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NATURAL TRANSFORMATIONS OF AFFINORS INTO FUNCTIONS AND AFFINORS

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presented by Jacek Gancarzewicz

An affinor on a manifold M is a tensor field of type (1,1) on M which can be interpreted as an endomorphism $TM \longrightarrow TM$ of the tangent bundle covering the identity on M.

In this paper we give a characterization of the natural transformations of affinors into functions and affinors. In section 2 we prove that all natural transformations of affinors on n-dimensional manifolds into functions are of the form $F(a_1(t), ..., a_n(t))$, where $a_1(t), ..., a_n(t)$ denote the coefficients of the characteristic polynomial of t and F is a smooth function on $\mathbf{R}^n$. In section 3 we prove that all natural transformations of affinors (on n-dimensional manifolds) into itself are of the form

$$t \longrightarrow \sum_{i=1}^{n} F_i(a_1(t), ..., a_n(t)) \cdot t^{n-i}$$

where $F_1, ..., F_n$ are smooth functions on $\mathbf{R}^n$.

All manifolds and maps are assumed to be infinitely differentiable.

1. Natural transformations of tensor fields.

Let p, q, r, s, n be positive integers. Let M be an n-dimensional manifold. We denote by $\mathcal{X}^p_q M$ the space of tensor fields of type (p, q) on M.

A family of maps $T_M : \mathcal{X}^p_q M \longrightarrow \mathcal{X}^r_s M$ is called a natural transformation of tensor fields if :

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

(1) for any M, any open $U \subset M$ and all $t_1, t_2 \in \mathcal{X}^p_q M$ the following implication

$$t_1|U = t_2|U \Longrightarrow T_M t_1|U = T_M t_2|U$$

is true,

(2) for any two n-dimensional manifolds M, N and for every injective immersion $\varphi : M \longrightarrow N$ we have

$$\varphi_* \circ T_M = T_N \circ \varphi_*$$

Using Borel's lemma in a standard way (see [3]) we can easily verify that for tensor fields $t_1, t_2 \in \mathcal{X}^p_q M$ and a point $x \in M$ we have

$$j^\infty_x t_1 = j^\infty_x t_2 \Longrightarrow (T_M t_1)(x) = (T_M t_2)(x)$$

Let k be either a positive integer or ∞ and let L^{k+1}_n be the group of $(k+1)$-jets of local diffeomorphisms of $\mathbf{R}^n$ with source and target $0 \in \mathbf{R}^n$.

We denote $V_{p,q} = \bigotimes^p \mathbf{R}^n \otimes \bigotimes^q (\mathbf{R}^n)^*$. The linear group $GL(n, \mathbf{R})$ acts on $V_{p,q}$ in the natural way.

Let $V^k_{p,q} = J^k_0(\mathbf{R}^n, V_{p,q})$. If $X = j^k_0 t$ then $(t^{(0)}, t^{(1)}, t^{(2)}, ...)$ are coordinates of X, where

$$t^{(s)} = \{\frac{\partial^s t^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial u^{k_1} \ldots \partial u^{k_s}} : i_1, ..., i_p, j_1, ..., j_q, k_1, ..., k_s = 1, ..., n\}$$

The group L^{k+1}_n acts on $V^k_{p,q}$ in the natural way: if $\xi = j^{k+1}_0 \varphi$, $X = j^k_0 t$ then $\xi \cdot X$ is the k-jet at 0 of

$$\mathbf{R}^n \ni u \longrightarrow J_u(\varphi) \cdot t(u) \in V_{p,q}$$

where $J_u(\varphi)$ is the Jacobi matrix of φ at u.

It is easy to verify that for a homothety

$$\kappa_c(u) = \frac{1}{c} \cdot u$$

where $c \in \mathbf{R} \backslash \{0\}$, the coordinates $(\bar{t}^{(0)}, \bar{t}^{(1)}, \bar{t}^{(2)}, ...)$ of $(j^{k+1}_0 \kappa_c) \cdot X$ are given by $\bar{t}^{(s)} = c^s \cdot t^{(s)}$ for $s = 0, 1, 2, ...$.

