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SPECIAL CONNECTIONS ON SMOOTH 3-WEB MANIFOLDS

ALENA VANŽUROVÁ

ABSTRACT. In this paper we will investigate connections useful by studying 3-webs on manifolds, especially Chern connections and parallelizing connections.

All objects under consideration (manifolds, bundles, vector or tensor fields etc.) are supposed to be of the class C^∞ (smooth). We will denote by M a manifold, by TM the corresponding tangent bundle, and by $\mathfrak{X}(M)$ the family of all vector fields on M.

1. THREE-WEBS AND PROJECTOR TENSOR FIELDS

The geometry of three systems of curves in a plane studied by W. Blaschke and his co-workers in 1930' has a natural generalization. The investigation of higher-dimensional cases was initiated by G. Bol and is due especially to S.S. Chern [8], M.A. Akivis and V.V. Goldberg.

A differentiable 3-web $\mathcal{W}$ of dimension n (of codimension n) in a $2n$-dimensional manifold M is a triple $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ of n-codimensional foliations on an open subset U of M which are in general position. More precisely, at any point $x \in U$ there is exactly one leaf $L^{(\alpha)}$ of $\mathcal{F}_\alpha$ through x ($\alpha = 1, 2, 3$), the tangent spaces to the leaves of different foliations have only zero vector in common, $T_xL^{(\alpha)} \cap T_xL^{(\beta)} = 0$, and the tangent space is their sum, $T_xL^{(\alpha)} \oplus T_xL^{(\beta)} = T_xM$.

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In most papers on this subject [1, 5, 6, 7, 8, 9], the web foliations are given by involutive systems of differential forms (completely integrable systems of Pfaffian equations),

$$\underset{1}{\omega}^i = 0, \quad \underset{2}{\omega}^i = 0, \quad \underset{3}{\omega}^i = 0, \quad \underset{1}{\omega}^i + \underset{2}{\omega}^i + \underset{3}{\omega}^i = 0, \quad i = 1, \dots, n$$

which admit coordinate transformations of the type $\underset{\alpha}{\omega}^{i'} = A_i^{i'} \underset{\alpha}{\omega}^i$ only. In this case each family of 1-forms $(\underset{\alpha}{\omega}^i \,|\, \underset{\beta}{\omega}^i)$, $\alpha \neq \beta$ constitues an adapted unholonomic co-frame [7]. Evaluations with respect to a co-frame $(\underset{1}{\omega}^i \,|\, \underset{2}{\omega}^i)$ and Cartan methods are usually used to investigate invariants of a 3-web under foliations-preserving local diffeomorphisms.

We will apply here a dual approach making use of adapted holonomic frames, adapted frame bundles, invariant projector tensor fields associated with a web, and Nijenhuis tensors. It anables us to work on the tangent bundle only, or on the bundle of adapted frames. This view-point appears in [10, 11, 13, 14, 15, 16], recently also in [12].

Definition 1.1. For our purpose let us introduce an *ordered (unordered) 3-web* on a manifold M of dimension $2n$ as an ordered (unordered) triple (D_1, D_2, D_3) of (smooth) n-dimensional distributions each couple of which forms an almost product structure $[D_\alpha, D_\beta]$; that is, the intersections $D_\alpha \cap D_\beta$ are trivial at each point, and the tangent bundle is a Whitney sum $D_\alpha \oplus D_\beta = TM$.

Two 3-webs $\mathcal{W}$, $\mathcal{W}'$ are *locally equivalent* if there exists a local diffeomorphism of their definition domains such that its tangent map respects web distributions: $Tf(D_\alpha) = D'_\alpha$, $\alpha = 1, 2, 3$; f is called a *local equivalence*. Equivalences (global diffeomorphisms saving web-distributions) can be introduced in a similar way but it is rather uneasy to obtain global results. The web geometry is usually concerned with investigations of invariants under the family (pseudogroup) of local equivalences of webs.

