3-MANIFOLDS AND RELATIVISTIC KINKS

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§1. Introduction

By M we shall mean an orientable, compact, connected 3-manifold without boundary. We say that M has type 1 if it admits a degree 1 map onto $\mathbb{R}P^3$ (written P^3), otherwise M is said to have type 2. For example P^3 has type 1 and S^3 type 2. Type was first studied in [9] in connection with the classification of relativistic kinks, and we explain this connection briefly in §2. We then summarize a few more recent results which help to determine the type of M. These were obtained by A.R. Shastri and the author [10], and are illustrated in §3 by determining the types of many 3-manifolds (Table 3.1). The main theorem (2.5) relates the type of M to $H_1(M;Z)$. Details of the proofs are largely omitted and can be found in [9] and [10].

Most of this material was presented in a lecture at the 11th Winter School in Geometry and Physics in Srní, Czecho-Slovakia. A number of interesting questions were posed there. In §4 these questions are mentioned and in most cases answered.

§2. Relation of Type with $H_1(M;Z)$ and Relativistic Kinks

In relativity theory one is interested in the homotopy classification of Lorentz metrics over the space-time manifold $M \times \mathbb{R}$. Finkelstein and Misner were the first to consider this [4] (for $M = S^3$) and used the term "kinks" to represent non-homotopic classes. It is not difficult to show, using the parallelizability of M [12] and hence also of $M \times \mathbb{R}$, that the homotopy classes of Lorentz metrics are equivalent to $[M,P^3]$, the homotopy classes of (pointed) maps of M into P^3.

Since P^3 is a topological group $[M,P^3]$ has a natural group structure. One then can take advantage of the fibration $Z_2 \to S^3 \xrightarrow{\eta} P^3$, the inclusion $P^3 \xrightarrow{i} P^\infty$, and prove the following.

2.1 Theorem:

(a) There is a short exact sequence of groups

$$E: [M,S^3] \rightarrowtail^{\eta\#} [M,P^3] \xrightarrow{i\#} \twoheadrightarrow [M,P^\infty],$$

(b) $[M,S^3] \approx Z$, classified by Brouwer degree, and

$[M,P^\infty] \approx H^1(M;Z_2)$ since P^∞ is an Eilenberg-MacLane space.

(c) $[M,P^3]$ is abelian.

It follows that the calculation of $[M,P^3]$ depends only whether E splits, and this is where the type of M is critical.

2.2 Theorem: The following are equivalent:

(i) E is does not split.

(ii) M is of type 1.

(iii) There is an element $x \in H^1(M;Z_2)$ with $x^3 \neq 0$.

2.3 Corollary: Let $H^1(M;Z_2) \approx Z_2^m$, $m \geq 0$. Then

$$[M,P^3] \approx \begin{cases} Z \oplus Z_2^{m-1}, & M \text{ type } 1 \\ Z \oplus Z_2^m, & M \text{ type } 2 \end{cases}.$$

As mentioned in §1, details of 2.1 – 2.3 all appear in [9]. We now turn to results that help determine the type of M, for details of which we refer to [10].

2.4 Proposition: For the connected sum $M_1 \# M_2$, one has
type $(M_1 \# M_2) = \min\{\text{type } M_1, \text{type } M_2\}$.

It follows that it suffices to determine the type of irreducible 3–manifolds.

2.5 Theorem:

(a) If M has type 1 then $H_1(M;Z)$ admits Z_2 as a direct summand.

(b) If $H_1(M;Z)$ admits Z_2 but not Z_2^2 as a direct summand then M has type 1.

Our final result helps to determine the type of $M = S^3/G$, where G is a finite subgroup of S^3 (in case Theorem 2.5 does not suffice).

2.6 Proposition: $M = S^3/G$ has type 1 if and only if the cohomology of the group G contains a class $y \in H^1(G;Z_2)$ with $y^3 \neq 0$ (here $H^1(G;Z_2)$ is with the trivial action on Z_2).

§3. Applications to Specific Manifolds

We now list a number of examples of 3–manifolds M (or families of such) together with $\pi_1(M)$, $H_1(M;Z) = \pi_1(M)_{ab}$, and type M. Examples (1)–(10) are taken from a list of 3–manifolds of interest in relativity theory [5]. In every case the knowledge of $H_1(M;Z)$ together with Theorem 2.5 suffices to easily determine the type, except for Example (7) which is dealt with by Proposition 2.6. Before listing these examples the notations and definitions of the manifolds will be specified, and following the examples some further details as to the proofs are given. It is also worth noting, in case $H_1(M;Z) \approx Z_n$, that Z_2 will be a direct summand if and only if $n \equiv 2 \pmod 4$, in which case Z_2^2 will not be a direct summand.

