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HIGHER ORDER GEODESIC SYMMETRIES

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§I. Introduction

Let $(M, <\ ,\ >_0)$ be a connected, complete C^∞-Riemannian manifold. A symmetry at a point x in M is an isometry $s_x : M \to M$ that has x as an isolated fixed point. An s-structure (of order k, $k \geqq 2$) on M is a family of symmetries (of order k), (s_x), $x \in M$. The main purpose of this note is to announce various results of manifolds that admit s-structures of order k. These manifolds are called pointwise Riemannian k-symmetric spaces. The final details will appear elsewhere. (For generalities on the subject see [Ko].)

§II. The homogeneous structure of pointwise Riemannian k-symmetric spaces

1. Pointwise Riemannian k-symmetric spaces are homogeneous manifolds (see e.g. [Ko]). However, their homogeneous structure is, in general, more complicated than that of the symmetric spaces of Cartan. To be more precise, an ordinary symmetric space M can be represented as a homogeneous space $M = G/H$, where G is a transitive group of isometries that admits an automorphism σ of order two such that

$$(G^\sigma)_0 \subset H \subset G^\sigma , \tag{1.1}$$

where $(G^\sigma)_0$ denotes the identity component of the fixed point set (G^σ) of σ. It is natural to ask if a pointwise Riemannian k-symmetric space admits such a representation with σ an automorphism of order k of G. It turns out that the answer is no. In fact, O. Kowalski [Ko] has shown S^{4n+1} with the metric induced as a geodesic sphere in complex projective space admits an s-structure of order four for which the above property fails. This leads us to introduce the following:

2. Let $(M, \langle\ ,\ \rangle_0)$ be a pointwise k-symmetric space of order k. The s-structure (s_x), $x \in M$, is said to be regular, if the symmetries satisfy the relation:

$$s_x \circ s_y = s_p \circ s_x ,$$

where x and y are any points in M, and $p = s_x(y)$. In this case M is called a (regular) Riemannian k-symmetric space.

It turns out that Riemannian k-symmetric spaces are precisely the pointwise Riemannian k-symmetric spaces whose homogeneous structure is analogous to that of the symmetric spaces of Cartan. More precisely (see e.g. [Ko]): let G denote the identity component of the closure in $I(M)$ (the group of isometries of M) of the group generated by the symmetries (s_x). Then G acts transitively on M. Furthermore, let H denote the isotropy group of G at a point o in M, so that $M = G/H$. Then $\sigma : G \to G$, defined by $\sigma(g) = s_o \circ g \circ s_o^{-1}$, is an automorphism of G of order k that satisfies (1.1). Conversely, given a triple (G,H,σ) with σ an automorphism of order k of G that satisfies (1.1), then G/H can be made, in a natural fashion, into a k-symmetric space.

In the next section we state various results concerning the nontrivial existence of pointwise and regular Riemannian k-symmetric spaces. Denote these two classes of spaces by $(\mathcal{P})$ and $(\mathcal{R})$ respectively. We shall see that the inclusion $(\mathcal{R}) \subset (\mathcal{P})$ is strict from a differentiable viewpoint.

§III. Existence of (pointwise) Riemannian k-symmetirc spaces

1. The most readily available examples are those of regular Riemannian k-symmetric spaces: ordinary Riemannian symmetric spaces are clearly regular 2-symmetric. 3-symmetric spaces of reductive type have been classified by J. A. Wolf and A. Gray [W-G]. k-symmetric spaces of dimension $\leqq 5$ have been classified by O. Kowalski [Ko]. Compact simply connected 4-symmetric spaces have been classified in [J-1] (see also [G-J] for the reductive case). Furthermore, a large class of examples is

obtained as follows:

Theorem A ([J-2]). (i) Let $M = G/H$ be a Kähler $\mathbb{C}$-space, then there exists an integer $k_0 \geqq 2$, such that for any $k \geqq k_0$, the space admits a regular s-structure of order k that makes the space into an Hermitian k-symmetric space.

