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NATURAL TRANSFORMATIONS OF LAGRANGIANS

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Introduction.

Let M be a differentiable manifold and let TM be the tangent bundle. A smooth function $L : TM \longrightarrow \mathbf{R}$ is called a *Lagrangian* on M. We denote by $\mathcal{C}^\infty(TM)$ the set of all Lagrangians on M and by $\Omega^p(TM)$ the set of all p-forms on TM.

In this paper we will study natural transformations of Lagrangians into p-forms on the tangent bundle for $p = 0, 1, 2$. Such a natural transformation A over n-dimensional differentiable manifolds M is a family of maps $A_M : \mathcal{C}^\infty(TM) \longrightarrow \Omega^p(TM)$ such that for every embedding $\varphi : M \longrightarrow N$ and for each Lagrangian $L \in \mathcal{C}^\infty(TN)$ the p-forms $A_M(L \circ T\varphi) \in \Omega^p(TM)$ and $A_N(L) \in \Omega^p(TN)$ are $T\varphi$-related.

Some natural transformations of Lagrangians into p-forms on the tangent bundle are considered in physics.

Example 1. For an n-dimensional differentiable manifold M let us denote by C_M the Liouville vector field on TM. Thus for every Lagrangian L on M the energy given by

$$E_M(L) = C_M(L) - L$$

is a natural transformation of Lagrangians into 0-forms on the tangent bundle i. e. a natural transformation of Lagrangians into itself.

Example 2. For an n-dimensional differentiable manifold M let us denote by J_M the canonical tangent structure on TM. Thus for every Lagrangian L on M the Poincaré-Cartan 1-form given by

$$\alpha_M(L) = dL \circ J_M$$

is a natural transformation of Lagrangians into 1-forms on the tangent bundle.

Example 3. For every Lagrangian L on M the Poincaré-Cartan 2-form is given by

$$\omega_M(L) = d(\alpha_M(L)) = d(dL \circ J_M).$$

Obviously ω is a natural transformation of Lagrangians into 2-forms on the tangent bundle. Poincaré-Cartan 2-forms are very important for theoretical mechanics (see [5]).

It is of interest to know all natural transformations of Lagrangians into p-forms on the tangent bundle for $p = 0, 1, 2$. In our paper we will study this problem.

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Basic definitions.

All manifolds and maps are assumed to be infinitely differentiable.

Let n be a fixed positive integer. A family of maps $A_M : \mathcal{C}^\infty(TM) \longrightarrow \Omega^p(TM)$, where M is an arbitrary n-dimensional manifold, is called a ***natural transformation*** of Lagrangians into p-forms on the tangent bundle if two following conditions hold:

(1) *The naturality condition.* For every injective immersion $\varphi : M \longrightarrow N$ of two n-dimensional manifolds M, N and for every Lagrangian $L \in \mathcal{C}^\infty(TN)$ we have

$$\bigwedge{}^p(T(T\varphi^{-1}))^* \circ A_M(L \circ T\varphi) = A_N(L) \circ T\varphi.$$

(2) *The regularity condition.* For all manifolds M, N such that $\dim N = n$ and for every smooth function $L : M \times TN \ni (t, v) \longrightarrow L_t(v) \in \mathbf{R}$ the map

$$M \times TN \ni (t, v) \longrightarrow A_N(L_t)(v) \in \bigwedge{}^p T^*(TN)$$

is also smooth.

Let A be a natural transformation of Lagrangians into p-forms on the tangent bundle. If U is an open subset of M and if $\varphi : U \longrightarrow M$ is the inclusion then from the naturality condition we obtain the following implication

$$K|TU = L|TU \Longrightarrow A_M(K)|TU = A_M(L)|TU$$

for all Lagrangians K, L on M. We say that the natural transformation A satisfies the locality condition if for every n-dimensional manifold M, for every open subset V of TM and for all Lagrangians K, L on M the following implication

$$K|V = L|V \Longrightarrow A_M(K)|V = A_N(L)|V$$

holds. The following examples show that there are natural transformations of Lagrangians into itself which don't satisfy the locality condition (see [1]).

Example 4. Let $f : [0,1] \longrightarrow \mathbf{R}$ be a continuous function. We define

$$A_M(L)(v) = \int_0^1 L(f(t)v)dt.$$

Example 5. We put

$$A_M(L) = d(L \circ 0_M)$$

where 0_M is the zero section of TM and $d(L \circ 0_M)$ is considered as a function on TM.

We say that the natural transformation A is *of order* r if for every n-dimensional manifold M, for every $v \in TM$ and for all Lagrangians K, L on M the following implication

$$j^r_v K = j^r_v L \Longrightarrow A_M(K)(v) = A_M(L)(v)$$

holds.