1.1. Recall that a distribution D of dimension n on M is an n-dimensional subbundle $D : M \to TM$ (assigning to each $x \in M$ an n-dimensional subspace D_x of the tangent space $T_x M$). Equivalently, a smooth distribution can be characterized by the local existence of a frame field (a linear basis constituted by an n-tuple of smooth vector fields on M). D is called *integrable* if for any couple of sections (vector fields) $X : M \to D$, $Y : M \to D$ their Lie bracket $[X, Y]$ is also a section of D. Shortly, $X, Y \in D \Longrightarrow [X, Y] \in D$. The integrability of D is equivalent with the existence of a local chart $(U; x^1, \dots, x^m)$ about each point $x \in M$ such that the first n-tuple of coordinate fields $\partial/\partial x^1, \dots, \partial/\partial x^n$ span D in a nbd U of x. Another characterization is: D is integrable if and only if for any projector P projecting onto D, the Nijenhuis bracket[2] vanishes on D. A *projector*[3] (projection tensor) is a $(1,1)$-tensor field P on M satisfying $P^2 = P$.

1.2. In the case of a 3-web manifold, the decompositions $TM = D_\alpha \oplus D_\beta$ of the tangent bundle yield almost-product structures $[D_\alpha, D_\beta]$, and six projectors P^β_α on M associated with the web such that $\ker P^\alpha_\beta = \operatorname{im} P^\beta_\alpha = D_\alpha$. Since the almost product structure $[D_1, D_2]$ is integrable[4] there exist local coordinates $(x^i \mid x^{n+i})$ in a neighborhood of any point such that $\partial_i = \partial/\partial x^i$ span D_1, and $\partial_{n+i} = \partial/\partial x^{n+i}$ span D_2, $i = 1, \dots, n$. Such coordinate systems will be called *adapted* to the structure $[D_1, D_2]$. By the above, $[D_1, D_2]$ is integrable iff both D_1 and D_2 are integrable which can be expressed by a single condition on the Nijenhuis bracket of the projector, $[P^2_1, P^2_1] = 0$.

The web is fully determined by a (suitably chosen) couple of the web-associated projectors [11, 13], and local equivalences are exactly such (local) diffeomorphisms f of the underlying manifolds the tangent map Tf of which is interchangeable with projectors, $Tf \circ P^\beta_\alpha = {P'}^\beta_\alpha \circ Tf$.

The web-projectors are characterized by the properties

(i) $P^\alpha_\beta P^\gamma_\beta = P^\gamma_\beta$ for $\alpha \neq \beta \neq \gamma \neq \alpha$,
(ii) $P^\alpha_\beta P^\gamma_\alpha = 0$ for $\gamma \neq \alpha \neq \beta$,
(iii) $P^\beta_\alpha + P^\alpha_\beta = I$,
(iv) $[P^\beta_\alpha, P^\beta_\alpha] = 0$.

Web-associated (1,1)-tensor fields B_γ are introduced by $B_\gamma = P^\beta_\alpha P^\alpha_\gamma + P^\alpha_\beta P^\beta_\gamma = P^\alpha_\gamma - P^\gamma_\beta$. These associated tensor fields are characterized by the conditions

(a) $B_1 + B_2 + B_3 = 0$,
(b) $B_\alpha B_\beta + B_\beta B_\alpha = -I$,
(c) ${B_\alpha}^2 - I = 0$,
(d) $[B_\alpha, B_\alpha] \mid D_\alpha \times D_\alpha = 0$ where $D_\alpha = \{X \mid B_\alpha X = X\} = \ker(B_\alpha - I)$.

They satisfy

(e) $B_\gamma P^\beta_\alpha = P^\alpha_\beta B_\gamma$,
(f) $B_\gamma = B_\gamma P^\beta_\alpha + P^\alpha_\beta B_\gamma$,
(g) $\{X + B_\gamma X; X \in D_\alpha\} = D_\gamma$.

[2] The Nijenhuis bracket of two $(1,1)$-tensor fields P, Q on M is a skew-symmetric $(1,2)$-tensor field, symmetric in P and Q and linear in P, Q over reals, given by the formula $[P,Q](X,Y) = [PX,QY] + [QX,PY] + PQ[X,Y] + QP[X,Y] - P[X,QY] - Q[X,PY] - P[QX,Y] - Q[PX,Y]$. If $PQ = QP$ we define $\{P.Q\}(X,Y) = [PX,QY] + PQ[X,Y] - P[X,QY] - Q[PX,Y]$ which is again a tensor field.