A map $E : V^k_{p,q} \longrightarrow V_{r,s}$ is called equivariant if

$$E((j^{k+1}_0 \varphi) \cdot X) = J_0(\varphi) \cdot E(X)$$

for $j^{k+1}_0 \varphi \in L^{k+1}_n$, $X \in V^k_{p,q}$.

We have the following:

Proposition 1.1. *There is a one-to-one correspondence between natural transformations of tensor fields of type (p,q) into tensor field of type (r,s) and equivariant maps $E : V^{\infty}_{p,q} \longrightarrow V_{r,s}$ which satisfies the condition :*

(3) for every open subset $\Omega \subset \mathbf{R}^n$ and every smooth $\gamma : \Omega \longrightarrow V_{p,q}$ the map

$$\Omega \ni x \longrightarrow E(j^{\infty}_0(\gamma \circ \tau_x)) \in V_{r,s}$$

is smooth, where $\tau_x : \mathbf{R}^n \ni y \longrightarrow y + x \in \mathbf{R}^n$ is the translation by vector x.

If T is a natural transformation then the corresponding equivariant map E_T is defined by

$$E_T(j^{\infty}_0 t) = T_{\mathbf{R}^n}(t)(0)$$

If E is an equivariant map, then the corresponding natural transformation T^E is defined by

$$(T^E_M t)(x) = (T^r_s \varphi^{-1})(E(j^{\infty}_0(\varphi_* t)))$$

where φ is a local system of coordinates on M such that $\varphi(x) = 0$.

The one-to-one correspondence between natural transformations and equivariant maps is formulated in Krupka's theorem [2]. We prove only that for a natural transformation T the corresponding equivariant map E_T satisfies the condition (3) and that for every equivariant map E which satisfies the condition (3) and for every tensor fields t of type (p,q) on an n-dimensional manifolds M the map $T^E_M(t)$ is smooth.

We have

$$E_T(j^{\infty}_0(\gamma \circ \tau_x)) = E_T(j^{\infty}_0((\tau_{-x})_* \gamma))$$
$$= T_{\mathbf{R}^n}((\tau_{-x})_* \gamma)(0) = (\tau_{-x})_* T_{\mathbf{R}^n}(\gamma)(0) = T_{\mathbf{R}^n}(\gamma)(x)$$

Since $T_{\mathbf{R}^n}(\gamma)$ is smooth, E_T satisfies the condition (3).

Now let us suppose that an equivariant map E satisfied (3) and that $\varphi : U \longrightarrow \mathbf{R}^n$ is a local system of coordinates on M such that $\varphi(x) = 0$. For every $y \in U$ the composition $\tau_{\varphi(y)} \circ \varphi$ is a local system of coordinates on M and $(\tau_{-\varphi(y)} \circ \varphi)(y) = 0$. We have

$$(T^E_M t)(y) = T^r_s(\tau_{-\varphi(y)} \circ \varphi)^{-1}(E(j^{\infty}_0((\tau_{-\varphi(y)} \circ \varphi)_* t))) = T^r_s \varphi^{-1}(E(j^{\infty}_0((\varphi_* t) \circ \tau_{\varphi(y)})))$$

By (3) the map $f(z) = E(j^{\infty}_0(\varphi_* t \circ \tau_z))$ is smooth, hence $T^E_M t = \varphi^{-1}_* f$ is smooth.

A natural transformation T of tensors of type (p,q) into tensors of type (r,s) has order k if for any n-dimensional manifold M, any $x \in M$ and all $t_1, t_2 \in \mathcal{X}^p_q M$ the following implication

$$j^k_x t_1 = j^k_x t_2 \Longrightarrow (T_M t_1)(x) = (T_M t_2)(x)$$

holds.

From Proposition 1.1 we deduce that T is of order k if and only if the following implication

$$j_0^k s_1 = j_0^k s_2 \Longrightarrow E_T(j_0^\infty s_1) = E_T(j_0^\infty s_2)$$

holds for every smooth $s_1, s_2 : \mathbf{R}^n \longrightarrow V_{p,q}$.

