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§1. Introduction

The homotopy classification of (based) maps $[X,Y]$ from a space X to a space Y takes on added complexity when Y is not simply connected and $\pi_1(Y)$ acts non-trivially on the higher homotopy groups of Y. A typical example of this is $Y = \mathbb{R}P^{2n}$ (we shall henceforth write P^m for $\mathbb{R}P^m$). In this note we are interested in the case $X = V$, a closed surface (meaning compact without boundary), and $Y = P^2$. This case arises in the homotopy classification of Lorentz metric tensors over the $(2 + 1)$–dimensional space–time manifold $\mathscr{M} = V \times \mathbb{R}$. Distinct homotopy classes are said to determine distinct relativistic kinks on $\mathscr{M}$.

The structure of $[V,P^2]$ (for V orientable) is completely determined in Theorems 1,2 of [8], which in turn is based on work of Olum ([4], [5], and [6]), and a note of Adams [1] (see also Eells–Lemaire [2] for an exposition of Olum's work as well as applications to the existence of harmonic and holomorphic maps). Since the Olum papers are lengthy, while the Adams note is sketchy (being actually a letter to Eells), it may be useful to give a short self-contained account of the portions of Olum's work needed for the case at hand, at the same time filling in details of Adams' proof, and attempting to give some of the geometrical intuition behind these results. In §2 we define orientation–true maps, compressible maps, and illustrate their meaning. The main structure theorems for $[V,P^2]$ (V oriented or non-oriented) are stated and proved in §3.

§2. The Classification Theorems For $[V,P^2]$

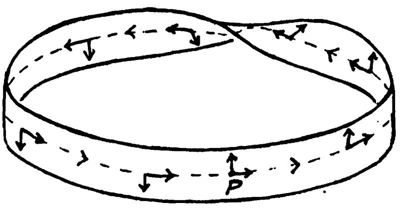
We first give two basic definitions, and then attempt to give some insight into their geometrical meaning. Recall that the first Stiefel–Whitney class $w_1(M) \in H^1(M;Z_2)$ for any smooth manifold M, and that M is orientable if and only if $w_1(M) = 0$ ([3], p.148, or [7], p.199).

2.1 DEFINITION: Let $f\colon M \to N$ be a smooth map (of smooth manifolds). We say f is orientation true if $f^*(w_1(N)) = w_1(M)$.

2.2 DEFINITION: Let V be a closed surface. A map $f\colon V \to P^2$ is compressible if $f \simeq g$ for some map $g = V \to P^1$, where $i = P^1 \hookrightarrow P^2$ is the standard inclusion $i[x_0,x_1] = [x_0,x_1,0]$.

To understand 2.1 from a more intuitive viewpoint, we shall relate it to loops on a manifold. Recall that given any two bases of $\mathbb{R}^n$, they are said to have the same or opposite orientation according to whether the determinant of the matrix expressing one in terms of the other is respectively positive or negative. The same clearly applies to two bases for the tangent space at a point P of a smooth manifold M, or even for two bases at two points $P,Q \in M$ provided they both lie in a coordinate neighborhood. Given a loop $\lambda\colon I \to M$ in M, one can cover it with a finite number of coordinate neighborhoods and use these to "transport" a given orientation at $P = \lambda(0)$ once around the loop. The result may be the same or the opposite orientation, in which case the loop is called respectively orientation preserving or orientation reversing, and clearly depends only on the homotopy class $[\lambda] \in \pi_1(M)$ of λ. Since M is evidently orientable if and only if all loops are orientation preserving, this suggests a relation between this idea and $w_1(M)$, which we make precise in 2.6 below.

The standard example of an orientation reversing loop is the central meridian of a Möbius band, as shown in Figure 2.3.



2.3 FIGURE

Now consider a map f: $M \to N$ and a loop λ: $I \to M$. Then $f\lambda$ is a loop in N. There are four possibilities: λ may be orientation preserving or reversing, and similarly for $f\lambda$. It will be proved in 2.6 that f is orientation true if and only if λ and $f\lambda$ have the same behaviour with respect to orientation, for all $[\lambda] \in \pi_1(M)$. First let us give two simple examples.

2.4 EXAMPLES:

(a) The identity map id: $M \to M$ is orientation true.

(b) If f is homotopically trivial, then f is orientation true if and only if M is orientable (since $f\lambda \simeq *$ is orientation preserving for any λ), or equivalently since $f^* w_1(N) = 0$ no matter what $w_1(N)$ is).

