_F A$.

Example 2.5. Let L be a simple Lie algebra over $\mathbb{C}$ and A be an arbitrary symmetric algebra over $\mathbb{C}$, for instance we can put $A=\mathbb{C}+\mathbb{C}\varepsilon$, where $\varepsilon^2=0$ and $f(a+b\varepsilon,c+d\varepsilon)=ad+bc$. Then the results of [S1,S2] provide infinitely many solutions of CYBE in $L\otimes_{\mathbb{C}} A$.

We will now consider an example of this type in the next section.

3. Rational solutions of CYBE and the quasi-maximal orders in $s\ell(n,\mathbb{C}[\varepsilon])$.

Let $\mathbb{C}[\varepsilon]$ be the symmetric associative algebra defined in Example 2.5. It is easily seen that $s\ell(n,\mathbb{C}[\varepsilon])$ is a Manin triple $(s\ell(n,\mathbb{C}[\varepsilon]),s\ell(n,\mathbb{C}),s\ell(n,\mathbb{C})^*)$ with the trivial Lie algebra structure on $s\ell(n,\mathbb{C})^*$.

Definition 3.1. Let L be a symmetric Lie algebra. We say that the subalgebra $W\subset L((u^{-1}))$ is an order if $u^{-N}L[[u^{-1}]]\subseteq W\subseteq u^kL[[u^{-1}]]$ for some integer n, K.

Lemma 3.2. Let L be a symmetric Lie algebra, $r(u,v)=\frac{t}{u-v}+p(u,v)$ be a rational solution of CYBE and W be the corresponding subalgebra. Then W is an order in $L((u^{-1}))$.

Proof. We know that $W=W^\perp$ and $W\supseteq u^{-N}L[[u^{-1}]]$ for some N. This implies that $W^\perp\subseteq(u^{-N}L[[u^{-1}]])^\perp=u^{N-1}L[[u^{-1}]]$. Thus the lemma is proved.

Example 3.3. Let $L=s\ell(n,\mathbb{C}[\varepsilon])$. Then $W_k=s\ell(n,\mathbb{C}[[u^{-1}]])+u^ks\ell(n,\mathbb{C}[[u^{-1}]])\cdot\varepsilon$ is an order for any k.

For the simple complex Lie algebras the theory of orders was developed in [S1, S2, S3]. The point is that any order in this case can be embedded into a maximal order and the set of maximal orders can be described. More exactly, there is a one-to-one correspondence between the maximal orders in $g((u^{-1}))$ and the co-root lattice in $H_{\mathbb{R}}$, where H is the Cartan subalgebra of g.

If we consider an order W corresponding to a rational solution then $W\subset\mathbb{O}_\alpha$ for some maximal order $\mathbb{O}_\alpha$. Such a maximal order $\mathbb{O}_\alpha$ satisfies $\mathbb{O}_\alpha+g[u]=g((u^{-1}))$ (***) because of 2.1 (ii).

The set of the maximal orders satisfying $(***)$ turns out to be finite and is in a one-to-one correspondence with the vertices of the extended Dynkin diagram for g. This is the basis for classification of rational solutions of CYBE for simple complex Lie algebras (see [S1, S2, S3] for details).

For $L = s\ell(n,\mathbb{C}[\varepsilon])$ unfortunately there are no maximal orders as Example 3.3 shows. However the following result is valid.

Theorem 3.4. Any order W in $s\ell(n,\mathbb{C}[\varepsilon])((u^{-1}))$ can be embedded into an order of the form gW_kg^{-1}, where $g \in GL(n,\mathbb{C}((u^{-1})))$ and W_k is as in Example 3.3.

Moreover, there exists a gauge transformation $g_1 \in GL(n,\mathbb{C}[u])$ such that $g_1Wg_1^{-1} \subseteq dW_kd^{-1}$ where

$$d = diag(u^{k_1},\cdots,u^{k_n}) \quad with \quad k_1 \leq k_2 \cdots \leq k_n \,.$$

If W satisfies $W + s\ell(n,\mathbb{C}[\varepsilon])[u] = s\ell(n,\mathbb{C}[\varepsilon])((u^{-1}))$, then $k_i = 0, 1$.