We prove now the following:

Proposition 1.2. *If $p = r$ and $r = s$ then every natural transformation of tensors of type (p,q) into tensors of type (r,s) has order zero.*

To prove this proposition we need the following:

Lemma 1.3. *Let $f : \mathbf{R}^n \longrightarrow V_{p,q}$ be a smooth map such that support f is compact. Then there is a smooth map $F : \mathbf{R}^n \longrightarrow V_{p,q}$ such that for any $i \in \mathbf{N}$ and for any $\alpha \in \mathbf{N}^n$*

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} F(\frac{1}{i}, 0, \dots, 0) = \frac{1}{(2^{i-1})^{|\alpha|}} \cdot \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(0) \tag{4}$$

This lemma implies immediately that $F(0) = f(0)$ and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} F(0) = 0$$

for $|\alpha| > 0$.

Proof: Let $\varphi : \mathbf{R} \longrightarrow [0,1]$ be a smooth function such that $\varphi(x) = 0$ if $x < -1 + \varepsilon$ and $\varphi(x) = 1$ if $x > -\varepsilon$ for some $\varepsilon > 0$. For $i \in \mathbf{N}$, $x \in \mathbf{R}^n$ we denote

$$\varphi_i(x) = \begin{cases} \varphi(i \cdot (i+1) \cdot (x_1 - \frac{1}{i})) & \text{if } x_1 \le \frac{1}{i} \\ 1 - \varphi((i-1) \cdot i \cdot (x_1 - \frac{1}{i-1})) & \text{if } x_1 \ge \frac{1}{i} \text{ and } i > 1 \\ 1 & \text{if } x_1 \ge 1 \text{ and } i = 1 \end{cases}$$

$$f_i(x) = f(\frac{1}{2^{i-1}} \cdot (x_1 - \frac{1}{i}), \frac{1}{2^{i-1}} \cdot x_2, \dots, \frac{1}{2^{i-1}} \cdot x_n)$$

and

$$F(x) = \begin{cases} \sum_{i=1}^\infty \varphi_i(x) \cdot f_i(x) & \text{if } x_1 > 0 \\ f(0) & \text{if } x_1 \le 0 \end{cases}$$

The proof that F is smooth and satisfies (4) is standard.

Proof of Proposition 1.2: Let T be a natural transformation of tensors of type (p,q) into tensors of type (r,s) and let f be a smooth map $\mathbf{R}^n \longrightarrow V_{p,q}$ with compact support. For every $c \in \mathbf{R} \backslash \{0\}$ we have

$$E_T(f^{(0)}, c \cdot f^{(1)}, c^2 \cdot f^{(2)}, \dots) = E_T((j_0^\infty \kappa_c) \cdot (j_0^\infty f)) = J_0(\kappa_c) \cdot E_T(j_0^\infty f) = E_T(j_0^\infty f)$$

Let F be the function from Lemma 1.3. Then we have

$$E_T(j_0^\infty(F \circ \tau_{(\frac{1}{i},0,\dots,0)})) = E_T(f^{(0)}, \frac{1}{2^{i-1}} \cdot f^{(1)}, (\frac{1}{2^{i-1}})^2 \cdot f^{(2)}, \dots) = E_T(j_0^\infty f)$$

for every $i \in \mathbf{N}$. Since the map $x \longrightarrow E_T(j_0^\infty(F \circ \tau_x))$ is smooth, we obtain that

$$E_T(j_0^\infty f) = \lim_{i \to \infty} E_T(j_0^\infty f)$$

$$= \lim_{i \to \infty} E_T(j_0^\infty(F \circ \tau_{(\frac{1}{i},0,\dots,0)})) = E_T(j_0^\infty F) = E_T(f^{(0)}, 0, 0, \dots)$$

Hence for all smooth $t_1, t_2 : \mathbf{R}^n \longrightarrow V_{p,q}$ such that $j_0^0 t_1 = j_0^0 t_2$ we have

$$E_T(j_0^\infty t_1) = E_T(t_1^{(0)}, 0, 0, \dots) = E_T(t_2^{(0)}, 0, 0, \dots) = E_T(j_0^\infty t_2)$$