[3] Any projector P on M yields a couple of complementary distributions $D = \operatorname{im} P$, $\tilde{D} = \ker P$ which form an almost product structure $[D, \tilde{D}]$ on M. Conversely, with a couple D, $\tilde{D}$ of complementary distributions on M, we can associate projectors P, $\tilde{P}$ so that $P + \tilde{P} = I$, $P\tilde{P} = \tilde{P}P = 0$, $D = \ker \tilde{P} = \operatorname{im} P$, $\tilde{D} = \ker P = \operatorname{im} \tilde{P}$.

[4] An almost-product structure is integrable if each of the distributions is integrable. Integrability of an almost product structure can be also introduced by existence of local coordinates such that the first family of coordinate fields spans the first distribution, and the second family determines the second one.

Proposition 1.1. [11, 13] *Let P be a projector and B be a $(1,1)$-tensor field on a $2n$-dimensional manifold M such that*

(i) $B^2 = I$,
(ii) $BP = (I-P)B$, *or equivalently* $PB + BP = B$,
(iii) $[P, P] = 0$, $[B, B] \mid \ker(B-I) \times \ker(B-I) = 0$.

Let us denote $D_1 = \ker(I-P)$, $D_2 = \ker P$, $D_3 = \ker(B-I)$. Then (D_1, D_2, D_3) is a three-web on M with $P_1^2 = P$, $B_3 = B$.

1.3. In [11], a $\{P, B\}$-structure on a manifold M was introduced as a couple of $(1,1)$-tensor fields P, B which satisfy on M the polynomial equations

$$P^2 = P, \qquad B^2 = I, \qquad PB + BP = B. \tag{1}$$

The last condition can be written as $PB = (I-P)B$. It follows that M is even-dimensional.

The following equalities can be verified:

$$PB = (I-P)B, \quad B(I-P) = PB, \quad PBP = (I-P)B(I-P) = 0,$$
$$BPB = I - P, \quad B(I-P)B = P.$$

Vector fields (or vectors) which belong to the distribution $D_1 = \ker(I-P)$ or $D_2 = \ker P$ of a $\{P, B\}$-structure will be called *homogeneous*.

A $\{P, B\}$-structure is called *integrable* if all three distributions $D_1 = \ker(I-P)$, $D_2 = \ker P$, and $D_3 = \ker(B-I)$ (all of constant dimension n) are integrable. If this is the case the distributions form an ordered 3-web (D_1, D_2, D_3) on M with $\{P_1^2, B_3\} = \{P, B\}$, and vice versa, any ordered 3-web gives rise to an integrable $\{P, B\}$-structure.

To simplify the notation we will write $\tilde{P}$ instead of $I - P$.

2. The Chern connection of a 3-web

2.1. For any $\{P, B\}$-structure there is a unique connection [11] ∇ such that the fields P and B are covariantly constant with respect to ∇, and each couple of homogeneous vectors at the same point $X \in \ker \tilde{P}$, $Y \in \ker P$ is conjugated with respect to the torsion tensor of ∇, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y]$:

$$\nabla P = 0, \qquad \nabla B = 0, \qquad T(PX, \tilde{P}Y) = 0.$$

This connection is called the *canonical connection* of a $\{P, B\}$-structure, and is given by

$$\begin{aligned}\nabla_X Y =& P(B[PX, BPY] + [\tilde{P}X, PY]) \\ &+ \tilde{P}(B[\tilde{P}X, B\tilde{P}Y] + [PX, \tilde{P}Y]).\end{aligned} \tag{2}$$

An evaluation on homogeneous vector fields gives

$$\begin{aligned} \nabla_{PX}PY &= PB[PX, BPY], \qquad & \nabla_{PX}\tilde{P}Y &= \tilde{P}[PX, \tilde{P}Y], \\ \nabla_{\tilde{P}X}\tilde{P}Y &= \tilde{P}B[\tilde{P}X, B\tilde{P}Y], \qquad & \nabla_{\tilde{P}X}PY &= P[\tilde{P}X, PY]. \end{aligned} \tag{3}$$

The corresponding torsion tensor is given by

$$\begin{aligned} T(X,Y) =&PB\Big([PX, BPY] + [BPX, PY]\Big) + \tilde{P}B\Big([\tilde{P}X, B\tilde{P}Y] \\ &+ [B\tilde{P}X, \tilde{P}Y]\Big) + [\tilde{P}X, PY] + [PX, \tilde{P}Y] - [X,Y]. \end{aligned} \tag{4}$$

The canonical connection of the integrable $\{P, B\}$-structure is called the *Chern connection* of the corresponding ordered 3-web (D_1, D_2, D_3) (or of the underlying 3-web manifold).