Proof. Let us consider any order $W \subset u^k sl(n,\mathbb{C}[\varepsilon])([[u^{-1}]]$. Since $\varepsilon \cdot sl(n,\mathbb{C}((u^{-1})))$ is an ideal in $sl(n,\mathbb{C}[\varepsilon])((u^{-1}))$, we have a canonical projection $j : sl(n,\mathbb{C}[\varepsilon])((u^{-1})) \to sl(n,\mathbb{C}((u^{-1})))$.

Clearly $j(W)$ is an order in $sl(n,\mathbb{C}((u^{-1})))$ and $W \subseteq j(W) + \varepsilon \cdot u^k sl(n,\mathbb{C}[[u^{-1}]])$.

It was proved in [S1] that $j(W) \subseteq g \; sl(n,\mathbb{C}[[u^{-1}]])g^{-1}$ for some $g \in GL(n,\mathbb{C}((u^{-1}))$. Hence

$$W \subseteq g \; sl(n,\mathbb{C}[[u^{-1}]])g^{-1} + \varepsilon g\{u^k \cdot g^{-1} \cdot sl(n,\mathbb{C}[[u^{-1}]])g\}g^{-1} \,.$$

Clearly, $g^{-1}sl(n,\mathbb{C}[[u^{-1}]])g \subseteq u^{k_1} sl(n,\mathbb{C}[[u^{-1}]])$. Thus, $W \subseteq gW_{k_2}g^{-1}$ for some k_2.

It follows from the results of [S1] that there exists $g_1 \in GL(n,\mathbb{C}[u])$ $g_2 \in GL(n,\mathbb{C}[[u^{-1}]])$ such that $g = g_1dg_2$ with $d = diag(u^{k_1},\cdots,u^{k_n})$, $0 = k_1 \leq k_2 \cdots \leq k_n$.

Finally, let $W + sl(n,\mathbb{C}[\varepsilon])[u] = sl(n,\mathbb{C}(\varepsilon))((u^{-1}))$ and let $W \subseteq gW_kg^{-1} = g_1dW_kd^{-1}g_1^{-1}$ since $g_2W_kg_2^{-1} = W_k$. Therefore $g_1dW_kd^{-1}g_1^{-1} + sl(n,\mathbb{C}[\varepsilon])[u] = sl(n,\mathbb{C}[\varepsilon])((u^{-1}))$ and hence,

$$dW_kd^{-1} + sl(n,\mathbb{C}[\varepsilon])[u] = sl(n,\mathbb{C}[\varepsilon])((u^{-1}))$$

since

$$g_1^{-1}sl(n,\mathbb{C}[\varepsilon])[u]g_1 = sl(n,\mathbb{C}[\varepsilon])[u] \,.$$

Clearly, the latter implies that

$$d \; sl(n,\mathbb{C}[[u^{-1}]])d^{-1} + sl(n,\mathbb{C}[u]) = sl(n,\mathbb{C}((u^{-1})))$$

that provides the required form for d. Thus the theorem is proved.

Remark 3.5. Solutions of CYBE constructed in Corollary 2.4 correspond to the orders imbedded into W_0. We will call W_k quasi-maximal orders.

Now we are going to describe the solutions of CYBE such that the corresponding orders can be embedded into W_k, $k \geq 1$.

Lemma 3.6.

$$W_k/W_k^{\perp} \cong sl(n,\mathbb{C}) + \omega^{-1}\cdot sl(n,\mathbb{C}) + \cdots + \cdots \omega^{-k-1} sl(n,\mathbb{C}) +$$

$$+\varepsilon sl(n,\mathbb{C})\cdot \omega^k \cdots + \varepsilon\ sl(n,\mathbb{C})\cdot \omega^{-1} .$$

Here ω is the image of u and informally saying $\omega^{-k-2} = \omega^k = 0$. Further $W_k/W_k^{\perp} \cong S + \varepsilon S^*$, where S is a graded Lie algebra $S = sl(n,\mathbb{C}[\omega^{-1}]/(\omega^{-k-2})) = sl(n,\mathbb{C}) + \omega^{-1} sl(n,\mathbb{C}) \cdots + \omega^{-k-1} sl(n,\mathbb{C})$ and S^* is dual to S. The proof is straightforward.