As far as the spaces, $S(n) = S^3\#(S^1 \times S^2)\#\ldots\#(S^1 \times S^2)$ is a sphere with n handles, L(m,n) is a lens space (cf.[11],p.88), Σ represents a mod 2 homology 3-sphere and T a fixed point free involution on Σ, and M_g is the total space of a principal S^1-bundle $S^1 \to M_g \to V$, where V is an orientable surface of genus g. As far as the groups, F_n is the free group on n generators, Q_{4n}, P_{24}, P_{48}, P_{120}, Z_n, $D_{(2n+1)2^k}$ $(k \geq 2)$, and $P'_{g\cdot 3^k}$ $(k \geq 1)$ are well known finite subgroups of S^3 (cf.[7]) of order indicated by the subscript and with presentations as follows:

$Q_{4n} = < x,y : x^n = y^2, xyx = y >$;

$P_{24},P_{48},P_{120} = < x,y : x^2 = (xy)^3 = y^m, x^4 = 1 >$, m = 3,4,5 respectively;

$D_{(2n+1)2^k} = < x,y : x^{2^k} = y^{2^{n+1}} = 1, xyx^{-1} = y^{-1} >$

$P'_{8\cdot 3^k} = < x,y,z : x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 >$; and

$H_g = < z,x_1,\ldots,x_g,y_1,\ldots,y_g : [z,x_i] = [z,y_i] = 1, \prod_{1\leq i\leq g} [x_i,y_i] = z^n >$,

where n is the chern number of bundle. From this table and using Corollary 2.3 the classification of the kinks $[M,P^3]$ is immediate.

3.1 TABLE

	M	$\pi_1(M)$	$H_1(M)$	Type
(1)	S^3	0	0	2
(2)	P^3	Z_2	Z_2	1
(3)	$S^1 \times S^2$	Z	Z	2
(4)	$T^3 = S^1 \times S^1 \times S^1$	Z^3	Z^3	2
(5)	$S(n)$	F_n	Z^n	2
(6)	$L(m,n)$, $m \equiv 1(\text{mod}2)$	Z_m	Z_m	2
	$m \equiv 2(\text{mod}4)$	Z_m	Z_m	1
	$m \equiv 0(\text{mod}4)$	Z_m	Z_m	2
(7)	S^3/Q_{4n}, $n \equiv 1(\text{mod}2)$		Z_4	2
	$n \equiv 2(\text{mod}4)$	Q_{4n}	Z_2^2	2
	$n \equiv 0(\text{mod}4)$		Z_2^2	1
(8)	S^3/P_{24}	P_{24}	Z_3	2
(9)	S^3/P_{48}	P_{48}	Z_2	1
(10)	S^3/P_{120}	P_{120}	0	2
(11)	$S^3/D_{(2n+1)2^k}$, $k \geq 2$	$D_{(2n+1)2^k}$	Z_{2^k}	2
(12)	$S^3/P'_{8\cdot 3^k}$, $k \geq 1$	$P'_{8\cdot 3^k}$	Z_{3^k}	2
(13)	Σ/T	not unique	$Z_2 \oplus$ finite abelian group of odd order	1
(14)	M_g	H_g	$Z_n \oplus Z^{2g}$	$\begin{cases} 1, n \equiv 2(\text{mod}4) \\ 2, n \not\equiv 2(\text{mod}4) \end{cases}$

Remarks

To handle example (7) Theorem 2.5 suffices when n is odd but not when n is even. Here we apply Proposition 2.6 and compute the ring structure of $H^*(Q_{4n};Z_2)$ using first an explicit free resolution X of Z over $\Lambda = ZQ_{4n}$ given in [2], p.253, and then computing a Λ–chain map $X \longrightarrow X \otimes X$ to determine products. The details appear in [10]. Example (13) arises in the thesis of Medrano [3], who shows that there are infinitely many non–homeomorphic such 3–manifolds and claims (without proof) they are all type 1. As in all the examples except (7), Theorem 2.5 proves this claim immediately. Example (14) arises in the study of Yang–Mills equations over a Riemann surface [1].

