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DERIVED ALGEBRA OF THE FRÖLICHER-NIJENHUIS BRACKET ALGEBRA

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All structures appearing in this note are of class C^∞. Let M be a connected paracompact manifold, $\dim M = m$. Let TM and T^*M denote the tangent and cotangent bundle of M respectively. We denote by $\Lambda^i T^*M$ the i-th exterior power of T^*M. We set

$$L_i = \Gamma(\Lambda^i T^*M \otimes TM), \quad i = 0, 1, \ldots, m,$$

where Γ denotes the functor of sections. To complete our notation we set

$$L_i = 0 \quad \text{for } i < 0 \text{ and } i > m,$$
$$L = \sum_{i=-\infty}^{\infty} L_i.$$

We recall that for every $i,\, j \in \mathbb{Z}$ there is a bilinear mapping

$$[\ ,\ ]\colon L_i \times L_j \to L_{i+j}$$

called the Frölicher-Nijenhuis bracket (see e.g. [1]). Endowed with this bracket L is a graded Lie algebra. Similarly we define the Frölicher-Nijenhuis bracket algebra with compact supports L^c. Instead of the functor Γ we use the functor Γ^c of sections with compact supports. Obviously L^c is an ideal in L. We are going to prove the following theorem.

Theorem. *For any $i,\, j \in \mathbb{Z}$ satisfying $0 \le i, j,\, i + j \le m$ there is*

$$[L_i, L_j] = L_{i+j}, \quad [L_i^c, L_j^c] = L_{i+j}^c.$$

First we shall need the following lemma.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Lemma. *On an m-dimensional paracompact manifold M there exist open coverings*

$$\{U_{k\sigma}\},\ \{V_{k\sigma}\},\ k=0,\,1,\ldots,m,\ \sigma\in\Gamma_k$$

with the following properties

(i) $\bar{U}_{k\sigma}\subset V_{k\sigma}$ *for every* $k=0,\,1,\ldots,m$ *and every* $\sigma\in\Gamma_k$.

(ii) $V_{k\sigma}\cap V_{k\tau}=\emptyset$ *for every* $k=0,\,1,\ldots,m$ *and* $\sigma,\,\tau\in\Gamma_k,\ \sigma\neq\tau$.

(iii) Each $V_{k\sigma}$ *is a domain of a chart* $(V_{k\sigma},\varphi_{k\sigma})$ *such that* $\varphi_{k\sigma}(U_{k\sigma})=I^m\subset\mathbb{R}^m$, *where* $I=(0,1)$.

For the proof of this lemma see [2].

Proof of the Theorem: We shall prove the theorem for the algebra L only. The reader will easily find that the proof for the algebra L^c requires only minor modifications.

Let $i,\,j\in\mathbb{Z}$ be such that $0\leq i,\,j,\,i+j\leq m$, and let $\alpha\in L_{i+j}$ be arbitrary. We set

$$U_k=\cup_{\sigma\in\Gamma_k}U_{k\sigma},\quad k=0,\,1,\ldots,m.$$

Because the covering $\{U_k\}$, $k=0,\,1,\ldots,m$ is locally finite, we can find a partition of unity $\{\rho_k\}$, $k=0,\,1,\ldots,m$ subordinate to this covering. We can write

$$\alpha=\sum_{k=0}^{m}\rho_k\alpha.$$

Obviously it suffices to prove that for every $k=0,\,1,\ldots,m$ there is $\rho_k\alpha\in[L_i,L_j]$.

Let us assume that k is fixed. From now on we shall work in fact on each open set $U_{k\sigma}$ separately. For the sake of simplicity we shall often identify $U_{k\sigma}$ with $\varphi_{k\sigma}(U_{k\sigma})=I^m$. Let $(x_1^{(k\sigma)},\ldots,x_m^{(k\sigma)})$ be the coordinates on $U_{k\sigma}$ determined by the chart $(U_{k\sigma},\varphi_{k\sigma})$. On $U_{k\sigma}$ we can write

$$\rho_k\alpha=\sum_{1\leq r_1<\cdots<r_{i+j}\leq m}\ \sum_{s=1}^{m}{}^{\sigma}f^s_{r_1\ldots r_{i+j}}\,dx^{(k\sigma)}_{r_1}\wedge\cdots\wedge dx^{(k\sigma)}_{r_{i+j}}\otimes\frac{\partial}{\partial x^{(k\sigma)}_s}.$$

