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THE CYCLIC HOMOLOGY OF P(G)

Bohumil Cenkl and Micheline Vigué-Poirrier

Let $P(G) = \oplus_{n\geq 0} P^n(G)$ be the free associative differential graded algebra, over a commutative ring K, associated with the data of a finitely generated torsion free nilpotent group G as in [1]. More precisely $P(G) = (T(V), d^*)$, where V is a free $\mathbb{Z}$ - module, graded in degree one, i.e. $V = V^{-1}$, and $d^* : T(V)^n \to T(V)^{n+1}$. To such a cochain algebra corresponds a negatively graded chain algebra $P_{-*} = P_{-*}(G) = T(V_*)$ with the differential $d_* = d^*$ and $V_{-1} = V^1$. Recall that the total Hochschild complex $\mathcal{C}_*$ of $(P_{-*}(G), d_*)$ is negatively graded. By definition $HH_*(P(G), d^*) = HH_*(P_{-*}(G), d_*) = H_*(\mathcal{C}_*, d+b)$ and $\mathcal{C}_* = \oplus_{n\geq 0}\mathcal{C}_{-n}$. The definition of the Connes boundary $B : \mathcal{C}_{-n} \to \mathcal{C}_{-n+1}$ can be found in [2] and [3]. Thus we have a bicomplex:

$$
\begin{array}{ccccccccccccc}
 & & 0 & \xleftarrow{B} & \mathcal{C}_0 & \xleftarrow{B} & \cdots & \xleftarrow{B} & \mathcal{C}_{-n+1} & \xleftarrow{B} & \mathcal{C}_{-n} & \xleftarrow{B} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle b+d} & & & & \downarrow{\scriptstyle b+d} & & \downarrow{\scriptstyle b+d} & & \\
0 & \longleftarrow & \mathcal{C}_0 & \xleftarrow{B} & \mathcal{C}_{-1} & \xleftarrow{B} & \cdots & \xleftarrow{B} & \mathcal{C}_{-n} & \xleftarrow{B} & \mathcal{C}_{-n-1} & \xleftarrow{B} & \cdots
\end{array}
$$

for $n\geq 0$.

The total complex $(\text{Tot } \mathcal{C})_* = \oplus_{n\geq 0}(\text{Tot } \mathcal{C})_{-n}$ is negatively graded. We have $(\text{Tot } \mathcal{C})_{-n} = \mathcal{C}_{-n} \oplus \mathcal{C}_{-n+2} \oplus \mathcal{C}_{-n+4} \oplus \cdots$. This complex will be denoted by $K[u]\otimes_B \mathcal{C}_*$, where $K[u]$ is the polynomial algebra on the generator u of degree -2. The differential is the operator $b + d + uB$.

Definition. The homology $HC_*(P^*, d^*)$ of the complex $(K[u]\otimes_B\mathcal{C}_*, b + d + uB)$, is called the cyclic homology of $(P(G), d^*)$.

The homology $HC_*(P^*, d^*)$ is negatively graded. Using the terminology introduced by Jones, it is called the negative cyclic homology. Then it is denoted by $HC_*^-(P_{-*}(G), d_*)$.

Let $\bar{V} = V_{-1} = P^1(G)$. Then according to the Theorem 1.5 in [2]

$$HH_*(P_{-*}, d_*) = H_*(P_{-*} \oplus P_{-*} \otimes \bar{V}, \delta),$$

where $\delta = d + \delta' + \delta''$, $d = d_*$ on P, $\delta'(a \otimes \bar{v}) = (-1)^{|a|}(av - (-1)^{|a|}va)$, $\delta''(a \otimes \bar{v}) = da \otimes \bar{v} - S(a, dv)$ for $a \in P, v \in V$. δ' and δ'' are both zero maps on P,(see [1], page 6). When the complex $(T(V), d)$ is negatively graded, we get similar results as thouse stated in [2], Theorem 2.4. Let

$$K_* = (K[u] \otimes (P_{-*} \oplus P_{-*} \otimes \bar{V}), D),$$

where $|u| = -2, D = 0$ on $K[u]$, and

$$D(u^n \otimes (a + b\bar{v})) = u^n \otimes \delta(a + b\bar{v}) + u^{n+2}\beta(a)$$

when $a \in P_{-*}$, $b \in P_{-*}$, $\bar{v} \in \bar{V}$, and

$$\beta(v_1 \dots v_p) = v_1 \dots v_{p-1} \otimes \bar{v}_p + \sum_{i=1}^{p-1}(-1)^{[|v_{i+1}|+\dots+|v_p|][|v_1|+\dots+|v_i|]} v_{i+1} \dots v_p v_1 \dots v_{i-1} \otimes \bar{v}_i.$$

