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THE MAXIMAL FUNCTION OF A COMPLEX MEASURE

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ABSTRACT. The Marcinkiewicz Interpolation Theorem is proved in the setting of the unit sphere in euclidean space of arbitrary dimension. This leads to a key result in the study of the maximal function of a complex measure and, in particular, of an integrable function on this unit sphere.

1. INTRODUCTION

In the theory of the H^p spaces of monogenic functions [2], which is developed in the framework of Clifford analysis [1], an important role is played by the results concerning the boundary behaviour of Poisson integrals of complex measures and, in particular, of integrable functions on the unit sphere.

Let P denote the Poisson kernel in $\mathring{B}_m$, the unit ball in $\mathbb{R}^m$; it is given by

$$P(x,y) = \frac{1-r^2}{|x-y|^m}$$

where $r^2 = |x|^2$ and y is on the unit spere S^{m-1}.

<u>Definition 1.1.</u>

If μ is a complex measure on S^{m-1}, then its Poisson integral $\mathcal{P}[\mu]$ is defined as

$$\mathcal{P}[\mu](x) = \int_{S^{m-1}} P(x,y)\, d\mu(y) \;, \qquad x \in \mathring{B}_m \;.$$

In the special case where the measure μ is derived from an L_1-function $\tilde{f}$, this definition rewrites as

$$\mathcal{P}[\tilde{f}](x) = \int_{S^{m-1}} P(x,y)\, \tilde{f}(y)\, d\sigma(y) \;, \qquad x \in \mathring{B}_m \;,$$

σ being the normalized Lebesgue measure on S^{m-1}, $\sigma(S^{m-1}) = 1$.

The main result about the boundary behaviour of Poisson integrals is the so called Koranyi theorem, relating the maximum value of the continuous function $\mathcal{P}[\mu]$ in a cone D_α, to the value of the maximal function of μ at the top of the cone. Let us first introduce these notions explicitly before stating the theorem.

Definition 1.2.

Let $\xi \in S^{m-1}$ and $c_\alpha > 1$. The conical region D_α with top ξ is defined by

$$D_\alpha(\xi) = \{y \in \mathbb{R}^m : |\xi-y| < c_\alpha(1 - |y|)\}$$

Definition 1.3.

If μ is a complex measure on S^{m-1}, then its maximal function $M\mu : S^{m-1} \to [0,+\infty]$ is given by :

$$M\mu(\xi) = \sup_{\psi>0} \frac{|\mu|(bk(\xi,\psi))}{\sigma(bk(\xi,\psi))}$$

where $|\mu|$ is the total variation of μ and $bk(\xi,\psi)$ is a sphere segment centered at $\xi \in S^{m-1}$ with half a solid angle ψ.

Theorem 1.4. (Koranyi)

Given a complex measure μ on S^{m-1}, for each conical region D_α there exists a constant A_α such that on S^{m-1} :

$$\sup_{x\in D_\alpha(\xi)} |\mathcal{P}[\mu](x)| \le A_\alpha\, M\mu(\xi) .$$

Proof : see [2].

The aim is to study, in the third section, this maximal function of a complex measure, especially when that measure is derived from an integrable function on S^{m-1}. The key result on the maximal function is strongly related to the so called Marcinkiewicz Interpolation Theorem, which we treat of first in the second section.

2. THE MARCINKIEWICZ INTERPOLATION THEOREM.

Let Σ denote a positive measure on S^{m-1}. We consider an operator T, acting on functions $\tilde{f} \in L_1(S^{m-1})$, and mapping them into Σ-measurable functions $T\tilde{f} : S^{m-1} \to [0,+\infty]$. Moreover we assume T to be subadditive :

$$T(\tilde{f} + \tilde{g}) \le T\tilde{f} + T\tilde{g} \quad \text{for all} \quad \tilde{f},\tilde{g} \in L_1(S^{m-1}) .$$

Next we define numbers c_r, $1 \le r \le \infty$, as to be the smallest constants for which the estimates

$$\Sigma(\{T\tilde{f} > t\}) \le c_r\, t^{-r} \int_{S^{m-1}} |\tilde{f}|^r d\Sigma \qquad \text{and} \qquad \|T\tilde{f}\|_\infty \le c_\infty \|\tilde{f}\|_\infty$$

hold over the whole of S^{m-1} and for all t, $0 < t < \infty$. Notice that the constants c_r might be ∞. Finally we introduce the notation $K(a,b,c,\dots)$ for a constant K, depending upon the parameters a,b,c,..., and which is finite whenever all the parameters are finite.