It is interesting to ask whether necessary and sufficient conditions for type 1 can be determined from $H_1(M;Z)$ using Theorem 2.5. This is not the case, for Example (7) with $n \equiv 2(\mathrm{mod}4)$ shows 2.5(a) is not sufficient, while the case $n \equiv 0(\mathrm{mod}4)$ shows 2.5(b) is not necessary.

§4. Further Questions and Observations

4.1 Question: Can M be generalized from a 3–manifold to a more general space.

We answer this by first observing that the exact sequence E in Theorem 2.1(a) is valid for M replaced by any connected CW–complex of dimension ≤ 3 (cf. [9], Prop. 4.2 and its Corollary). For 2.1(b), (c), the definition of type, and Theorems 2.2 – 2.5 to hold, it should suffice that M is a 3–dimensional Poincaré duality space. The manifold case is of course the interesting one for physical applications.

Before turning to the next question, note that the definition of type generalizes to any connected, closed, orientable odd dimension manifold M^{2n+1} with P^3 replaced by P^{2n+1} (P^{2n} is non–orientable so leads to a very different situation). The case P^7 is clearly of special interest.

4.2 Question: What happens if P^3 is replaced by P^7?

The answer is that for M a closed connected orientable 7–dimensional manifold, all of 2.1 – 2.5(a) hold with the same proofs as in the 3–dimensional case, except that it must also be shown $[M,P^7]$ is associative since P^7 is only an H–space with inverses (and not homotopy associative [6], [8]).

To see this one uses the same idea as in proving $[M,P^7]$ (or $[M^3,P^3]$) commutative. Namely, consider E: $Z = [M,S^7] \overset{\eta\#}{\rightarrowtail} [M,P^7] \overset{i\#}{\twoheadrightarrow} [M,P^\infty] = H^1(M;Z_2)$ as a short exact sequence of monoids with inverses. Let $d:[M,P^7] \longrightarrow Z$ be the degree homomorphism, and note $d\eta\#:Z \longrightarrow Z$ is multiplication by 2.

If $\alpha,\beta,\gamma \in [M,P^7]$, then $j\#((\alpha\beta)\gamma) = j\#(\alpha(\beta\gamma))$, so $((\alpha\beta)\gamma)(\alpha(\beta\gamma))^{-1} = z^n$ for some n, where $z = \eta\#(1)$ generates $\mathrm{Im}\eta\# = \mathrm{Ker}j\#$. Then $0 = d[((\alpha\beta)\gamma)(\alpha(\beta\gamma))^{-1}] = d(z^n) = 2n$ gives $n = 0$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

For the 7–dimensional case, however, 2.5(b) no longer is true. An obvious counterexample would be $M = P^5 \times S^2$ (or $P^3 \times S^4$). Here $H_1(M;Z) = Z_2$, but M clearly is not type 1.

4.3 Question: What happens in Example (14) if "principal" is omitted in the bundle $\xi:S^1 \to M \to V$.

In this case we lose the exact knowledge of the fundamental group, for the extension

$$\pi_1(S^1) = Z \rightarrowtail \pi_1(M) \twoheadrightarrow \pi_1(V)$$

need no longer be central. However, the same result connecting type M with $c(\xi)$ is still true, as we now prove.

4.4 Proposition: Let $\xi:S^1 \to M \to V$ be a fibre bundle over a closed orientable surface V of genus g. Then $H_1(M) \approx Z_n \oplus Z^{2g}$, where $n = c(\xi)$ (Z_0 is to be interpreted as Z here).

Proof: Consider the homology spectral sequence, which will have simple coefficients since M is orientable (it is an orientable fibration in the sense of [11], p.476). The only possible non-zero differential is

$$d_2:Z = H_2(V;Z) = E^2_{2,0} \to E^2_{0,1} = H_0(V;H_1(S^1)) = Z.$$

Then d_2 is multiplication by n, and clearly $n = c(\xi)$. One then has $E^\infty_{0,1} = Z_n$ and $E^\infty_{0,1} = Z^{2g}$, from which $H_1(M) \approx Z_n \oplus Z^{2g}$.

Corollary: type M = 1 iff $n \equiv 2 \pmod 4$.

4.5 Question: Does $\pi_1(M)$ determine type M?

A partial answer is that type M = 1 implies the existence of $y \in H^1(\pi_1(M);Z_2)$ with $y^3 \neq 0$. This is readily proved by attaching cells to M in dimensions 3,4,... so as to obtain an inclusion $M \xrightarrow{i} Y = K(\pi_1(M),1)$, and noting that i^* is an isomorphism in dimension 1.

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