It is clear that if a natural transformation has a order r then it satisfies the locality condition. Using the standard methods and so-called Borel Lemma (see [6]) we can prove that if a natural transformation of Lagragians into p-forms on the tangent bundle satisfies the locality condition then this natural transformation is of order ∞. The following example shows that there are natural transformations of Lagrangians into itself which satisfy the locality condition and have no finite order (see [1]).

Example 6. Suppose that $f : \mathbf{R} \longrightarrow \mathbf{R}$ is a smooth function such that $f|(-\infty, 0] = 0$ but $f \neq 0$. We define

$$A_M(L)(v) = \sum_{i=1}^{\infty} f(L(v) - \sum_{j=1}^{i}(1 + (C_M^j(L)(v))^2))$$

where $C_M^j = (C_M \circ \ldots \circ C_M)(L)$ (j times). It is seen at once that this definition makes sense because in a neighbourhood of arbitrary $v \in TM$ the first sum has a finite number of non-zero therms.

Natural transformations of Lagrangians into itself.

We have the following characterization of natural transformations of Lagrangians into itself (see [1]).

THEOREM 1. *Let $n \geq 2$ and let r be a positive integer. If A is a natural transformation of order r of Lagrangians into itself then there is one and only one smooth function $f : \mathbf{R}^{r+1} \longrightarrow \mathbf{R}$ such that $A_M(L) = f(L, C_M^1(L), \ldots, C_M^r(L))$ for every n-dimensional manifold M and for every Lagrangian L on M.*

The following example shows that the assumption that $n \geq 2$ in Theorem 1 is necesseary.

Example 7. Let φ be a local coordinate system on a 1-dimensional manifold M and let L be a Lagrangian on M. Setting

$$\begin{aligned}(A_M(L) \circ T\varphi^{-1})(x, v) =& \frac{\partial^2(L \circ T\varphi^{-1})}{\partial x \partial v}(x, v)\frac{\partial(L \circ T\varphi^{-1})}{\partial v}(x, v)v^3 \\ & - \frac{\partial^2(L \circ T\varphi^{-1})}{\partial v^2}(x, v)\frac{\partial(L \circ T\varphi^{-1})}{\partial x}(x, v)v^3 \\ & - \frac{\partial(L \circ T\varphi^{-1})}{\partial x}(x, v)\frac{\partial(L \circ T\varphi^{-1})}{\partial v}(x, v)v^2\end{aligned}$$

we obtain a natural transfotmation of Lagrangians into itself which has the order two. It is clear that A is not of the form discribed in Theorem 1.

Natural transformations of Lagrangians into 1-forms on the tangent bundle.

Let R^r denotes the set of all natural transformations of order r of Lagrangians into itself. It is evident that R^r with the sum and product

$$(A + B)_M(L) = A_M(L) + B_M(L),$$
$$(A \cdot B)(L) = A_M(L)B_M(L)$$

is a ring. Let M_p^r denotes the set of all natural transformations of order r of Lagrangians into p-forms on the tangent bundle. It is evident that M_p^r is a module over R^r if we define

$$(A+B)_M(L) = A_M(L) + B_M(L),$$
$$(\Gamma \cdot A)_M(L) = \Gamma_M(L) A_M(L)$$

for all $A, B \in M_p^r$, $\Gamma \in R^r$, for every n-dimensional manifold M and for every Lagrangian L on M. We can verify (see [4]) that M_p^r is a free module and we have

THEOREM 2. *Let $n \geq 3$ and let r be a positive integer. The natural transformations given by formulas*

$$d(C_M^i(L)) \qquad \text{for } i = 0, \dots, r-1,$$

$$d(C_M^i(L)) \circ J_M \qquad \text{for } i = 0, \dots, r-1,$$

for every n-dimensional manifold M and for every Lagrangian L on M, form a basis of the module M_1^r.

Natural transformations of Lagrangians into 2-forms on the tangent bundle.

Using the similar methods as in [1], [2] and [4] we can prove

THEOREM 3. *Let $n \geq 4$ and let r be a positive integer. The natural transformations given by formulas*

$$d(C_M^i(L)) \wedge d(C_M^j(L)) \qquad \text{for } 0 \leq i < j \leq r-1,$$

$$d(C_M^i(L)) \wedge (d(C_M^j(L)) \circ J_M) \qquad \text{for } i, j = 0, \dots, r-1,$$

$$(d(C_M^i(L)) \circ J_M) \wedge (d(C_M^j(L)) \circ J_M) \qquad \text{for } 0 \leq i < j \leq r-1,$$

$$d(d(C_M^i(L)) \circ J_M) \qquad \text{for } i = 0, \dots, r-2,$$

for every n-dimensional manifold M and for every Lagrangian L on M, form a basis of the module M_2^r.

We can show this theorem also for $n = 3$ but the method of verification is more complicated.

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