2.2. In the following, let us assume an integrable $\{P, B\}$-structure corresponding to some 3-web (D_1, D_2, D_3) formed by integrable distributions. Integrability of $D_3 = \ker(B - I) = \{PX + BPX\,;\, X \in \mathfrak{X}(M)\}$ is equivalent with each of the following conditions:

$$B\{B, B\}(PX, PY) = \{B, B\}(PX, PY), \tag{5}$$

$$B\{B, B\}(\tilde{P}X, \tilde{P}Y) = \{B, B\}(\tilde{P}X, \tilde{P}Y). \tag{6}$$

Moreover $\{B, B\}(BX, BY) = \{B, B\}(X, Y)$ for X, $Y \in \mathfrak{X}(M)$.

Let us evaluate the torsion tensor of the Chern connection of the 3-web on couples of homogeneous vector fields belonging to the first (or second, or third) distribution.

Proposition 2.1. *The torsion T of the Chern connection of a 3-web satisfies the following identities (X, Y being arbitrary vector fields on M):*

$$\begin{aligned} PT(\tilde{P}X, \tilde{P}Y) = 0, \qquad & \tilde{P}T(PX, PY) = 0, \\ T(PX, PY) &= -P\{B, B\}(PX, PY) = BT(BPX, BPY), \\ T(\tilde{P}X, \tilde{P}Y) &= -\tilde{P}\{B, B\}(\tilde{P}X, \tilde{P}Y) = BT(B\tilde{P}X, B\tilde{P}Y), \\ T(PX + BPX, PY + BPY) &= T(PX, PY) + BT(PX, PY). \end{aligned} \tag{7}$$

Proof. Integrability of D_1, or of D_2 respectively, is equivalent with

$$\tilde{P}T(PX, PY) = 0, \quad \text{or} \quad PT(\tilde{P}X, \tilde{P}Y) = 0, \text{ respectively.}$$

Using integrability conditions, (5), (6) and the formulae established in **1.3.** we obtain for any X, $Y \in \mathfrak{X}(M)$:

$$\begin{aligned}
T(PX,PY) &= PB[PX,BPY] + PB[BPX,PY] - [PX,PY]\\
&= PT(PX,PY) = -P\{B,B\}(PX,PY) =\\
&= - PB\{B,B\}(PX,PY) = -P\{B,B\}(BPX,BPY),\\
T(\tilde{P}X,\tilde{P}Y) &= \tilde{P}B[\tilde{P}X,B\tilde{P}Y] + \tilde{P}B[B\tilde{P}X,\tilde{P}Y] - [\tilde{P}X,\tilde{P}Y]\\
&= \tilde{P}T(\tilde{P}X,\tilde{P}Y) = -\tilde{P}\{B,B\}(\tilde{P}X,\tilde{P}Y),\\
T(BPX,BPY) &= \tilde{P}B[BPX,PY] + \tilde{P}B[PX,BPY] - [BPX,BPY]\\
&= \tilde{P}T(BPX,BPY) = -\tilde{P}\{B,B\}(PX,PY)\\
&= - \tilde{P}B\{B,B\}(PX,PY) = -\tilde{P}\{B,B\}(BPX,BPY),\\
BT(PX,PY) &= - \tilde{P}B\{B,B\}(PX,PY) = -\tilde{P}\{B,B\}(PX,PY)\\
&= T(BPX,BPY),\\
BT(\tilde{P}X,\tilde{P}Y) &= - P\{B,B\}(\tilde{P}X,\tilde{P}Y) = T(B\tilde{P}X,B\tilde{P}Y).
\end{aligned}$$

Corollary. *The torsion T satisfies*

$$X,Y \in D_\alpha \Rightarrow T(X,Y) \in D_\alpha.$$

2.3. It is well known that if a 3-web is parallelizable then its Chern connection is symmetric, that is, vanishing of the torsion tensor of the Chern connection is a necessary (but not sufficient) condition for parallelizability of the given 3-web. In some considerations, e. g. concerning parallelizability conditions, the torsion tensor T can be substituted by other tensors constructed through projectors, especially by $[P,B]$. It was proved in [15] that

$$T = 0 \quad \text{if and only if} \quad [P,B] = 0.$$

It follows by the equalities $T(PX,\tilde{P}Y) = B[P,B](PX,\tilde{P}Y) = 0$,

$$T(PX,PY) = B[P,B](PX,PY), \quad T(\tilde{P}X,\tilde{P}Y) = -B[P,B](\tilde{P}X,\tilde{P}Y).$$

The tensor field T is called *a torsion* of a given 3-web. It can be proved that the Chern connection is symmetric if and only if all Nijenhuis brackets of projectors vanish.