2. Classification of natural transformations of affinors into functions.

If $L : V \longrightarrow V$ is an endomorphism of an n-dimensional vector space V then $a_1(L), \dots, a_n(L)$ denote the coefficients of the characteristic polynomial

$$W_L(\lambda) = \det(\lambda \cdot id_V - L) = \lambda^n + a_1(L)\lambda^{n-1} + \dots + a_n(L) id_V$$

Theorem 2.1 *There is a one-to-one correspondence between natural transformations of affinors into functions and all smooth functions $F : \mathbf{R}^n \longrightarrow \mathbf{R}$. The natural transformation corresponding to a function F is defined by*

$$(T_M t)(x) = F(a_1(t_x), \dots, a_n(t_x))$$

for every an n-dimensional manifold M, $t \in \mathcal{X}_1^1, x \in M$.

Propositions 1.1 and 1.2 ensure that Theorem 2.1 is equivalent to the following:

Proposition 2.2. *There is a one-to-one correspondence between all smooth functions $F : \mathbf{R}^n \longrightarrow \mathbf{R}$ and functions $G : \mathbf{R}^n \otimes (\mathbf{R}^n)^* \longrightarrow \mathbf{R}$ such that*

(5) for every open set $\Omega \subset \mathbf{R}^n$ and for every smooth map $\gamma : \Omega \longrightarrow \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ the composition $G \circ \gamma$ is smooth,

(6) for all matrices $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$, $A \in GL(n, \mathbf{R})$ we have $G(A \cdot X \cdot A^{-1}) = G(X)$.

The function G corresponding to the function F is defined by

$$G(X) = F(a_1(X), \dots, a_n(X)) \tag{7}$$

for every matrix $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$.

Proof: It is clear that for any smooth function $F : \mathbf{R}^n \longrightarrow \mathbf{R}$ the formula (7) defines a function such that the conditions (5) and (6) hold.

We need to show that for a function $G : \mathbf{R}^n \otimes (\mathbf{R}^n)^* \longrightarrow R$ we can construct a function F for which the equality (7) holds. It suffices to prove that for all matrices $X_1, X_2 \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ the following implication

$$W_{X_1} = W_{X_2} \Longrightarrow G(X_1) = G(X_2) \tag{8}$$

holds. The condition (6) says that the function G is constant on orbits of the group $GL(n, \mathbf{R})$. Let J_i be Jordan's matrices equivalent to X_i for $i = 1, 2$. The matrices J_i are of the form

$$\begin{bmatrix}
\lambda_1 &&&&&&&&&&&&& \\
\varepsilon^i_{11} & \lambda_1 &&&&&&&&&&&& \\
& \varepsilon^i_{12} & \ddots &&&&&&&&&&& \\
&&& \lambda_2 &&&&&&&&&& \\
&&& \varepsilon^i_{21} & \lambda_2 &&&&&&&&& \\
&&&& \varepsilon^i_{22} & \ddots &&&&&&&& \\
&&&&&& \ddots &&&&&&& \\
&&&&&&& A_1 &&&&&& \\
&&&&&&& E^i_{11} & A_1 &&&&& \\
&&&&&&&& E^i_{12} & \ddots &&&& \\
&&&&&&&&&& A_2 &&& \\
&&&&&&&&&& E^i_{21} & A_2 && \\
&&&&&&&&&&& E^i_{22} & \ddots & \\
&&&&&&&&&&&&& \ddots
\end{bmatrix}$$

where $\varepsilon^i_{11}, \varepsilon^i_{12}, \ldots, \varepsilon^i_{21}, \varepsilon^i_{22}, \ldots, \ldots$ are either 0 or 1 and $E^i_{11}, E^i_{12}, \ldots, E^i_{21}, E^i_{22}, \ldots, \ldots$ are either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, A_2 = \begin{bmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix}, \ldots$$

The coefficients $\lambda_1, \lambda_2, \ldots$ and the matrices $A_1, A_2, \ldots$ are the same in both matrices J_1 and J_2 because $W_{J_1} = W_{X_1} = W_{X_2} = W_{J_2}$ and $\lambda_1, \lambda_2, \ldots, \alpha_1 - i\beta_1, \alpha_1 + i\beta_1, \alpha_2 - i\beta_2, \alpha_2 + i\beta_2, \ldots$ are the eigenvalues. Let us denote

$$P : \mathbf{R}^n \ni t \longrightarrow t_1 \cdot J_1 + (1 - t_1) \cdot J_2 \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$$