Let P_i be a parabolic subalgebra of $sl(n)$ corresponding to the i-th simple root α_i, i.e. is generated by all root vectors corresponding to the simple roots $\alpha_1, \cdots, \alpha_{k-1}$ and their opposite exect for $(-\alpha_i)$.

Let $d_i = diag(\underbrace{1,\cdots,1}_{i}, u, \cdots, u) \in GL(n,\mathbb{C}((u^{-1})))$.

Lemma 3.7.

$$T_{k,i} = W_k \cap d_i^{-1} sl(n,\mathbb{C}[\varepsilon])[u] d_i$$

$$= (P_i + u^{-1}P_i^{\perp}) + \varepsilon P_i^{\perp}\cdot u^{-1} + \varepsilon P_i + \varepsilon\cdot sl(n,\mathbb{C})\cdot u + \cdots + \varepsilon\cdot sl(n,\mathbb{C}) u^k .$$

Here $P_i^{\perp}$ is the orthogonal complement to P_i in $sl(n,\mathbb{C})$ with respect to the Killing form.

Proof: Direct computations.

Now let $r(u,v)$ be a rational solution of CYBE for $sl(n,\mathbb{C}[\varepsilon])$ and W the corresponding order. Using a gauge transformation from $GL(n,\mathbb{C}[u])$ we can consdier W as a subalgebra in $d_i W_k d_i^{-1}$ for some i and k.

Lemmas 3.6 and 3.7 show that we can view $T_{k,i}$ as a subalgebra of $S + \varepsilon S^*$, where S was defined in Lemma 3.6.

Note that $S + \varepsilon S^*$ has a natural invariant symmetric non-degenerate bilinear form induced by $(a, \varepsilon b) = b(a)$. Moreover, $T_{k,i}$ is a Lagrangian subalgebra with respect to this symmetric form, because it is easy to see that $(x,y) = 0$ for any $x, y \in T_{k,i}$ and $dim\ T_{k,i} = \frac{1}{2}\ dim(S + \varepsilon S^*)$.

Proposition 3.8. Let W satisfy the conditions of Theorem 2.4 and $W \subset d_i W_k d_i^{-1}$. Then the image of $d_i^{-1} W d_i$ in $W_k/W_k^{\perp} = S + \varepsilon S^*$ is a Lagrangian subalgebra $\tilde{W}$ such that $\tilde{W} \oplus T_{k,i} = S + \varepsilon S^*$.

Conversely, if $\tilde{W}$ is a Lagrangian subalgebra of $S + \varepsilon S^*$ such that $\tilde{W} \oplus T_{k,i} = S + \varepsilon S^*$ and W_0 is the preimage of $\tilde{W}$ in W_k, then $d_i W_0 d_i^{-1} \subset d_i W_k d_i^{-1}$ satisfies all the conditions of Theorem 2.4 and therefore provides a rational solution of CYBE for $sl(n,\mathbb{C}[\varepsilon])$.

Proof. All the statements follow from Lemmas 3.6 and 3.7.

Theorem 3.9. Let $\tilde{W}$ be a Lagrangian subalgebra of $S + \varepsilon S^*$ such that

$$\tilde{W} \oplus T_{ik} = S + \varepsilon S^* .$$

Then there exists a one-to-one correspondence between the set of such a $\tilde{W}$ and the set of the subalgebras M of S such that

(i) $M + (P_i + \omega^{-1} P_i^{\perp}) = S$

(ii) There exists a 2-cocycle B on M that non-degenerate on $M \cap (P_i + \omega^{-1}P_i^{\perp})$.

Note that $P_i + \omega^{-1}P_i^{\perp}$ is a subalgebra of S since $P_i^{\perp}$ is a commutative subalgebra of $sl(n,\mathbb{C})$.