Because $\bar{U}_{k\sigma}\subset V_{k\sigma}$, and $(V_{k\sigma},\varphi_{k\sigma})$ is a chart it is easy to see that $\operatorname{supp}{}^{\sigma}f^s_{r_1\ldots r_{i+j}}\subset U_{k\sigma}$ is compact. For every $(i+j)$-tuple $1\leq r_1<\cdots<r_{i+j}\leq m$ and every $1\leq s\leq m$ we can define $\beta^s_{r_1\ldots r_{i+j}}$ by the formula

$$\beta^s_{r_1\ldots r_{i+j}}|U_{k\sigma}={}^{\sigma}f^s_{r_1\ldots r_{i+j}}\,dx^{(k\sigma)}_{r_1}\wedge\cdots\wedge dx^{(k\sigma)}_{r_{i+j}}\otimes\frac{\partial}{\partial x^{(k\sigma)}_s}$$

$$\beta^s_{r_1\ldots r_{i+j}}|(M\setminus U_k)=0.$$

We have

$$\rho_k\alpha = \sum_{1\leq r_1<\cdots<r_{i+j}\leq m} \sum_{s=1}^{m} \beta^s_{r_1\dots r_{i+j}},$$

and therefore it suffices to prove that for every $(i+j)$-tuple $1 \leq r_1 < \cdots < r_{i+j} \leq m$ and every $1 \leq s \leq m$ there is $\beta^s_{r_1\dots r_{i+j}} \in [L_i, L_j]$. Now we shall divide the proof into two parts.

(1) We shall assume here that $s \neq r_1, \dots, r_{i+j}$. We shall abbreviate $\beta = \beta^s_{r_1\dots r_{i+j}}$, ${}^\sigma f = {}^\sigma f^s_{r_1\dots r_{i+j}}$. In our considerations we shall use an auxiliary function χ defined on $I = (0,1)$ which has the following properties

(i) suppχ is compact
(ii) $\chi \geq 0$ on I
(iii) $\sqrt{\chi}$ is a C^∞-function on I
(iv) $\int_0^1 \chi(t)dt = 1$.

We define ${}^\sigma\chi = \chi \circ \mathrm{pr}^s \circ \varphi_{k\sigma}$, where $\mathrm{pr}^s \colon I^m \to I$ denotes the s-th projection. (In the sequel we again identify $U_{k\sigma}$ with $\varphi_{k\sigma}(U_{k\sigma}) = I^m$.) We set

$$\begin{aligned} {}^\sigma\psi(x_1, \dots, \hat{x}_s, \dots, x_m) &= \int_0^1 {}^\sigma f(x_1, \dots, x_m)dx_s \\ {}^\sigma\tilde{g}(x_1, \dots, x_m) &= {}^\sigma\chi(x_s)\, {}^\sigma\psi(x_1, \dots, \hat{x}_s, \dots, x_m) \\ {}^\sigma g &= {}^\sigma f - {}^\sigma\tilde{g}. \end{aligned}$$

Obviously supp ${}^\sigma g$, supp ${}^\sigma\tilde{g} \subset U_{k\sigma}$ are compact and

$$\int_0^1 {}^\sigma g(x_1, \dots, x_m)dx_s = 0.$$

Furthermore we define.

$${}^\sigma G(x_1, \dots, x_m) = \int_0^{x_s} {}^\sigma g(x_1, \dots, t, \dots, x_m)dt.$$

${}^\sigma G$ is a C^∞-function on $U_{k\sigma}$, and the vanishing of the above integral implies that supp ${}^\sigma G \subset U_{k\sigma}$ is compact. We define an element $\gamma \in L_{i+j}$ by the formula

$$\begin{aligned} \gamma|U_{k\sigma} &= {}^\sigma g dx^{(k\sigma)}_{r_1} \wedge \cdots \wedge dx^{(k\sigma)}_{r_{i+j}} \otimes \frac{\partial}{\partial x^{(k\sigma)}_s} \\ \gamma|(M \setminus U_k) &= 0. \end{aligned}$$

Similarly we define $\tilde{\gamma} \in L_{i+j}$. Because $\beta = \gamma + \tilde{\gamma}$ it will be sufficient to prove that both γ, $\tilde{\gamma} \in [L_i, L_j]$. Before proceeding further we shall recall one formula for the Frölicher-Nijenhuis bracket (see [1]). If ω and ω' are p-form and q-form on M respectively, and

X, $X' \in L_0$ are vector fields, then

$$[\omega \otimes X, \omega' \otimes X'] = (\omega \wedge \omega') \otimes [X, X'] + (\omega \wedge \mathcal{L}_X \omega') \otimes X' - (\mathcal{L}_{X'} \omega \wedge \omega') \otimes X + (-1)^p \big((d\omega \wedge \iota_X \omega') \otimes X' + (\iota_{X'} \omega \wedge d\omega') \otimes X\big),$$

where $\mathcal{L}_X$ denotes the Lie derivative and ι_X the inner product operator.