3. The curvature of a web

The curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ of the Chern connection can be given explicitely by the formula

$$
\begin{aligned}
R(X,Y)Z = P\Big(&[\tilde{P}X, PB[PY, BPZ]] - [\tilde{P}Y, PB[PX, BPZ]] \\
&+ [\tilde{P}X, P[\tilde{P}Y, PZ]] - [\tilde{P}Y, P[\tilde{P}X, PZ]] - [\tilde{P}[X,Y], PZ]\Big) \\
&+ \tilde{P}\Big([PX, \tilde{P}B[\tilde{P}Y, B\tilde{P}Z]] - [PY, \tilde{P}B[PX, B\tilde{P}Z]] \\
&+ [PX, \tilde{P}[PY, \tilde{P}Z]] - [PY, \tilde{P}[PX, \tilde{P}Z]] - [P[X,Y], \tilde{P}Z]\Big) \\
&+ PB\Big([PX, \tilde{P}[PY, BPZ]] - [PY, \tilde{P}[PX, BPZ]] \\
&- [P[X,Y], BPZ] + [PX, BP[\tilde{P}Y, PZ]] - [PY, BP[\tilde{P}X, PZ]]\Big) \\
&+ \tilde{P}B\Big([\tilde{P}X, P[\tilde{P}Y, B\tilde{P}Z]] - [\tilde{P}Y, P[\tilde{P}X, B\tilde{P}Z]] \\
&- [\tilde{P}[X,Y], B\tilde{P}Z] + [\tilde{P}X, B\tilde{P}[PY, \tilde{P}Z]] - [\tilde{P}Y, B\tilde{P}[PX, \tilde{P}Z]]\Big).
\end{aligned} \tag{8}
$$

On horizontal and vertical leaves, the curvature is zero:

$$R \,|\, D_1 \times D_1 = 0, \qquad R \,|\, D_2 \times D_2 = 0.$$

Proposition 3.1. *The curvature R of the Chern connection satisfies* [11]

(i) $RP = PR, \qquad R\tilde{P} = \tilde{P}R,$
(ii) $BR = RB,$
(iii) $R(PX, PY) = 0, \quad R(\tilde{P}X, \tilde{P}Y) = 0$ *for* $X,\ Y \in \mathfrak{X}(M)$.

Proof. The verification of (i) can be done by a direct evaluation if we substitute correspondingly PZ or $\tilde{P}Z$ into the furmula (8). We obtain $R(X,Y)PZ = PR(X,Y)Z$ and similarly for $\tilde{P}$. To verify (ii) we can either use the formula (8), or the equalities

$$B\nabla_X Y = \nabla_Y BZ, \qquad \nabla_X Y = B\nabla_Y BZ \tag{9}$$

which follow by $\nabla B = 0$. We obtain

$$\nabla_X \nabla_Y Z = B\nabla_X B\nabla_Y Z = B\nabla_X \nabla_Y BZ,$$

$$-\nabla_Y \nabla_X Z = -B\nabla_Y \nabla_X BZ, \quad -\nabla_{[X,Y]} Z = -B\nabla_{[X,Y]} BZ.$$

Together, $R(X,Y)Z = BR(X,Y)BZ$ and further $R(X,Y)BZ = BR(X,Y)Z$ for X, Y, $Z \in \mathfrak{X}(M)$. Let us evaluate $R(PX, PY)$:

$$
\begin{aligned}
R(PX,PY)\tilde{P}Z &= \tilde{P}[PX,[PY,\tilde{P}Z]] - \tilde{P}[PY,[PX,\tilde{P}Z]] - \tilde{P}[[PX,PY],\tilde{P}Z] \\
&= \tilde{P}([PX,[PY,\tilde{P}Z]] + [PY,[\tilde{P}Z,PX]] + [\tilde{P}Z,[PX,PY]]) = 0, \\
R(PX,PY)PZ &= BR(PX,PY)BPZ = 0.
\end{aligned}
$$

A similar evaluation for $R(\tilde{P}X, \tilde{P}Y)$ completes the proof.