Proof. Consider the image of $\tilde{W}$ under the canonical projection $j : S + \varepsilon S^* \to S$ (εS^* is an ideal in $S + \varepsilon S^*$). Let $M = j(\tilde{W})$. Since $j(T_{k,i}) = P_i + \omega^{-1}P_i^{\perp}$ we have

$$M + (P_i + \omega^{-1}P_i^{-1}) = S .$$

Let $M^{\perp} \subset S^*$ be the set $\{s \in S : s(M) = 0\}$. Clearly $\tilde{W} \subset M + \varepsilon S^*$ and since $\tilde{W}^{\perp} = \tilde{W}$ ($\tilde{W}$ is Lagrangian with respect to the canonical form in $S + \varepsilon S^*$), $\tilde{W} \supset (M + \varepsilon S^*)^{\perp} = \varepsilon M^{\perp}$. Therefore, $\tilde{W}$ is uniquely defined by its image W_0 in $(M + \varepsilon S^*)/\varepsilon M^{\perp} \cong M + \varepsilon M^*$. It is not difficult to see that $M + \varepsilon M^*$ has a canonical invariant symmetric non-degenerate form induced by $(m, \varepsilon a) = a(m)$ and W_0 is a Lagrangian subalgebra with respect to this form. We have: $dim\ W_0 = dim\ M = \frac{1}{2} dim(M + \varepsilon M^*)$ and W_0 is projected onto M under the canonical projection $M + \varepsilon M^* \to M$. This implies that $W_0 = \{a + \varepsilon f(a) : a \in M\}$, where f is a linear map $f : M \to M^*$. Then B is the image of f under the canonical isomorphism $Hom(M, M^*) \xrightarrow{\sim} M^* \otimes M^*$. B is skew-symmetric since W_0 is Lagrangian. The fact that B is a 2-cocycle i.e., satisfies $B([x,y],z) + B([z,x],y) + B([y,z],x) = 0$ for all $x, y, z \in M$ can be established exactly in the same way as Corollary 2.2.

It remains to prove that B is non-degenerate on $(P_i + \omega^{-1}P_i^{\perp}) \cap M$.

First we recover $\tilde{W}$ from (M, B) or which is the same from the pair $(M,\ f : M \to M^*)$, namely

$$\tilde{W} = \{a + \varepsilon f(a) + \varepsilon m^{\perp} : a \in M, m^{\perp} \in M^{\perp}\} .$$

If B is degenerate on $M \cap (P_i + t^{-1}P_i^{\perp})$, there exists $0 \neq a \in M \cap (P_i + \omega^{-1}P_i^{\perp})$ such that $f(a) = 0$. Then $a \in \tilde{W}$ (we just put $m^{\perp} = 0$) and simultaneously $a \in P_i + \omega^{-1}P_i^{\perp} \subset T_{k,i}$. Hence, $\tilde{W} \cap T_{k,i} \neq (0)$ which is a contradiction. The theorem is proved.

Now we are ready to construct a series of examples of the rational solutions.

The idea is to set $M = S$ in notations of the Theorem 3.9. Then the condition (i) of Theorem 3.9 is satisfied and all we need to the existence of B on $M = S^i$ such that B is non-degenerate on $M \cap (P_i + \omega^{-1}P_i^{\perp}) = P_i + \omega^{-1}P_i^{\perp}$.

Now we would like to remind a definition of the Frobenius Lie algebra.

Definition 3.10. A Lie algebra L is called Frobenius if there exists $f \in L^*$ such that $f([x,y])$ is a non-degenerate bilinear form on L.

For a general discussion on the Frobenius Lie algebras see [E, S1]. Note that over $\mathbb{R}, \mathbb{C}$ one has another definition for the Frobenius Lie algebras.

Definition 3.11. A Lie algebra L over $\mathbb{R}$ or $\mathbb{C}$ is called Frobenius if the coadjoint representation ad^* is locally transitive (see [E, 0]).

A bilinear form of the type $f([x,y])$ is a 2-cocycle and even a 2-coboundary.

Theorem 3.9 implies the following result.