(ii) Let $M = SU(k_0)/T$, where $k_0 \geqq 2$, and T is a maximal torus of $SU(k_0)$. Then M is a Riemannian k-symmetric space for any $k \geqq k_0$, and with respect to any $SU(k_0)$-invariant metric. Furthermore, M cannot be endowed with a Riemannian metric that admits a regular s-structure of order $< k_0$.

Remark. Part (ii) shows that for each $k \geqq 2$, the classes $(\mathcal{R}_k)$ of regular Riemannian k-symmetric spaces are essentially distinct from a differentiable viewpoint. Our next theorem states that the inclusion $(\mathcal{R}) \subset (\mathcal{P})$ is strict from a differentiable viewpoint.

2. Theorem B ([J-3]). Let $M = V_{n,n-2}$ be the real Stiefel manifold of orthonormal 2-frames in Euclidean n-space. Regard M as the homogeneous space $SO(n)/SO(n-2)$ endowed with the $SO(n)$-invariant metric $< , >_0$ induced by the negative of the Killing form of $SO(n)$. Assume that n is even, $n = 2q$, and that $n > 30$. Then $(M, < , >_0)$ is a pointwise Riemannian k-symmetric space for any $k = 2m$, $m \geqq 2$. Furthermore, M is not diffeomorphic to the underlying manifold of a regular Riemannian h-symmetric space for any $h \geqq 2$.

Remark. It should be pointed out that the underlying manifold of most Stiefel manifolds (real, complex, or symplectic) can not be endowed with the structure of a pointwise Riemannian h-symmetric space for $h \geqq 2$ (see loc. cit.). In this context, a natural problem is that of determining for a given differentiable manifold M all possible Riemannian metrics on M that admit (regular) s-structures, and of what orders. This brings us to our next subject matter.

§IV. Regular s-structures of finite order on compact Riemannian symmetric spaces

Let $(M, <\ ,\ >_0)$ be a Riemannian manifold. The degree of symmetry of the metric can be measured by determining whether or not the metric admits regular s-structures and if so, of what order. For example, in this setting, Hermitian symmetric spaces of Cartan are the most symmetric since they admit regular s-structures of order k for all $k \geqq 2$. It is natural to ask if non-Hermitian symmetric spaces admit regular s-structures of order k for some $k > 2$. This was done for spheres by the third named author in [Sa] by means of the theory of extrinsic k-symmetric spaces. The next theorem gives a more general answer in the case of positive Euler characteristic.

Theorem C ([J-S]). Let $(M, <\ ,\ >_0)$ be a compact simply connected irreducible Riemannian symmetric space with positive Euler characteristic. Then,

(i) if M is not a Hermitian symmetric space, the Riemannian metric $<\ ,\ >_0$ is the only metric on M (up to homothety) that admits regular s-structures of order $k \geqq 2$. Furthermore, save for $M = S^6$, the geodesic involutions are the only possible regular s-structure on M. If $M = S^6$, then the representation of S^6 as the coset space $G_2/SU(3)$ renders S^6 as a 3-symmetric space.

(ii) if M is a Hermitian symmetric space, the metric $<\ ,\ >_0$ admits regular s-structures of order k, for all $k \geqq 2$. Furthermore, save in the following cases, the metric $<\ ,\ >_0$ is, up to homothety, the only metric on M that admits regular s-structures:

(a) $M = \mathbb{C}P^{2n+1}$ represented as $Sp(n+1)/Sp(n)\times U(1)$, or

(b) $M = SO(2n)/U(n)$ represented as $SO(2n-1)/U(n-1)$, or

(c) $M = SO(7)/SO(5)\times SO(2)$ represented as $G_2/U(2)$.

In these last three cases, M has a one-parameter family of non-symmetric metrics $<\ ,\ >_t$, $t > 0$, that admit regular s-structures of order k, for any $k \geqq 3$.

Remark. In the case of zero Euler characteristic, the topological classification is far from complete. Thus our results are not as definite as in the case of positive Euler characteristic. We can prove the following

Theorem D ([J-S]). Let $(M, < , >_0)$ be a compact simply connected irreducible Riemannian symmetric space with zero Euler characteristic. Assume that $M = G/H$ with G simple, and $G = I(M)_0$. Then the only regular s-structure that $(M, < , >_0)$ admits is precisely that obtained from the geodesic involutions.