Let us simplify the formula (8). We evaluate R on the remaining triples of homogeneous vectors:

$$
\begin{aligned}
R(PX,\tilde{P}Y)PZ &= - P[\tilde{P}Y, PB[PX,BPZ]] - P[\tilde{P}[PX,\tilde{P}Y],PZ] \\
&\quad - PB[P[PX,\tilde{P}Y],BPZ] + PB[PX,BP[\tilde{P}Y,PZ]], \\
R(PX,\tilde{P}Y)\tilde{P}Z &= + \tilde{P}[PX,\tilde{P}B[\tilde{P}Y,B\tilde{P}Z]] - \tilde{P}[P[PX,\tilde{P}Y],\tilde{P}Z] \\
&\quad - \tilde{P}B[\tilde{P}[PX,\tilde{P}Y],B\tilde{P}Z] - \tilde{P}B[\tilde{P}Y,B\tilde{P}[PX,\tilde{P}Z]].
\end{aligned}
\tag{10}
$$

Now

$$
\begin{aligned}
R(X,Y)Z &= (R(PX,\tilde{P}Y) - R(PY,\tilde{P}X))\, PZ \\
&\quad + (R(PX,\tilde{P}Y) - R(PY,\tilde{P}X))\, \tilde{P}Z
\end{aligned}
$$

and via (10), we obtain a bit simple formula for R (with 16 members, instead of 20 in (8)).

Remark. The formulas equivalent with the first and second Bianchi identities were established in [11].

4. Connections parallelizing web distributions

4.1. Parallel distributions. A distribution D is *parallel* with respect to a connection ∇ if

$$\forall X \in \mathfrak{X}(M) \quad \forall Y \in D \qquad \nabla_X Y \in D.$$

Proposition 4.1. *Web-distributions are parallel with respect to the Chern connection of a 3-web (D_1, D_2, D_3).*

Proof. Since $(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y$ and $\nabla P = 0$ we obtain

$$\nabla_X PY = P\nabla_X Y.$$

So $\tilde{P}(\nabla_X PY) = 0$, that is, $\nabla_X PY \in \ker \tilde{P} = D_1$. Similarly by $\tilde{P} = 0$ follows that $\nabla_X \tilde{P}Y$ belongs to D_2. Now let $Y \in D_3$. Then $BY = Y$. We use $\nabla B = 0$ to obtain

$$
\begin{aligned}
(B - I)(\nabla_X Y) &= B(\nabla_X Y) + (\nabla_X B)Y - \nabla_X Y \\
&= \nabla_X BY - \nabla_X Y = 0
\end{aligned}
$$

which proves $\nabla_X Y \in D_3$.

Corollary 4.1. *Any web-distribution is parallel relative any other,*

$$\forall X \in D_\beta \quad \forall Y \in D_\alpha \qquad \nabla_X Y \in D_\alpha.$$

Especially, each distribution is autoparallel ($\alpha = \beta$).

Corollary 4.2. *The Chern connection is reducible to the subbundle $D_\alpha \to M$, $\alpha = 1,2,3$. The reduced connection will be denoted ∇_α; $\nabla = \nabla_\alpha \oplus \nabla_\beta$. Moreover, ∇ can be reduced to any integral submanifold $L^{(\alpha)}$ (=leaf of the foliation) D_α.*

4.2. Induced connections on the coordinatizing loops. Any 3-web on a manifold can be locally coordinatized by a local Lie loop. Let us fix a point $p \in M$, and denote by L the integral submanifold od D_1 through p. Then in some nbd U_p of p, a differentiable loop multiplication $x \circ y$ with unit e can be introduced on L so that M is diffeomorphic to the direct product $L \times L$; p corresponds to (e,e), and the integral submanifolds of the distributions D_1, D_2, D_3 correspond to the foliations

$$L \times \{g\}, \quad \{g\} \times L, \quad \{(x,y) \in L \times L\,;\, x \circ y = g\}, \quad g \in L.$$