Corollary 3.12. Let $P_i + \omega^{-1}P_i^{\perp}$ be a Frobenius Lie algebra with the linear form f. Extend f to S in any way and put $B(x,y) = f([x,y])$ on S. Then the pair

(S,B) satisfies the conditions (i), (ii) of theorem 3.9 and thus provides a rational solution of CYBE.

Example 3.13. Let $i=1$. Then

$$P_1+\omega^{-1}P_1^{\perp}\subset S=sl(n,\mathbb{C}[\omega^{-1}]/(\omega^{-k-2}))$$

is isomorphic to the semidirect product

$$gl(n-1,\mathbb{C})\oplus(\mathbb{C}^{n-1}\oplus\mathbb{C}^{n-1})\ .$$

Here $gl(n-1,\mathbb{C})$ acts on each copy of $\mathbb{C}^{n-1}$ in the standard way. It was proved in [E] that $gl(r,\mathbb{C})\oplus\underbrace{(\mathbb{C}^r\oplus\cdots\oplus\mathbb{C}^r)}_{p\ times}$ is Frobenius if and only if p divides r.

Hence, $P_1+\omega^{-1}P_1$ is Frobenius if and only if n is odd.

Conclusion. We have constructed rational solutions for any odd n and any $k\geq 1$.

Remark 3.14. Since it is possible to embed $sl(n,\mathbb{C}[\varepsilon])\to sl(2n,\mathbb{C})$, one can pose a natural question whether the solutions constructed above fall into the classification theory obtained in [BD, S1, S2]? The answer is negative because these solutions are not "non-degenerate" in the sense of [BD].

4. Rational solutions of CYBE for gl(n).

Now $gl(n,F)$ is a symmetric Lie algebra with respect to the form $Tr(xy)$. Here F is any field. Let e_{ij} be the matrix units. Then the corresponding invariant element $t=\sum_{i,j}e_{ij}\otimes e_{ji}\in gl(n,F)\otimes gl(n,F)$ acts on $F^n\otimes F^n$ as the twist map $t(a\otimes b)=b\otimes a$. This twist map usually is denoted by P and we write P instead of t in this case. The theory of the rational solutions of CYBE for $gl(n,\mathbb{C})$ is very similar to those of $sl(n,\mathbb{C}[\varepsilon])$. Since the proofs are also rather similar, we omit them and make just corresponding statements.

Theorem 4.1. Let z be a central element of $gl(n,\mathbb{C})$. Then any order in $gl(n,\mathbb{C}((u^{-1})))$ can be embedded into an order of the form $g(sl(n,\mathbb{C}[[u^{-1}]]+z\cdot u^k\cdot\mathbb{C}[[u^{-1}]])g^{-1}$ for some $g\in GL(n,\mathbb{C}((u^{-1})))$.

Applying a gauge transformation (an element from $GL(n,\mathbb{C}[u])$) we can achieve that $g=diag(u^{k_1},\cdots,u^{k_n})$ with $0=k_1\leq\cdots\leq k_n$.

Proposition 4.2. Let the order W correspond to a rational solution of CYBE by Theorem 2.1. Then for a maximal order, which contains W, we have $k_i=0,1$ and $k\geq -1$ in the notations of Theorem 4.1.

Theorem 4.3. Let W be the order corresponding to a rational solution of CYBE for $gl(n)$, which is embedded into $d_i(sl(n,\mathbb{C}[[u^{-1}]]+z\cdot u^k\mathbb{C}[[u^{-1}]])d_i^{-1}$, where

$$d_i=diag(\underbrace{1\cdots 1}_{i},u\cdots u)\ .$$

Let $S=sl(n,\mathbb{C})+z\cdot\mathbb{C}[u]/(u^{k+1})$ and P_i be the same as in Lemma 3.7. Then the set of such W is in a one-to-one correspondence with the set of pairs (M,B) satisfying the following two conditions

(i) $M + (P_i + z \cdot \mathbb{C}[u]/(u^{k+1}) = S$
(ii) B is a 2-cocycle on M and non-degenerate on $M \cap (P_i + z \cdot \mathbb{C}[u]/(u^{k+1}))$.