Using the norm $\|\cdot\|$ on P (see [1], page 3), we define a filtration on the complex K_* by setting

$$F_i = \{c = u^n \otimes (a + b\bar{v}) | \max(\|a\|, \|b\| + \|v\|) \le i\}.$$

It is obvious that the filtration is an ascending filtration $F_i \subset F_{i+1}$. Then from the construction of the differential D on K it follows that

$$DF_i \subset F_i$$

Let $\{E^r, d^r\}$ be the spectral sequence corresponding to the filtration $\{F_i\}$. Let $F_i = F_i^{'} \oplus F_i^{''}$, where

$$F_i^{'} = \{w \otimes a \in K[u] \otimes P | \; \|w \otimes a\| \le i\},$$
$$F_i^{''} = \{w \otimes (b \otimes \bar{v}) \in K[u] \otimes (P \otimes \overline{V}) | \; \|w \otimes (b \otimes \bar{v})\| \le i\},$$

and let $p^{'} : F_i^{'} \longrightarrow E_i^0$, $p^{''} : F_i^{''} \longrightarrow E_i^0, p = p^{'} + p^{''} : F_i \longrightarrow E_i^0$ be the projections.

Next consider the maps $d, \delta^{'}$ and $\delta^{''}$ (page 5 of [1])

$$d = 1 \otimes d : K[u] \otimes P \longrightarrow K[u] \otimes P,$$
$$\delta^{'} = 1 \otimes \delta' : K[u] \otimes (P \otimes \overline{V}) \longrightarrow K[u] \otimes P,$$
$$\delta^{''} = 1 \otimes \delta'' : K[u] \otimes (P \otimes \overline{V}) \longrightarrow K[u] \otimes (P \otimes \overline{V}),$$
$$\delta^{'''} \quad : K[u] \otimes P \longrightarrow K[u] \otimes (P \otimes \overline{V}),$$

where

$$\delta^{'''}(u^n \otimes (v_1 \cdots v_p)) = u^{n+1} \otimes (v_1 \cdots v_{p-1} \otimes \overline{v_p} + \sum_{i=1}^{p-1}(-1)^{\epsilon_i} v_{i+1} \cdots v_p v_1 \cdots v_{i-1} \otimes \overline{v_i}),$$

$$\epsilon_i = (|v_{i+1}| + \cdots + |v_p|)(|v_1| + \cdots + |v_i|).$$

Then from (pages 6 and 7 of [1]) it follows that on the elements $u^n \otimes (a \oplus (b \otimes \bar{v}) \in K[u] \otimes (P^1 \oplus (P^1 \otimes \overline{V}))$ of norm equal to i,

$$d(u^n \otimes a) = -u^n \otimes \sum a^t \cdot t + \cdots,$$
$$\delta^{'}(u^n \otimes b \otimes \bar{v}) = u^n \otimes (bv + vb),$$
$$\delta^{''}(u^n \otimes b \otimes \bar{v}) = -u^n \otimes \sum b^t \cdot t \otimes \bar{v} + u^n \otimes S(b, v^t \cdot t) + \cdots,$$
$$\delta^{'''}(u^n \otimes a) = u^{n+1} \otimes \bar{a}.$$

Here $\cdots$ stands for the terms of filtration $\le i - 1$. This proves

Lemma 1. *Let $u^n \otimes (a \oplus (b \otimes \overline{v}))$ be an element of $P^1 \oplus (P^1 \otimes \overline{V})$ of norm i then*

$$pD(u^s \otimes (a \oplus (b \otimes \overline{v}))) = p^{'}\{u^s \otimes (bv + vb - \sum a^t \cdot t)\}$$

$$+p^{''}\{u^s \otimes (S(b, v^t \cdot t) - \sum b^t \cdot t \otimes \overline{v}) + u^{n+1} \otimes \overline{a}\}.$$