Let $\mathcal{L}_x$ ($\mathcal{R}_x$) denote the left, or right multiplication respectively,

$$\mathcal{L}_x(y) = x \circ y, \qquad \mathcal{R}_x(y) = y \circ x.$$

Denote by Λ^l (Λ^r) the left-canonical (right-canonical) connection on L defined by left-invariant (right-invariant) vector fields

$$X^l_x = (T\mathcal{L}_x)X_e, \qquad X^r_x = (T\mathcal{R}_x)X_e.$$

The Chern connection on $L \times L$ induces the left-canonical connection Λ^l of L on the subnmanifold $L^{(1)} = L \times \{e\} \approx L$, $\nabla_1 \,|\, L^{(1)} = \Lambda^l$. Similarly, the right-canonical connection $\Lambda^r = \nabla_2 \,|\, L^{(2)}$ is induced on $L^{(2)} = \{e\} \times L \approx L$, [11].

4.3. Parallelizing connections. The Chern connection is not the unique connection with respect to which the web-distributions are parallel. In fact, there is a $2n^3$-parameter family of such connections [13]. If ∇ is one of them, then the other are of the form $\nabla + S$ where S is a $(1,2)$-tensor field satisfying

$$P^\alpha_\beta\, S(X, P^\beta_\alpha) = 0, \qquad \alpha,\beta = 1,2,3, \;\; \beta \neq \alpha, \;\; X, Y \in \mathfrak{X}(M), \tag{11}$$

that is, $S(X,Y) \in D_\alpha$ whenever $Y \in D_\alpha$. All tensor fields S satisfying this condition can be constructed as follows [13]. Let us choose a differentiable map

$$\Phi_1 : M \to \mathrm{Hom}\,(TM, \mathrm{End}\, D_1);$$

then for $p \in M$ and $X_p \in T_pM$, $\Phi_{1,p}X_p \in \mathrm{End}\, D_{1,p}$. We can extend Φ_1 via B into

$$\Phi : M \to \mathrm{Hom}\,(TM, \mathrm{End}\, TM)$$

by the formula

$$(\Phi X)Y = (\Phi_1 X)(P^2_1 Y) + (B\,\Phi_1\, BX)(P^1_2 Y); \tag{12}$$

(ΦX) is a $(1,1)$-tensor field for $X \in \mathfrak{X}(M)$, and depends linearly on X. Now S defined by

$$S(X,Y) = (\Phi X)Y$$

is a $(1,2)$-tensor field which satisfies the condition (11). On the other hand, any S satidfying (11) induces, for each vector field X, a $(1,1)$-tensor field $\Phi X = S(X,-)$ commuting with projectors; its restriction Φ_1 onto D_1 gives an endomorphism of the subspace $D_{1,p}$ at any point, and the formula (12) is true.

The torsion tensors T, $\overset{\circ}{T}$ of the connections ∇ and $\overset{\circ}{\nabla}$ are related by

$$\overset{\circ}{T}(X,Y) = T(X,Y) + S(X,Y) - S(X,Y),$$

the curvature tensors satisfy

$$\begin{aligned}\overset{\circ}{R}(X,Y)Z =& R(X,Y)Z + S(T(X,Y),Z)\\ &+ (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\\ &+ S(X,S(Y,Z)) - S(Y,S(X,Z)).\end{aligned}$$

5. Chern connections for unordered 3-webs

By the above results any 3-web gives rise to a family of integrable $\{P_\alpha^\beta, B_\gamma\}$-structures with $(\alpha,\beta,\gamma) \in S_3$. If we consider *unordered* 3-webs then any 3-web manifold is endowed with three Chern connections $\overset{\gamma}{\nabla}$, $\gamma = 1,2,3$ corresponding to $\{P_\alpha^\beta, B_\gamma\}$-structures (the expression is symmetric in α, β):

$$\begin{aligned}\overset{\gamma}{\nabla}_X Y =& P_\alpha^\beta B_\gamma[P_\alpha^\beta X, B_\gamma P_\alpha^\beta Y] + P_\beta^\alpha B_\gamma[P_\beta^\alpha X, B_\gamma P_\beta^\alpha Y]\\ &+ P_\alpha^\beta[P_\beta^\alpha X, P_\alpha^\beta Y] + P_\beta^\alpha[P_\alpha^\beta X, P_\beta^\alpha Y]\end{aligned} \tag{13}$$