Moreover, since S is a splitting central extension of $sl(n)$, we see that $M = M_1 \oplus M_2$, where M_1 is a subalgebra of $sl(n,\mathbb{C})$ such that $M_1 + P_i = sl(n,\mathbb{C})$ and $M_2 \subseteq z \cdot \mathbb{C}[u]/(u^{k+1})$.

Remark 4.4. The pairs (M_1, B) are classified in [S1] if B is non-degenerate on $M_1 \cap P_i$.

Theorem 4.3 provides a classification of rational solutions of CYBE for $gl(n,\mathbb{C})$. For instance the following statement is obtained:

Corollary 4.5. Consider $gl(2,\mathbb{C})$. Then all rational solutions fall into the following classes:

I. $W \subset d_i(sl(2,\mathbb{C}[[u^{-1}]] + z \cdot u^k\mathbb{C}[[u^{-1}]])d_1^{-1}$. Then $M_1 = sl(2,\mathbb{C})$, M_2 is any even-dimensional subspace of $\mathbb{C}[u]/(u^{k+1})$
$B = B_1 \otimes B_2$, where B_i is defined on $M_i (i = 1, 2)$ as follows:
$B_1(x, y) = f([x, y])$, where $f(A) = a_{12}$ for any $A \in gl(2)$.
B_2 is any non-degenerate skew-symmetric form on M_2.

II. $W \subset sl(2,\mathbb{C}[[u^{-1}]]) + z \cdot u^k\mathbb{C}[[u^{-1}]]$.
Then either:

$$\begin{aligned} M_1 &= \{Borel\ subalgebra\ of\ \ sl(2,\mathbb{C})\} \\ M_2 &= \{any\ even\ dimensional\ subspace\ of\ \ \mathbb{C}[u]/(u^{k+1})\} \\ B|_{M_1} &= B_1\ \ from\ part\ I \\ B|_{M_2} &= B_2\ \ from\ part\ I \end{aligned}$$

B is subject to a cross-relation, namely $B(e_{12}, M_2) = 0$;
or

$$\begin{aligned} &M_1\ is\ a\ 1-dimensional\ subspace\ of\ sl(2,\mathbb{C}), \\ &M_2\ is\ any\ odd\ dimensional\ subspace\ in\ \mathbb{C}[u]/(u^{k+1}), \\ &B\ is\ any\ nondegenerate\ skewform\ on\ M_1 \oplus M_2 \end{aligned}$$

Proof. Consider Case I: i.e. $W \subset d_1(sl(2,\mathbb{C}[[u^{-1}]] + z \cdot u^k\mathbb{C}[[u^{-1}]])d_1^{-1}$.

It follows from [S1] that $M_1 = sl(2,\mathbb{C})$, otherwise there exists a gauge transformation from $GL(n,\mathbb{C}[u])$ mapping W to $sl(2,\mathbb{C}[[u^{-1}]]) + z \cdot u^k\mathbb{C}[[u^{-1}]]$.

Further, since $[M_1, M_2] = M_1$, we see that $B(M_1, M_2) = 0$ because B is a 2-cocycle and M_2 is in the center. Now it is clear that $B = B_1 \oplus B_2$ and B_1 is non-degenerate on $M_1 \cap P_1 = P_1$ while B_2 is non-degenerate on M_2. This suffices to establish all the claims in the case I. Now we go on to the case II. We have just two possibilities mentioned above for M_1. Let us assume that $M_1 = \{Borel\ subalgebra\ of\ sl(2)\}$.

We would like to prove that B, which is non-degenerate on $M_1 \oplus M_2$, is non-degenerate on each M_i. The Borel subalgebra has two generators h and b satisfying $[h, b] = b$. Then $B(b, M_2) = 0$ (again because B is a 2-cocycle and M_2 in the center) and if B was degenerate on M_1, we get that $B(b, M) = 0$ which contradicts to the non-degeneracy of B on M.

Suppose now that B is degenerate on M_2 and since B is skew-symmetric, there exists at least two independent vectors d_1, $d_2 \in M_2$ such that $B(b, d_i) = 0$. Let

$B(d_i, h) = \lambda_i$. Then $B(\lambda_1 d_2 - \lambda_2 d_1, h) = 0$ and $B(\lambda_1 d_2 - \lambda_2 d_1, M) = 0$. Hence, we have checked all the claims for B in this case.