Suppose that the group G, which is the fundamental group of a k- dimensional nilmanifold, is a free abelian group. Then a simple verification of the computation preceeding Lemma 1 gives

Lemma 2. *The E^1-term of the spectral sequence $\{E^r, d^r\}$ is isomorphic to the cyclic homology*

$$HC_*(P(H)),$$

where H is a free abelian group on k generators.

Lemma 3. *If H is a free abelian group on k generators,then the cochain algebra $(P^* = P(H), d^*)$ is quasi-isomorphic to the exterior algebra on the free $\mathbb{Z}$-module generated by k elements of degree one, with zero differential.*

Proof. See [1].

Lemma 4. *Let K be a commutative ring containing $\mathbb{Q}$, let $\wedge(f_1, \cdots, f_k)$ be the exterior algebra on the K-free module $\oplus_{i=1}^k K f_i$, with $|f_i| = -1$, and let $K[e_1, \cdots, e_k]$ be the polynomial algebra on the K-free module $\oplus_{i=1}^k K e_i$ with $|e_i| = 0$. Then ,*

$$HC^-_{-n}(\wedge(f_1, \cdots f_k), d = 0) = HC^-_{-n}(K) \oplus \beta(\wedge^{n+1}(f_1, \cdots f_k)) \cdot K[e_1, \cdots e_k]$$

for $n \geq 0$; β is the algebra derivation defined by $\beta(f_i) = e_i$, $\beta(e_i) = 0$, and $\wedge^i(f_1, \cdots f_k)$ denotes the K-vector space generated by words of length i in the variables $f_1, \cdots, f_k$.

Remark. Since $HC^-_{-n}(K) = K$ for n even and $HC^-_{-n} = 0$ for n odd, it follows that for $n \geq k$, $HC_{-n}(\wedge(f_1, \cdots, f_k)) = 0$ for n odd and $HC_{-n}(\wedge(f_1, \cdots, f_k)) = K$ for n even.

Proof of Lemma 4. A modification of the proof of the Theorem 2.4 in [3] shows that the map

$$\theta : (C_*, b, B) \to (\wedge(f_i) \otimes K[e_i], 0, \beta),$$

$$\theta(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \frac{(-1)^{\epsilon_p(a)}}{p!} a_0 \beta(a_1) \cdots \beta(a_p)$$

if $a_0 \in \wedge(f_i), a_i \in \wedge^+(f_i), 1 \leq i \leq p$ satisfies

1 . $\theta \circ b = 0, \theta \circ B = \beta \circ \theta$,

2 . $H_*(C_*, b) = \wedge(f_i) \otimes K[e_i]$,

3 . $HC_*(C_*, b, B) = HC_*(\wedge(f_i) \otimes K[e_i], 0, \beta)$. Here $HC_*(\wedge(f_i) \otimes K[e_i], 0, \beta)$ is the homology of the complex $L_* = \oplus_{n \geq 0} L_{-n}$,

$$L_{-n} = (\wedge(f_i) \otimes K[e_i])_{-n} \oplus (\wedge(f_i) \otimes K[e_i])_{-n+2} \oplus \cdots$$

with differential β^L,

$$\beta^L(a_{-n}, a_{-n+2}, \cdots) = (0, \beta(a_{-n}), \beta(a_{-n+2}), \cdots).$$

Since $H_*(\wedge(f_i) \otimes K[e_i], \beta)$ is equal to zero in non-zero degrees, and is equal to K in degree zero, it follows that

$$H_{-n}(L_*) = \begin{cases} K \oplus \beta(\wedge^{n+1}(f_i)) \cdot K[e_i] & \text{if } n \text{ is even,} \\ \beta(\wedge^{n+1}(f_i)) \cdot K[e_i] & \text{if } n \text{ is odd} \end{cases}$$