Let $\xi \in L_i$ be an element such that

$$\xi | U_{k\sigma} = dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}.$$

Further let $\eta \in L_j$ be defined by the formula

$$\eta | U_{k\sigma} = {}^\sigma G dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}$$
$$\eta | (M \setminus U_k) = 0.$$

We shall now compute the bracket $[\xi, \eta]$. At any point $x \notin U_k$ we have

$$[\xi, \eta]_x = 0 = \gamma_x.$$

Further we compute $[\xi, \eta]$ on $U_{k\sigma}$. For the sake of simplicity we denote $y_p = x_{r_p}^{(k\sigma)}$, $x_s = x_s^{(k\sigma)}$. The above formula for the Frölicher-Nijenhuis bracket gives

$$[\xi, \eta] | U_{k\sigma} = (dy_1 \wedge \cdots \wedge dy_i \wedge \mathcal{L}_{\frac{\partial}{\partial x_s}} ({}^\sigma G dy_{i+1} \wedge \cdots \wedge dy_{i+j})) \otimes \frac{\partial}{\partial x_s} =$$
$${}^\sigma g dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \gamma | U_{k\sigma}.$$

We have thus proved that $\gamma = [\xi, \eta] \in [L_i, L_j]$.

Further let $\lambda \in L_i$ be an element such that

$$\lambda | U_{k\sigma} = \sqrt{{}^\sigma \chi} dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}.$$

We define $\mu \in L_j$ by the formula

$$\mu | U_{k\sigma} = x_s^{(k\sigma)} \, {}^\sigma \psi \sqrt{{}^\sigma \chi} dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes \frac{\partial}{\partial x_s^{(k\sigma)}}$$
$$\mu | (M \setminus U_k) = 0.$$

Considering $[\lambda, \mu]$ we find again that for every $x \notin U_k$ there is

$$[\lambda, \mu]_x = 0 = \tilde{\gamma}_x.$$

It remains to compute $[\lambda,\mu]|U_{k\sigma}$. We write again y_p instead of $x_{r_p}^{(k\sigma)}$, and x_s instead of $x_s^{(k\sigma)}$. The formula for the Frölicher-Nijenhuis bracket gives

$$\begin{aligned}
&[\lambda,\mu]|U_{k\sigma} = \\
&\quad \left(\sqrt{{}^\sigma\chi}dy_1 \wedge \cdots \wedge dy_i \wedge \mathcal{L}_{\frac{\partial}{\partial x_s}}(x_s\, {}^\sigma\psi\sqrt{{}^\sigma\chi}dy_{i+1} \wedge \cdots \wedge dy_{i+j})\right) \otimes \frac{\partial}{\partial x_s} - \\
&\quad \left((\mathcal{L}_{\frac{\partial}{\partial x_s}}(\sqrt{{}^\sigma\chi}dy_1 \wedge \cdots \wedge dy_i)) \wedge x_s\, {}^\sigma\psi\sqrt{{}^\sigma\chi}dy_{i+1} \wedge \cdots \wedge dy_{i+j}\right) \otimes \frac{\partial}{\partial x_s} = \\
&\left(\sqrt{{}^\sigma\chi}\, {}^\sigma\psi\sqrt{{}^\sigma\chi} + \sqrt{{}^\sigma\chi}x_s\, {}^\sigma\psi\frac{\partial\sqrt{{}^\sigma\chi}}{\partial x_s} - \frac{\partial\sqrt{{}^\sigma\chi}}{\partial x_s}x_s\, {}^\sigma\psi\sqrt{{}^\sigma\chi}\right) \cdot dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \\
&\qquad {}^\sigma\chi\, {}^\sigma\psi dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = {}^\sigma\tilde{g} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial x_s} = \tilde{\gamma}|U_{k\sigma}.
\end{aligned}$$

We have thus proved that $\tilde{\gamma} = [\lambda,\mu] \in [L_i, L_j]$. Consequently $\beta = \gamma + \tilde{\gamma} \in [L_i, L_j]$. This finishes the first part of the proof.