The remaining case is obvious. The proof is complete

5. Rational solutions of QYBE and symmetric associative algebras

Definition 5.1. An associative finite dimensional algebra A over a field F is called symmetric if there exists $f \in A^*$ such that $f(xy)$ is a non-degenerate symmetric form on A.

Example 5.2. The set of all $n \times n$ matrices $M_n(F)$, $f(X) = tr(X)$. Note that we used the notation $gl(n, F)$ for the same object considered as a Lie algebra.

Example 5.3. Let H be a finite dimensional unimodular almost co-commutative Hopf algebra over F. Then H is symmetric.

Proof. It is known that H is a Frobenius associative algebra which means that there exists $f_1 \in H^*$ such that $f_1(xy)$ is a non-degenerate bilinear form (see [LS, P]). Moreover, there exists a so-called Nakayama automorphism $\alpha : H \to H$ such that $f_1(xy) = f_1(y\alpha(x))$ for all $x, y \in H$.

It was proved in ([LS, BF1]) that $\alpha = S^2$ for unimodular Hopf algebras, where S is the antipod in H and S^2 is an inner automorphism for almost co-commutative Hopf algebras ([D3, M]).

Let $S^2(a) = vav^{-1}$ for some invertible element $v \in H$. Then $f(xy) = f_1(xyv)$ is a non-degenerate symmetric bilinear form in x, y. Note that any Drinfeld's double is a Hopf algebra of this kind.

Lemma 5.4. Consider a symmetric associative algebra A and define the Lie algebra structure on A by $[a, b] = ab - ba$. (The standard notation is $A^{(-)}$). Then $A^{(-)}$ is a symmetric Lie algebra with respect to $(a, b) = f(ab)$.

The proof is straightforward.

Corollary 5.5. Let $t \in A^{(-)} \otimes A^{(-)}$ be the invariant element constructed in Lemma 1.5 and $r \in A^{(-)} \otimes A^{(-)}$. Then $r + \frac{t}{u-v}$ satisfies CYBE if and only if r satisfies CYBE.

We are going to quantize this rational solution of CYBE. This means that we are looking for a solution of the quantum Yang-Baxter equation (QYBE)

$$R^{12}(u_1 - u_2)R^{13}(u_1 - u_3)R^{23}(u_2 - u_3) = R^{23}(u_2 - u_3)R^{13}(u_1 - u_3)R^{12}(u_1 - u_2)$$

in the form $R(u) = 1 \otimes 1 + hR_1 + ... + h^n R_n + ..$ such that R_i are rational functions with values in $A \otimes A$ and $R_1(u) = r + \frac{t}{u}$.

Let us assume that the algebra A is defined over real or complex numbers. Since $r \in A^{(-)} \otimes A^{(-)}$ is a constant solution of CYBE, one can quantize it in the following sense: there exists

$$R_h = 1 \otimes 1 + \sum_{k=1}^{\infty} h^k R_k \in U(A^-)[[h]] \otimes U(A^-)[[h]]$$

satisfying QYBE with $R_1 = r$ and $R_h \cdot R_h^{21} = 1 \otimes 1$ (see [D]). According to the PBW-theorem there is a canonical homomorphism $\alpha : U(A^-)[[h]] \to A[[h]]$. Clearly, $R = \alpha(R_h) \in A \otimes A$ satisfies QYBE and is of the form $R = 1 \otimes 1 + hr + ...$

In what follows we need a result concerning the element t obtained in [BF2].
Theorem 5.6. The element $t \in A \otimes A$ has the following properties:
(i) $(a \otimes b)t = t(b \otimes a)$ for all $a, b \in A$;
(ii) $t^{12}t^{13} = t^{23}t^{12} = t^{13}t^{23}$ *and* $t^{12}t^{23} = t^{23}t^{13} = t^{13}t^{12}$;
(iii) t satisfies QYBE without parameter.