For every $t \in \mathbf{R}^n$ the matrix $P(t)$ has the same characteristic polynomial as the matrices J_1, J_2. Clearly all matrices having the same characteristic polynomial are included in a finite number of orbits, because every orbit holds Jordan's matrix and there is a finite number of systems $\varepsilon_{11}, \varepsilon_{12}, ..., \varepsilon_{21}, \varepsilon_{22}, ..., ...,$ $E_{11}, E_{12}, ..., E_{21}, E_{22}, ..., ...$. Hence $(G \circ P)(\mathbf{R}^n)$ is a finite set. From (5) the composition $G \circ P$ is the continuous function. Hence $G \circ P$ is a constant function. In particular $(G \circ P)(1, 0, ..., 0) = (G \circ P)(0, 0, ..., 0)$ and the condition (8) is satisfied.

We denote

$$S : \mathbf{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & & & -x_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -x_2 \\ & & 1 & -x_1 \end{bmatrix} \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$$

It is easily seen that $a_i(S(x)) = x_i$ for $i = 1, ..., n$ and $F = G \circ S$. Hence F is unique and smooth, as the condition (5) is satisfied.

3. Classification of natural transformations of affinors into affinors.

Theorem 3.1. *There is a one-to-one correspondence between natural transformations of affinors into affinors and all systems of n smooth functions $F_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i = 1, ..., n$. The natural transformation corresponding to functions F_i is defined by*

$$(T_M t)(x) = \sum_{i=1}^{n} F_i(a_1(t_x), ..., a_n(t_x)) \cdot t_x^{n-i}$$

for every an n-dimensional manifold M, $t \in \mathcal{X}_1^1 M$, $x \in M$, where $t_x^k = t_x \circ ... \circ t_x$ (k times).

Propositions 1.1 and 1.2 ensure that Theorem 3.1 is equivalent to following:

Proposition 3.2 *There is a one-to-one correspondence between all systems of n smooth functions $F_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i = 1, ..., n$ and maps $G : \mathbf{R} \otimes (\mathbf{R}^n)^* \longrightarrow \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ such that*

(9) for every open set $\Omega \subset \mathbf{R}^n$ and every smooth map $\gamma : \Omega \longrightarrow \mathbf{R}^n \otimes (\mathbf{R}^n)^*$ the composition $G \circ \gamma$ is smooth,

(10) for all matrices $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$, $A \in GL(n, \mathbf{R})$ we have $G(A \cdot X \cdot A^{-1}) = A \cdot G(X) \cdot A^{-1}$.

The map G corresponding to functions F_i is defined by

$$G(X) = \sum_{i=1}^{n} F_i(a_1(X), ..., a_n(X)) \cdot X^{n-i} \tag{11}$$

for every matrix $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$.

Proof: It is sufficient to show that for every G satisfying (9) and (10) there are the unique smooth functions F_i for $i = 1, ..., n$ such that the equality (11) holds. If F_i satisfy (11) then for $x \in \mathbf{R}^n$ we have

$$G(S(x)) = \sum_{i=1}^{n} F_i(x) \cdot (S(x))^{n-1} = \begin{bmatrix} F_n(x) & \dots \\ \vdots & \\ F_1(x) & \dots \end{bmatrix}$$

because $(S(x))^i(e_1) = e_{i+1}$ for $i = 1, ..., n-1$ where $e_1, ..., e_n$ denotes the canonical basis in $\mathbf{R}^n$. Hence the functions F_i are unique. Let us define

$$F_i(x) = G(S(x))_1^{n-i+1} \tag{12}$$

for $i = 1, ..., n$. Clearly F_i are smooth from (9). We only need to show that F_i satisfy (11).