The next lemma will help to prove 2.6 and also gives another interpretation of $w_1(M)$. Note first that we may identify $H^1(M;Z_2) \approx \mathrm{Hom}(H_1M,Z_2) \approx \mathrm{Hom}(\pi_1M,Z_2)$, using the universal coefficient theorem and the fact Z_2 is abelian as well as $(\pi_1M)_{ab} \approx H_1M$.

2.5 LEMMA: *Let* $[\lambda] = \alpha \in \pi_1(M)$, *then* $w_1(\alpha)$ *equals* 0 or 1 *according to whether* λ *is respectively orientation preserving or reversing, where* $w_1 = w_1(M)$ *is regarded as an element of* $\mathrm{Hom}(\pi_1 M, Z_2)$.

PROOF: It is well known that there are just two stable classes of vector bundles ξ over S^1, and they are distinguished by their first Stiefel–Whitney class $w_1(\xi) \in H^1(S^1;Z_2) \approx Z_2$. Clearly λ is orientation preserving if and only if the induced vector bundle $\xi := \lambda^*(\tau_m)$ is trivial, where τ_m is the tangent bundle of M. The proof is now completed by the equalities

$$w_1(\alpha) = \lambda^*(w_1) = \lambda^* w_1(\tau_m) = w_1 \lambda^*(\tau_m) = w_1(\xi),$$

where the first equality simply expresses the identification $H^1(M;Z_2) \approx \mathrm{Hom}(\pi_1 M, Z_2)$.

2.6 PROPOSITION: $f\colon M \to N$ *is orientation true if and only if* λ *and* $f\lambda$ *have the same orientation character for any loop* λ *in* M.

PROOF: This is now trivial from 2.5. For example, if $f^* w_1(N) = w_1(M)$, then setting $\alpha = [f] \in \pi_1(M)$ and $\beta = f_*(\alpha) = [f\lambda] \in \pi_1(N)$, we have

$$w_1(M)(\alpha) = f^* w_1(N)(\alpha) = w_1(N) f_* \alpha = w_1(N)(\beta),$$

so λ and $f\lambda$ have the same orientation character by 2.5. The converse is similar.

Two other well known equivalent statements and Proposition 2.6 are summarized in the next result. We do not prove (iii) or (iv), but note that (iii) is an easy generalization of (part of) [7] Theorem 38.12, while (iv) is a restatement of (iii) in sheaf theoretic language.

2.7 PROPOSITION: *The following are equivalent:*

(*i*) $f\colon M \to N$ *is orientation true,*

(*ii*) *a loop* λ *in* M *is orientation preserving if and only if the loop* $f\lambda$ *in* N *is orientation preserving,* (*iii*) $f^{-1}\beta_N(\pi) = \beta_M(\pi)$, *where* $\beta_M(\pi)$ *is the coefficient bundle ("orientation bundle") associated to the tangent bundle of a smooth manifold* M *([7], p.200), and* f^{-1} *denotes its pull-back to* N,

(*iv*) *letting* $\mathscr{T}_M \to M$ *denote the orientation sheaf of any manifold, one has* $f^{-1}\mathscr{T}_N = \mathscr{T}_M$.

In our main application M will be a closed orientable surface V and N will be P^2. Since $w_1(P^2) = x$, the non-zero element of $H^1(P^2;Z_2)$, we have the following corollary:

2.8 COROLLARY: *For an orientable surface* V, $f\colon V \to P^2$ *is orientation true if and only if* $f^*(x) = 0$.

We now turn to the definition of compressibility. The geometric meaning of the definition is clear enough, and we will first illustrate it with a few examples. Note that points of P^2 are written $[x_0,x_1,x_2] = [-x_0,-x_1,-x_2]$ with $x_0^2 + x_1^2 + x_2^2 = 1$, $P^1 \subset P^2$ is determined by $x_2 = 0$, and points of $T^2 = S^1 \times S^1$ are written (θ,φ) where θ,φ are real numbers modulo 2π.

2.9 EXAMPLES:

(a) Any homotopically trivial $f\colon V \to P^2$ is compressible.

(b) The maps $\kappa\colon S^2 \twoheadrightarrow P^2$ (the usual double over) and id: $P^2 \to P^2$ are not compressible, since they are non-zero on the second homotopy group π_2.

(c) The map $f\colon T^2 \to P^2$ given by $F(\theta,\varphi) = [\cos\frac{\theta}{2},\sin\frac{\theta}{2},0]$ is clearly compressible.