It turns out that having R, which quanizes r and using the properties of t one can quantize $r + \frac{t}{u}$. More exactly:

Theorem 5.7. $R + \frac{ht}{u}$ satisfies QYBE with parameter and quantizes $r + \frac{t}{u}$.

Proof. Taking into account that R and t satisfy QYBE to prove that $R + \frac{ht}{u}$ is a solution of QYBE it suffices to verify the following equalities:
$R^{12}R^{13}t^{23} = t^{23}R^{13}R^{12}$;
$t^{12}R^{13}R^{23} = R^{23}R^{13}t^{12}$;
$R^{12}t^{13}R^{23} = R^{23}t^{13}R^{12}$;
$t^{12}t^{13}R^{23} = t^{23}t^{13}R^{12}$;
$t^{12}R^{13}t^{23} = R^{23}t^{13}t^{12}$;
$R^{12}t^{13}t^{23} = t^{23}R^{13}t^{12}$;
Let us check for instance the fourth one:

$$t^{12}t^{13}R^{23} = t^{12}R^{21}t^{13} = R^{12}t^{12}t^{13}$$

because of Theorem 5.6 (i).

By the same arguments $t^{23}t^{13}R^{12} = R^{12}t^{23}t^{13}$. Then Theorem 5.6 (ii) provides the required equality. All the other equalities can be established in a similar way. Obviously $R + \frac{ht}{u}$ quantizes $r + \frac{t}{u}$. The theorem is proved.

Final remarks. If $A = gl(n)$ and $r = 0$, then one gets exactly the Yang solution of QYBE (see **Introduction**). In this case (*i.e.* $A = gl(n)$) it is possible to employ Yangians to prove Theorem 5.7. Moreover, the representation theory of Yangians enables one to construct infinitely many rational solutions of QYBE having given r and the simpliest one is exactly that of Theorem 5.7 (see [KST]). Probably this indicates that there exist Yangians for symmetric Lie algebras.

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References.

[BD] Belavin, A., Drinfeld, V. *On classical Yang-Baxter equation for simple Lie algebras,* Funct. Anal. Appl., v. 16, 1982.

[BF1] Beidar, K., Fong, Y., Stolin, A. *On antipodes and integrals in Hopf algebras over rings and QYBE,* preprint, 1995.

[BF2] Beidar, K., Fong, Y., Stolin, A. *On Frobenius algebras and QYBE,* to appear in Trans. of Am. Math. Soc.

[D] Drinfeld, V. *On constant quasiclassical solutions of the quantum Yang-Baxter equation,* Sov. Math. Dokl., v. 28, 1983.

[D1] Drinfeld, V. *Quasi-Hopf algebras,* Leningrad Math. J.,v. 1, 1990.

[D2] Drinfeld, V. *Quantum Groups,* Proc. ICM-86, v. 1, 1987.

[D3] Drinfeld, V. *On almost cocommutative Hopf algebras,* Leningrad Math. J., v. 1, 1990.

[E] Elashvili, A. *Frobenius Lie algebras,* Funct. Anal. Appl., v. 16, 1982.

KST] Khoroshkin, S., Stolin, A., Tolstoy, V. *Rational solutions of QYBE and deformations of Yangians,* preprint, 1996.

[LS] Larson, R., Sweedler, M.*An associative orthogonal bilinear form for Hopf algebras,* Amer. J. Math., v. 91, 1969.

[M] Montgomery, S. *Hopf algebras and their action on rings,* Regional Conference Series in Math., v. 82, AMS, Providence, Rhode Island, 1993.

[S1] Stolin, A. *On rational solutions of Yang-Baxter equation for* $sl(n)$, Math. Scand., v. 69, 1991.

[S2] Stolin, A. *On rational solutions of Yang-Baxter equations. Maximal orders in loop algebras,* Comm. Math. Phys., v. 141, 1991.

[S3] Stolin, A. *Geometrical approach to rational solutions of CYBE,* Proc. of Symposia Gaussiana, Walter de Gruyter, 1995.

First two athors's address: National Cheng-Kung University, Taiwan

Third athor's address : Royal Institute of Technology, Sweden