At first we prove (11) for a matrix X which has n different eigenvalues. We need following:

Lemma 3.3. *Let us suppose that $X \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. If the matrix X has n different eigenvalues then X^{n-i} for $i = 1, ..., n$ are linearly independent.*

Proof: From Jordan's theorem the matrix

$$Y = \begin{bmatrix} 0 & & & -a_n(X) \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -a_2(X) \\ & & 1 & -a_1(X) \end{bmatrix}$$

is equivalent to the matrix X because X has n different eigenvalues. Assume $Y = A \cdot X \cdot A^{-1}$ where $A \in GL(n, \mathbf{R})$ and $\sum_{i=1}^{n} \lambda_i \cdot X^{n-i} = 0$, then

$$0 = A \cdot (\sum_{i=1}^{n} \lambda_i \cdot X^{n-i}) \cdot A^{-1} = \sum_{i=1}^{n} \lambda_i \cdot (A \cdot X \cdot A^{-1})^{n-i} = \sum_{i=1}^{n} \lambda_i \cdot Y^{n-i} = \begin{bmatrix} \lambda_n & \dots \\ \vdots & \\ \lambda_1 & \dots \end{bmatrix}$$

Hence $\lambda_i = 0$ for $i = 1, ..., n$. This prove the lemma.

If the matrix X has n different eigenvalues then from Jordan's theorem there exists $A \in GL(n, \mathbf{R})$ such that $X = A \cdot J \cdot A^{-1}$ where

$$J = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix} & & \\ & & & & \begin{bmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix}$$

Let us denote

$$K = \begin{bmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

Clearly $K^{-1} = K$ and $K \cdot J \cdot K^{-1} = J$. From (10) we have $K \cdot G(J) \cdot K^{-1} = G(K \cdot J \cdot K^{-1}) = G(J)$. Multiplying an arbitrary matrix by K on the left is equivalent to multiplying the first row of this matrix by -1. Multiplying an arbitrary matrix by K on the right is equivalent to multiplying the first column of this matrix by -1. Hence the terms of the first row and first column of matrix $G(J)$ are equal to zero except for the term in the (1,1) entry.

Suppose l denotes the integer such that the matrix

$$\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}$$

is on the lth and $(l+1)$th rows and the lth and $(l+1)$th columns in the matrix J. We denote

$$L = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

where the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

is on the lth and $(l+1)$th rows and the lth and $(l+1)$th columns in the matrix L. Clearly $L^{-1} = L$ and $L \cdot J \cdot L^{-1} = J$. From (10) we have $L \cdot G(J) \cdot L^{-1} = G(L \cdot J \cdot L^{-1}) = G(J)$. Multiplying an arbitrary matrix by L on the left is equivalent to multiplying the lth and $(l+1)$th rows of this matrix by -1. Multiplying an arbitrary matrix by L on the right is equivalent to multiplying the lth and $(l+1)$th columns of this matrix by -1. Hence the terms of lth and $(l+1)$th rows and the lth and $(l+1)$th columns of the matrix $G(J)$ are equal to zero except for the terms in the (l,l), $(l,l+1)$, $(l+1,l)$ $(l+1,l+1)$ entries.

Repeated application of the argument above enables us to write

$$G(J) = \begin{bmatrix} \eta_1 & & & & & \\ & \eta_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} \gamma_1 & \delta_1 \\ \varepsilon_1 & \zeta_1 \end{bmatrix} & & \\ & & & & \begin{bmatrix} \gamma_2 & \delta_2 \\ \varepsilon_2 & \zeta_2 \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix} \tag{13}$$

We denote

$$M = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ & & & & & & \ddots \end{bmatrix}$$

We see that for all $\alpha, \beta \in \mathbf{R}$ we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

It follows that $M \cdot J \cdot M^{-1} = N \cdot J \cdot N^{-1}$. From (10) we have $M \cdot G(J) \cdot M^{-1} = G(M \cdot J \cdot M^{-1}) = G(N \cdot J \cdot N^{-1}) = N \cdot G(J) \cdot N^{-1}$. We see that for all $\gamma, \delta, \varepsilon, \zeta \in \mathbf{R}$ we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \gamma & \delta \\ \varepsilon & \zeta \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \zeta & \varepsilon \\ \delta & \gamma \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma & \delta \\ \varepsilon & \zeta \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \gamma & -\delta \\ -\varepsilon & \zeta \end{bmatrix}$$