(d) The map $g\colon T^2 \to P^2$ given by $g(\theta,\varphi) = [\cos\frac{\theta}{2}\cos\varphi,\cos\frac{\theta}{2}\sin\varphi,\sin\frac{\theta}{2}]$ is not obviously compressible, but can be seen to be compressible using the homotopy

$$H(\theta,\varphi,t) = [\cos\tfrac{\theta}{2}\cos\varphi - \sin t\,\sin(\tfrac{\theta}{2})\sin\varphi, \cos(\tfrac{\theta}{2})\sin\varphi + \sin t\,\sin(\tfrac{\theta}{2})\cos\varphi, \sin(\tfrac{\theta}{2})\cos t\,],$$

with $H(\theta,\varphi,1) = [\cos(\frac{\theta}{2} + \varphi), \sin(\frac{\theta}{2} + \varphi), 0]$.

We now show that for $f\colon V \to P^2$, the compressibility of $[f]$ is nearly determined by $Sq^1 y \in H^2(V;Z_2)$, where $y = f^*(x)$ and as always $x \in H^1(P^2;Z_2)$ is the non-zero element (we also use x for the non-zero element of $H^1(P^\infty;Z_2)$).

2.10 Proposition: *Let* $f\colon V \to P^2$, $y = f^*(x)$. *If* f *is compressible then* $Sq^1 y = 0$, *whereas if* $Sq^1 y = 0$ *then there exists some compressible map* $g\colon V \to P^2$ *with* $y = g^*(x)$.

Proof: First, we set up notation $i\colon P^1 \hookrightarrow P^2$ and $j = P^2 \hookrightarrow P^\infty$ for the standard inclusions, and write $ji = k\colon P^1 \hookrightarrow P^\infty$. If f is compressible we have $f \simeq if_1$ for some $f_1\colon V \to P^1$, whence $Sq^1 y = Sq^1 f^*(x) = Sq^1 f_1^* i^*(x) = f_1^* Sq^1 i^*(x) = 0$ since $Sq^1 i^*(x) \in H^2(P^1;Z_2) = 0$. On the other hand, suppose $Sq^1 y = 0$. Since $Sq^1 = \beta$, the Bockstein homomorphism arising from the coefficient homomorphisms $0 \to Z_2 \to Z_4 \to Z_2 \to 0$, this is equivalent to y being the reduction of a Z_4 cohomology class. A quick look at the (well known for any surface V) cohomology $H^*(V;\Lambda)$ for $\Lambda = Z, Z_4$ shows that all classes in $H^1(V;Z_4)$ are in fact reductions of integral classes in $H^1(V;Z)$.

Hence, letting ρ be the coefficient homomorphism induced by $Z \twoheadrightarrow Z_2$, there exists $z \in H^1(V;Z)$ with $\rho(z) = y$. Since $P^1 \cong S^1$ is a $K(Z,1)$, there is a map $h\colon V \to P^1$ representing z. Let $g = ih$, which is evidently compressible. It is well known that $kh\colon V \to P^\infty = K(Z_2,1)$ represents on one hand the cohomology class $(kh)^*(x)$, and on the other hand also represents $\rho(z)$. Thus $y = \rho(z) = (kh)^*(x) = (jih)^*(x) = (jg)^*(x) = g^* j^*(x) = g^*(x)$, as required. □

§3. Structure of $[V,P^2]$

3.1 **Theorem:** *Let* V *be a closed surface,* $f\colon V \to P^2$, *and* $Y = f^*(x) \in H^1(V;Z_2)$. *We have*

(*a*) *If* f *is orientation true then there are countably many classes* $[g] \in [V,P^2]$ *with,* $g^*(x) = y$,

(*b*) *If* f *is non-orientation true then there are exactly two classes* $[g] \in [V,P^2]$ *with* $g^*(x) = y$,

(*c*) *In the non-orientation true case, with* V *orientable, both classes corresponding to the given* y *are compressible.*

Proof: The condition $f^*(x) = g^*(x)$ implies $i^*f^*(x) = i^*g^*(x)$, i.e. $(fi)^*(x) = (gi)^*(x)$, where $i\colon L \hookrightarrow V$ is the inclusion of the 1-skeleton of V. Letting $j\colon P^2 \hookrightarrow P^\infty$, we have $j_\#\colon [L,P^2] \to [L,P^\infty]$ and the previous equality is the same as $j_\#[fi] = j_\#[gi]$. But $\dim L = 1$ implies that $j_\#$ is an isomorphism, by the cellular approximation thoerem. So $[fi] = [gi]$, or equivalently $f|L \simeq g|L$. Conversely, $f|L \simeq g|L$ implies $i^*f^*(x) = i^*g^*(x)$, but $i^*\colon H^1(V;Z_2) \to H^1(L;Z_2)$ is monic so $f^*(x) = g^*(x)$. Thus, as mentioned in [1], $f^*(x) = g^*(x)$ is equivalent to $f|L \simeq g|L$.