Summarizing our results we get

Theorem. *Let G be a finitely generated torsion free nilpotent group, and let $k = dimK(G,1)$. Let $P(G)$ be the polynomial cochain algebra of G endowed with the norm $\|\cdot\|$ as defined in [1]. Let K be a commutative ring containing the rationals. Then the norm $\|\cdot\|$ induces filtrations on the Hochschild and Connes complexes such that*

1 . *There is a spectral sequence $E^r_{p,-q}$ converging to $HH_{-*}(P_{-*}(G))$ with the E^1-term isomorphic to $HH_{-*}(P_{-*}(H))$, where H is the free abelian group on k generators. In fact*

$$\bigoplus_{p-q=n} E^1_{p,-q} = HH_{-n}(P_{-*}(H)) \simeq \wedge^n(f_1, \cdots, f_k) \otimes K[e_1, \cdots e_k],$$

where $K[e_1, \cdots e_k]$ is the polynomial algebra on k generators in degree 0, and $\wedge^n(f_1, \cdots, f_k)$ is the vector space spanned by words of length n in the exterior algebra on the K-free module generated by k generators in degree one.

2 . *There is a spectral sequence ${}_cE^r_{p,-q}$ converging to $HC^-_{-*}(P_{-*}(G))$ with ${}_cE^1$-term isomorphic to $HC^-_{-n}(P_{-n}(H))$. We have*

$$\bigoplus_{p-q=n\geq 0} {}_cE^1_{p,-q} = HC^-_{-n}(P_{-n}(H)) = HC^-_{-n}(K) \oplus V_n$$

with

a) *$HC^-_{-n}(K) = 0$ if n is odd, and $HC^-_{-n}(K) = K$ if n is even,*

b) *$V_n = 0$ if $n \geq k$,*

c) *if $n < k$, then V_n is the $K[e_1, \cdots, e_k]$-module generated by elements of the form :*

$$\sum_{j=1}^{n+1} (-1)^{j-1} e_{i_j} de_{i_1} \wedge \cdots \wedge de_{i_{j-1}} \wedge de_{i_{j+1}} \wedge \cdots \wedge de_{i_{n+1}}$$

for all $\{i_1 < i_2 < \cdots < i_{n+1}\} \in [1, \cdots, k]$. de_i stands for the differential form of the i-th variable e_i of the polynomial algebra $K[e_1, \cdots e_k]$.

3 . *If K is of characteristic zero, then the Connes long exact sequence*

$$\cdots \to HC^-_{-n+2} \to HC_{-n} \to HH_{-n} \to HC^-_{-n+1} \to \cdots$$

induces the short exact sequences:

$$\begin{array}{ccccc} 0 \longrightarrow & HC^-_{-n}(P_{-*}(H))/HC^-_{-n}(K)) & \longrightarrow & HH_{-n}(P_{-*}(H)) \\ & \simeq\downarrow & & \simeq\downarrow \\ 0 \longrightarrow & \mathrm{Ker} \cap (\wedge^n_\cdot(f_i)\cdot K[e_i]) & \longrightarrow & \wedge^n(f_1,\cdots,f_k)\otimes K[e_1,\cdots e_k] \end{array}$$

$$\begin{array}{ccc} \longrightarrow & HC^-_{-n+1}(P_{-*}(H))/HC^-_{-n+1}(K) & \longrightarrow 0 \\ & \simeq\downarrow & \\ \xrightarrow{\beta} & \beta[\wedge^n(f_1,\cdots,f_k)]\cdot K[e_1,\cdots,e_k] & \longrightarrow 0 \end{array}$$

for $n \geq 1$.

Corollary. *Let G be a finitely generated torsion free nilpotent group and let $k = dimK(G,1)$. Let $P(G)$ be the cochain algebra of G with coefficients in a field of zero characteristic. Then*

1 . $HH_{-n}(P_{-*}(G)) = 0$ *if* $n > k$,

2 . $\widetilde{HC}^-_{-n}(P_{-*}(G)) = 0$ *if* $n \geq k$, *and where $\widetilde{HC}$ is the quotient of the cyclic homology of P_{-*} over the cyclic homology of the ground field.*

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