Theorem 2.1.

For all $\tilde{f} \in L_p(S^{m-1})$ and $1 < p < r \le \infty$,

$$\int_{S^{m-1}} (T\tilde{f})^p \, d\Sigma \le K_p(c_1, c_r) \int_{S^{m-1}} |\tilde{f}|^p \, d\Sigma .$$

Proof :

Let $F : S^{m-1} \to [0,+\infty]$ be a Σ-measurable function. It is then easily seen that

$$\int_{S^{m-1}} F(\eta)^p \, d\Sigma(\eta) = \int_0^\infty pt^{p-1} \Sigma(\{F(\eta) > t\})dt .$$

Given $t > 0$, consider the decomposition of $\tilde{f} \in L_1(S^{m-1})$:

$$\tilde{f} = \tilde{g}_t + \tilde{h}_t \quad \text{where} \quad \tilde{g}_t(\xi) = \begin{cases} 0 & \text{if } |\tilde{f}(\xi)| < t \\ \tilde{f}(\xi) & \text{if } |\tilde{f}(\xi)| \ge t \end{cases}$$

and where $\tilde{h}_t = \tilde{f} - \tilde{g}_t$. We define

$G(t) = \Sigma(\{T\tilde{g}_t > \frac{t}{2}\})$ and $H(t) = \Sigma(\{T\tilde{h}_t > \frac{t}{2}\})$.

Let $G_1(t)$ denote $\frac{2c_1}{t} \int_{|\tilde{f}|\ge t} |\tilde{f}| d\Sigma$, and, for given $r < \infty$,

$$H_1(t) = \frac{2^r c_r}{t^r} \int_{|\tilde{f}|<t} |\tilde{f}|^r d\Sigma .$$

It is then clear that

$$G(t) \le c_1 \frac{2}{t} \int_{S^{m-1}} |\tilde{g}_t| d\Sigma \le \frac{2c_1}{t} \int_{|\tilde{f}|\ge t} |\tilde{f}| d\Sigma = G_1(t) ,$$

and similarly, $H(t) \le H_1(t)$.

Now applying Fubini's Theorem on

$$\int_0^\infty pt^{p-1} G_1(t)dt = 2c_1 p \int_0^\infty \int_{|\tilde{f}|\ge t} t^{p-2} |\tilde{f}| d\Sigma \, dt$$

we find that

$$\int_0^\infty pt^{p-1} G_1(t)dt = \frac{2c_1 p}{p-1} \int_{S^{m-1}} |\tilde{f}|^p d\Sigma .$$

Similarly, one gets

$$\int_0^\infty pt^{p-1}H_1(t)dt = 2^r c_r p \int_{S^{m-1}} |\tilde{f}(\xi)|^r d\Sigma(\xi) \int_{|\tilde{f}(\xi)|}^\infty t^{p-r-1}\, dt$$

$$= \frac{2^r c_r p}{r-p} \int_{S^{m-1}} |\tilde{f}|^p d\Sigma$$

since $p < r$ by assumption.

In view of $T\tilde{f} \le T\tilde{g}_t + T\tilde{h}_t$, it follows that

$$\Sigma(\{T\tilde{f} > t\}) \le \Sigma(\{T\tilde{g}_t > \frac{t}{2}\}) + \Sigma(\{T\tilde{h}_t > \frac{t}{2}\}) \le G_1(t) + H_1(t) .$$

Combining the obtained results we find that

$$\int_{S^{m-1}} (T\tilde{f})^p d\Sigma = \int_0^\infty pt^{p-1}\, \Sigma(\{T\tilde{f} > t\})dt \le \int_0^\infty pt^{p-1}\, G_1(t)dt + \int_0^\infty pt^{p-1}\, H_1(t)dt$$

$$\le K_p(c_1, c_r) \int_{S^{m-1}} |\tilde{f}|^p d\Sigma$$

whence the desired result for $r < +\infty$.