By standard obstruction theory, the remaining obstruction to a homotopy on all of V $(= V^{(2)})$ lies in $H^2(V;\ \underline{\pi_2(P^2)})$, where $\pi_2(P^2)$ is the local coefficient system $\pi_2(P^2) = Z$ with the non-trivial action of $\pi_1(P^2) = Z_2$, pulled back by $f_* = \pi_1(V) \to \pi_1(P^2)$ to form a local coefficent system on V. Since $\underline{\pi_2(P^2)}$ is isomorphic as a local system over P^1 to the orientation bundle of coefficients $\beta_{P^2}(\pi)$, our obstruction lies in $H^2(V;f^{-1}\beta_{P^2}(\pi)) \approx Z$ (cf. [7], p.201). Otherwise, it is not hard to see that the local system of coefficients will have the effect of identifying the fundamental cocycle in $C^2(V,f^{-1}\beta_{P^2}(\pi))$ with its negative, so

$H^2(V; \underline{f^{-1}{}_{P^2}(\pi)}) \cong Z_2$ here. This proves (a), (b).

For (c) it will be useful to first observe that $\pi_2(P^2,P^1)$ may be identified with the group ring $Z[Z_2]$, where $Z_2 = \{1,t\}$ and t acts as $t(m + tn) = n + mt$, representing the action of $\pi_1(P^2) = Z_2$ on this group. This can be seen from the exact homotopy sequence

$$0 \to \pi_2(P^2) = Z \longrightarrow \pi_2(P^2,P^1) \xrightarrow{\partial} \pi_1(P^1) = Z \twoheadrightarrow \pi_1(P^2) = Z_2,$$

which shows $\mathrm{Im}\partial = 2Z \approx Z$. Thus $\pi_2(P^2,P^1)$ is an extension of Z by Z, and must be the non-abelian extension since the action of $\pi_1(P^2)$ on $\pi_2(P^2)$ is non-trivial. This is precisely $Z[Z_2]$.

Again following [1], we note that the obstruction to a compression into P^1 lies in $H^2(V; \underline{\pi_2(P^2,P^1)})$. This group will be (in the case f non-orientation true and V orientable)

$$Z[Z_2]/(1 - t) \approx Z.$$

It follows that the map

$$Z_2 \approx H^2(V; \underline{\pi_2(P^2)}) \to Z \approx H^2(V; \underline{\pi_2(P^2,P^1)})$$

is zero, which means that for the two classes [f], [g], $\in [V,P^2]$ corresponding to a given non-orientation true $y \in H^1(V;Z_2)$, the difference between their obstructions to compressibility is zero. Thus they are either both compressible or both incompressible. Combining this with 2.10 completes the proof of (c), since $Sq^1 = 0$ on $H^1(V;Z_2)$ when V is orientable.

3.2 Remark: In case (c) Adams does not distinguish between V orientable or non-orientable. However, his result certainly does not hold if V is non-orientable, an easy example being $V = P^2$ and $0 = y \in H^1(P^2;Z_2)$. Certainly $Sq^1y = 0$, and $y \neq w_1(P^2)$, so it represents a non-orientation true map, but $[P^2,P^1] \approx H^1(P^2;Z) = 0$ shows that there is at most one compressible homotopy class in $[P^2,P^2]$.

For an orientable surface V, further details giving the complete classification of the compressible maps of V into P^2 are given in [8]. The incompressible ones, of course, satisfy $f^*(x) = 0$ so lift to maps $V \to S^2$ and are readily classified by their Brouwer degree. In the applications to relativistic kinks, the Brouwer degree of these incompressible maps corresponds to a known physical invariant, the "kink number". For the compressible maps, it is not yet clear what the physical distinction between the two homotopy classes corresponding to a given $0 \neq y \in H^1(V;Z_2)$ represents.

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