(2) Here we shall assume that there exists q, $1 \le q \le i+j$ such that $s = r_q$. We shall consider only the case $1 \le q \le i$. The case $i+1 \le q \le i+j$ can be treated similarly. We shall abbreviate $\beta = \beta^{r_q}_{r_1 \ldots r_{i+j}}$, ${}^\sigma f = {}^\sigma f^{r_q}_{r_1 \ldots r_{i+j}}$. Let $\xi \in L_i$ be an element such that

$$\xi|U_{k\sigma} = dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge dx_{r_i}^{(k\sigma)} \otimes x_{r_q}^{(k\sigma)}\frac{\partial}{\partial x_{r_q}^{(k\sigma)}}.$$

Further let $\eta \in L_j$ be defined by the formula

$$\eta|U_{k\sigma} = -dx_{r_{i+1}}^{(k\sigma)} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)} \otimes {}^\sigma f\frac{\partial}{\partial x_{r_q}^{(k\sigma)}}$$

$$\eta|(M \setminus U_k) = 0.$$

We shall compute the difference $\beta - [\xi,\eta]$. At any point $x \notin U_k$ we have

$$\beta_x - [\xi,\eta]_x = 0.$$

Further we compute $\beta - [\xi,\eta]$ on $U_{k\sigma}$. For the sake of simplicity we denote $y_p = x_{r_p}^{(k\sigma)}$,

$x_n = x_n^{(k\sigma)}$.

$$(\beta - [\xi,\eta])|U_{k\sigma} =$$

$$^\sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - \left[dy_1 \wedge \cdots \wedge dy_i \otimes y_q \frac{\partial}{\partial y_q}, -dy_{i+1} \wedge \cdots \wedge dy_{i+j} \otimes {}^\sigma f \frac{\partial}{\partial y_q}\right] =$$

$$^\sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} + dy_1 \wedge \cdots \wedge dy_{i+j} \otimes [y_q \frac{\partial}{\partial y_q}, {}^\sigma f \frac{\partial}{\partial y_q}] -$$

$$\big((\mathcal{L}_{^\sigma f \frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}\big) \otimes y_q \frac{\partial}{\partial y_q} =$$

$$^\sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} + y_q \frac{\partial {}^\sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - {}^\sigma f dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} -$$

$$y_q \big((d\iota_{^\sigma f \frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}\big) \otimes \frac{\partial}{\partial y_q} =$$

$$y_q \frac{\partial {}^\sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} -$$

$$y_q \big((d{}^\sigma f \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}\big) \otimes \frac{\partial}{\partial y_q} =$$

$$y_q \frac{\partial {}^\sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} -$$

$$y_q \big((\frac{\partial {}^\sigma f}{\partial y_q} dy_q \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j}\big) \otimes \frac{\partial}{\partial y_q} -$$

$$y_q \left((\sum_{\substack{n=1 \\ n \neq r_q}}^{m} \frac{\partial {}^\sigma f}{\partial x_n} dx_n \wedge \iota_{\frac{\partial}{\partial y_q}}(dy_1 \wedge \cdots \wedge dy_i)) \wedge dy_{i+1} \wedge \cdots \wedge dy_{i+j} \right) \otimes \frac{\partial}{\partial y_q} =$$

$$y_q \frac{\partial {}^\sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} - y_q \frac{\partial {}^\sigma f}{\partial y_q} dy_1 \wedge \cdots \wedge dy_{i+j} \otimes \frac{\partial}{\partial y_q} +$$

$$\sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^{m} \big((-1)^q y_q \frac{\partial {}^\sigma f}{\partial y_q} dx_n \wedge dy_1 \wedge \cdots \wedge \widehat{dy_q} \wedge \cdots \wedge dy_{i+j}\big) \otimes \frac{\partial}{\partial y_q} =$$

$$\sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^{m} \big((-1)^q y_q \frac{\partial {}^\sigma f}{\partial y_q} dx_n \wedge dy_1 \wedge \cdots \wedge \widehat{dy_q} \wedge \cdots \wedge dy_{i+j}\big) \otimes \frac{\partial}{\partial y_q}.$$

Now we define for $n \neq r_1, \dots, r_{i+j}$ an element $\lambda_n \in L_{i+j}$ by the formula

$$\lambda_n | U_{k\sigma} = \big((-1)^q x_{r_q}^{(k\sigma)} \frac{\partial {}^\sigma f}{\partial x_{r_q}^{(k\sigma)}} dx_n^{(k\sigma)} \wedge dx_{r_1}^{(k\sigma)} \wedge \cdots \wedge \widehat{dx_{r_q}^{(k\sigma)}} \wedge \cdots \wedge dx_{r_{i+j}}^{(k\sigma)}\big) \otimes \frac{\partial}{\partial x_{r_q}^{(k\sigma)}}$$

$$\lambda_n | (M \setminus U_k) = 0.$$

We can easily see that

$$\beta - [\xi,\eta] = \sum_{\substack{n=1 \\ n \neq r_1, \dots, r_{i+j}}}^{m} \lambda_n.$$

But by virtue of the first part of the proof $\lambda_n \in [L_i, L_j]$. Thus we have $\beta - [\xi, \eta] \in [L_i, L_j]$, and consequently $\beta \in [L_i, L_j]$. This finishes the proof of the theorem.

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