If $r = +\infty$, we may assume without loss of generality that $c_\infty \le \frac{1}{2}$ (if not, consider $\frac{T}{2c_\infty}$).

It is then easily seen that

$$\|\tilde{h}_t\|_\infty = \sup_{\xi \in S^{m-1}} |\tilde{h}_t(\xi)| = \sup_{|\tilde{f}|<t} |\tilde{f}(\xi)| < t$$

and hence $\|T\tilde{h}_t\|_\infty \le c_\infty \|\tilde{h}_t\|_\infty \le \frac{t}{2}$.

As $T\tilde{h}_t \le \frac{t}{2}$ a.e. on S^{m-1}, it follows that $H(t) = \Sigma(\{T\tilde{h}_t > \frac{t}{2}\}) = 0$ leading to $\Sigma(\{T\tilde{f} > t\}) = G(t) \le G_1(t)$. Combining the obtained results, the desired result indeed follows since

$$\int_{S^{m-1}} (T\tilde{f})^p d\Sigma = \int_0^\infty pt^{p-1}\Sigma(\{T\tilde{f} > t\})dt$$

$$\le \int_0^\infty pt^{p-1}\, G_1(t)dt$$

$$= \frac{2pc_1}{p-1} \int_{S^{m-1}} |\tilde{f}|^p d\Sigma$$

$$= K_p(c_1, c_\infty) \int_{S^{m-1}} |\tilde{f}|^p d\Sigma .$$

3. THE MAXIMAL FUNCTION OF A COMPLEX MEASURE.

The notion of the maximal function of a complex measure μ on S^{m-1} was already introduced in the first section (definition 1.3). The following two propositions may be proved along classical lines.

<u>Proposition 3.1.</u>

The maximal function $M\mu$ of the complex measure μ on S^{m-1} is lower semi-continuous.

<u>Proposition 3.2.</u>

There exists a constant c_1 such that for all complex measures μ on S^{m-1} and all $\lambda > 0$:

$$\sigma(\{M\mu > \lambda\}) \le c_1 \frac{\|\mu\|}{\lambda} .$$

Now consider the special case where the complex measure μ is derived from an L_1-function : $\mu = \tilde{f} d\sigma$, $\tilde{f} \in L_1(S^{m-1})$. Its maximal function also reads

$$M\tilde{f}(\xi) = \sup_{\psi>0} \frac{1}{\omega(\xi,\psi)} \int_{bk(\xi,\psi)} |\tilde{f}(\eta)| d\sigma(\eta)$$

where $\omega(\xi,\psi) = \sigma(bk(\xi,\psi))$.

From proposition 3.2 it now follows readily that

<u>Corollary 3.3.</u>

For all $\tilde{f} \in L_1(S^{m-1})$ and all $\lambda > 0$: $\sigma(\{M\tilde{f} > \lambda\}) \le \frac{2m}{\lambda} \|\tilde{f}\|_1$.

The following proposition is also easily verified :

<u>Proposition 3.4</u>

The operator $M : \tilde{f} \in L_1(S^{m-1}) \mapsto M\tilde{f}$ is subadditive and moreover satisfies

$$M\tilde{f}(\xi) \le \|\tilde{f}\|_\infty , \quad \xi \in S^{m-1} .$$

Combining the above results with the Marcinkiewicz Interpolation Theorem of the second section, we finally arrive at the main result

<u>Theorem 3.5.</u>

Let $\tilde{f} \in L_p(S^{m-1})$, $1 < p < \infty$, then there exists a constant $A(p)$ such that

$$\int_{S^{m-1}} |M\tilde{f}|^p d\sigma \le A(p) \int_{S^{m-1}} |\tilde{f}|^p d\sigma .$$

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