From the equality $M \cdot G(J) \cdot M^{-1} = N \cdot G(J) \cdot N^{-1}$ and from (13) we conclude that

$$\begin{bmatrix} \zeta_1 & \varepsilon_1 \\ \delta_1 & \gamma_1 \end{bmatrix} = \begin{bmatrix} \gamma_1 & -\delta_1 \\ -\varepsilon_1 & \zeta_1 \end{bmatrix}$$

Hence $\zeta_1 = \gamma_1$ and $\varepsilon_1 = -\delta_1$. Repeated application of the above arguments enables us to write

$$G(J) = \begin{bmatrix} \eta_1 & & & & & \\ & \eta_2 & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} \gamma_1 & -\delta_1 \\ \delta_1 & \gamma_1 \end{bmatrix} & & \\ & & & & \begin{bmatrix} \gamma_2 & -\delta_2 \\ \delta_2 & \gamma_2 \end{bmatrix} & \\ & & & & & \ddots \end{bmatrix} \tag{14}$$

Let V be a set consisting of all matrices of the form (14) for $\eta_1, \eta_2, ..., \gamma_1, \delta_1, \gamma_2, \delta_2, ... \in \mathbf{R}$. The set V is an n-dimensional linear space. By induction $J^k \in V$ for every $k \in \mathbf{N}$. By Lemma 3.3 the matrices J^{n-i} for $i = 1, ..., n$ form a basis of V. Hence there is λ_i such that $G(J) = \sum_{i=1}^{n} \lambda_i \cdot J^{n-i}$ as $G(J) \in V$. By (10) we have

$$G(X) = G(A \cdot J \cdot A^{-1}) = A \cdot G(J) \cdot A^{-1}$$

$$= A \cdot (\sum_{i=1}^{n} \lambda_i \cdot J^{n-i}) \cdot A^{-1} = \sum_{i=1}^{n} \lambda_i \cdot (A \cdot J \cdot A^{-1})^{n-i} = \sum_{i=1}^{n} \lambda_i \cdot X^{n-i}$$

We need only show that $\lambda_i = F_i(a_1(X), ..., a_n(X))$ for $i = 1, ..., n$. From Jordan's theorem there exists $B \in GL(n, \mathbf{R})$ such that $S(a_1(X), ..., a_n(X)) = B \cdot X \cdot B^{-1}$ as matrix X has n different eigenvalues. Hence

$$G(S(a_1(X), ..., a_n(X))) = B \cdot G(X) \cdot B^{-1} = B \cdot (\sum_{i=1}^{n} \lambda_i \cdot X^{n-i}) \cdot B^{-1}$$

$$= \sum_{i=1}^{n} \lambda_i \cdot (B \cdot X \cdot B^{-1})^{n-i} = \sum_{i=1}^{n} \lambda_i \cdot (S(a_1(X), ..., a_n(X)))^{n-i} = \begin{bmatrix} \lambda_n & \dots \\ \vdots & \\ \lambda_1 & \dots \end{bmatrix}$$

From (12) we have that $\lambda_i = F_i(a_1(X), ..., a_n(X))$ for $i = 1, ..., n$. This proves (11) for the case of X with n different eigenvalues.

We next show (11) in general case. Let X be an arbitrary matrix and let Y be a matrix which has n different eigenvalues. Let P be an n-dimensional affine subspace in the $\mathbf{R}^n \otimes (\mathbf{R}^n)^*$ such that $X, Y \in P$. Suppose that W(Z) denotes the discriminant of the characteristic polynomial of a matrix $Z \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. Then W is a polynomial and $W(Z) \neq 0$ if and only if Z has n different eigenvalues. We have $W|P \neq 0$ because $W(Y) \neq 0$. Hence $Q = \{Z \in P | W(Z) \neq 0\}$ is a dense subset of P. We know that $G|Q = C|Q$ where $C(Z) = \sum_{i=1}^{n} F_i(a_1(Z), ..., a_n(Z)) \cdot Z^{n-i}$ for $Z \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$. Suppose D denotes an affine parametrization of P. From (9) $G \circ D$ is smooth and $G|P = G \circ D \circ D^{-1}$ is smooth too. If two smooth maps are equal on a dense set then these maps are equal. In particular $G(X) = C(X)$. This ends the proof.

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