§V. Differences between Cartan symmetric spaces and Riemannian k-symmetric spaces with $k > 2$

Although (regular) Riemannian k-symmetric spaces have a similar homogeneous structure to Riemannian symmetric spaces of Cartan, their theory has some major differences:

1. It is well known that the algebra of G-invariant differential operators $\mathcal{D}(G/H)$ on the homogeneous space G/H is commutative if (G,H) is a symmetric pair (see e.g. [H-2]). By contrast, $(SU(k),T)$, T a maximal torus of $SU(k)$ is a k-symmetric pair (see II above), and yet, $\mathcal{D}(SU(k)/T)$ is non-commutative for $k > 2$ (see e.g. [H-2, p. 243]). Furthermore, since (see e.g. [J-2]) $I(SU(k)/T)_0 = SU(k)/\mathbb{Z}_k$ with respect to any $SU(k)$-invariant metric, it follows that these spaces provide examples of non-commutative naturally reductive spaces (see loc. cit. for more details).

2. Duality fails. Given a compact Riemannian 4-symmetric space it is possible to associate, in a natural way, a non-compact dual. However, the classification in [G-J] shows that this does not exhaust all non-compact Riemannian 4-symmetric spaces.

3. A Riemannian k-symmetric space with $k > 2$ may have non-equivalent regular s-structures. For example, the 3-symmetric space $SO(2n+2)/U(n)\times U(1)$ admits two non-equivalent 4-symmetric structures whose fibrations are described as follows (see [J-2], and (b) and (c) below):

Base Space	Fiber Space
$SO(2n+2)/U(n+1)$	$U(n+1)/U(n)\times U(1)$
$SO(2n+2)/SO(2n)\times SO(2)$	$SO(2n)/U(n)$

4. The classification in [G-J] shows the existence of non-compact Riemannian 4-symmetric spaces G/H with G a non-compact simple Lie group that cannot be endowed with G-invariant naturally reductive Riemannian metrics.

5. Almost complex structures on k-symmetric spaces, invariant under the symmetries, are not necessarily integrable for $k > 2$.

On the positive side, we can say the following:

(a) Riemannian k-symmetric spaces, with k odd, have associated in a natural fashion an almost-complex structure. This structure was used by A. Gray in [Gr] to give a characterization of 3-symmetric spaces in terms of their curvature tensor

(b) Riemannian k-symmetric spaces, with k even, can be regarded, in a natural fashion, as fiber bundles over ordinary symmetric spaces whose fibers are complete totally geodesic submanifolds which are regular $\frac{k}{2}$-symmetric spaces as well (see e.g. [J-1] and [G-J] for a detailed description of these fibrations for $k = 4$). Furthermore, these spaces have associated an f-structure $(f^3 + f = 0)$ which plays the analogous role of a complex structure (see e.g., [Ra]).

(c) The fibrations in (b) provide a generalization of the twistor fibrations used to study harmonic maps (see e.g., [Bu], [S] and [Ra]).

§VI. Outline of the proofs

1. Theorem A part (i): In this case, H is the centralizer of a torus S of G (see [Be]). Thus, one has to chose an element $s \in S$ so that conjugation with respect to s in G defines an automorphism σ of order k that satisfies (1.1).

2. Theorems A (part (ii)), B, and C, are proved in a similar way. For a

given manifold M, one has to proceed as follows:

2.1. Determine all possible representations of M as a homogeneous space G/H, with G as a compact connected Lie group acting effectively and transitively on M. This part is clearly topological in nature. See [J-2], [H-S], [0,1], and [Sh].

2.2 Given any such a representation G/H, determine whether or not G has finite order automorphisms σ for which (1.1) holds true. If such automorphisms exist, determine all possible orders. For this part, the structure theory of finite order automorphisms of complex semisimple Lie algebras is of fundamental importance. See e.g. [H-1], and [W-G].

3. Theorem D is proved as Theorem C, but the topological part has to be replaced by its Riemannian counterpart. Namely, at this time, only the representations G/H of M with G a subgroup of $I(M, < , >_0)$ are available ([0-1]) for general M.

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