or equivalently, using the formulae

$$\begin{aligned}&P_\beta^\alpha \circ P_\gamma^\alpha = P_\beta^\alpha, \qquad P_\beta^\alpha \circ P_\beta^\gamma = P_\beta^\gamma,\\ &P_\beta^\alpha \circ P_\alpha^\gamma = 0, \qquad P_\beta^\gamma \circ P_\alpha^\beta = P_\alpha^\beta - P_\gamma^\beta,\\ &P_\gamma^\alpha - P_\beta^\alpha = P_\alpha^\beta - P_\alpha^\gamma,\\ &B_\gamma = P_\gamma^\alpha - P_\beta^\gamma = P_\gamma^\beta - P_\alpha^\gamma,\\ &B_\gamma P_\beta^\alpha = P_\beta^\alpha B_\gamma = P_\gamma^\alpha - P_\beta^\alpha\end{aligned}$$

we can write

$$\begin{aligned}\overset{\gamma}{\nabla}_X Y =& \sum_{(\alpha,\beta)\in S_2} (P_\gamma^\alpha - P_\beta^\alpha)[P_\alpha^\beta X, P_\gamma^\beta Y] + P_\alpha^\beta[P_\beta^\alpha X, P_\alpha^\beta Y] =\\ &= (P_\gamma^\alpha - P_\beta^\alpha)[P_\alpha^\beta X, P_\gamma^\beta Y] + (P_\gamma^\beta - P_\alpha^\beta)[P_\beta^\alpha X, P_\beta^\alpha Y]\\ &+ P_\alpha^\beta[P_\beta^\alpha X, P_\alpha^\beta Y] + P_\beta^\alpha[P_\alpha^\beta X, P_\beta^\alpha Y].\end{aligned} \tag{14}$$

The formula for the corresponding torsion tensor $\overset{\gamma}{T}$ is

$$\overset{\gamma}{T} = \sum_{(\alpha,\beta)\in S_2} (P^\alpha_\gamma - P^\alpha_\beta)\Big([P^\beta_\alpha X, P^\beta_\gamma Y] - [P^\beta_\gamma X, P^\beta_\alpha Y]\Big) + [P^\alpha_\beta X, P^\beta_\alpha Y] + [P^\beta_\alpha X, P^\alpha_\beta Y] - [X,Y]. \tag{15}$$

Especially ($\gamma = 3$),

$$\begin{aligned}\nabla = \overset{3}{\nabla} =&(P^1_3 - P^1_2)[P^2_1 X, P^2_3 Y] + (P^2_3 - P^2_1)[P^1_2 X, P^1_3 Y]\\ &+P^2_1[P^1_2 X, P^2_1 Y] + P^1_2[P^2_1 X, P^1_2 Y],\end{aligned}$$

$$\begin{aligned}T = \overset{3}{T} = &(P^1_3 - P^1_2)([P^2_1 X, P^2_3 Y] - [P^2_3 X, P^2_1 Y])\\ &+ (P^2_3 - P^2_1)([P^1_2 X, P^1_3 Y] - [P^1_3 X, P^1_2 Y])\\ &+ [P^2_1 X, P^1_2 Y] + [P^1_2 X, P^2_1 Y] - [X,Y].\end{aligned} \tag{16}$$

It can be proved that for any permutation (α,β,γ) of indexis, $\overset{\gamma}{T}(P^\beta_\alpha X, P^\alpha_\beta Y) = 0$, $P^\beta_\alpha \overset{\gamma}{T}(P^\alpha_\beta X, P^\alpha_\beta Y) = 0$, and

$$B_\gamma \overset{\gamma}{T}(P^\beta_\alpha X, P^\beta_\alpha Y) = -P^\alpha_\beta B_\gamma\{B_\gamma, B_\gamma\}(P^\beta_\alpha X, P^\beta_\alpha Y) = \overset{\gamma}{T}(B_\gamma P^\beta_\alpha X, B_\gamma P^\beta_\alpha Y).$$

Further, the equivalence

$$T = 0 \iff [P^\alpha_\beta, P^\gamma_\delta] = 0$$

holds. Consequently, if one of the Chern connections of the web is symmetric then all three are symmetric.

The explicit formula for $\overset{\gamma}{R}$ is rather complicated.

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