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Some theorems for holomorphic functions with proximate order $1 + log(logr)/logr$

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SOME THEOREMS FOR HOLOMORPHIC FUNCTIONS WITH PROXIMATE ORDER $1 + \log(\log r)/\log r$.

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1. Introduction.

In this paper we treat holomorphic functions, which are defined in the right half plane and satisfies the following estimate:
for any positive number ε and ε', there exists $C_{\varepsilon,\varepsilon'}$ such that

$$(*) \qquad |F(z)| \leq C_{\varepsilon,\varepsilon'} \exp(x \log x + k|y| + \varepsilon|z|)$$

for $x = Rez \geq \varepsilon'$.

The characterization of entire function with this kind of estimate (entire functions with proximate order $1+\log(\log r)/\log r$ is given by Palamodov in [8] by using the Fourier transforms of rapidly decreasing generalized function. We will give some theorems (for example, Carlson's theorem, Liouville type theorem) for holomorphic functions which satisfies the above estimate $(*)$. In section 2, we consider the Mellin transform of holomorphic functions with proximate order $1 + \log(\log r)/\log r$. We will deduce strong asymptotic expansion of the Mellin transform $MF(w)$. In section 3 we will give the integral representation of $F(z)$ by means of Mellin transform of $F(z)$. Finally in section 4, we will give some theorems for holomorphic functions defined in the direct product of the right half plane with proximate order $1 + \log(\log r)/\log r$. For the details of proximate order, we refer the reader to [3] and [4].

2. Mellin transform of holomorphic functions with proximate order $1 + \log(\log r)/\log r$.

In this section we will investigate the Mellin transform $MF(w)$ of holomorphic functions $F(z)$ with proximate order $1 + \log(\log r)/\log r$. Especially we deduce the strong asymptotic expansion of $MF(w)$. The strong asymptotic expansion of $MF(w)$ is given by Kubyshin in [2] and by the author in [11]. Now we define the

Mellin transform of holomorphic function defined in the right plane with proximate order $1+\log(\log r)/\log r$.

Definition.

Let $F(z)$ be holomorphic in the right half plane $\{z \in C; Re z > 0\}$ and satisfy the estimate $(*)$. The Mellin transform $MF(w)$ of the function $F(z)$ is defined as follows:

$$MF(w) = -(2i)^{-1} \int_{c-i\infty}^{c+i\infty} F(z)(-w)^z (\sin \pi z)^{-1} dz,$$

where c is an arbitrary number between 0 and 1.

$MF(w)$ has the following properties:

Proposition 1. (Kubyshin [2] and Yoshino [3])

(i) $MF(w)$ is holomorphic in $S_k = \{w \in C; k < |\arg w| \leq \pi\}$.

(ii) $MF(w)$ has the following asymptotic expansion in any subsector $S_{k+\epsilon}$ of S_k

$$MF(w) \sim \sum_{n=0}^{\infty} F(n) w^n.$$

More precisely, the following estimate is valid: for any $\epsilon > 0$ and $\epsilon' > 0$ and natural number N, there exist constants $C_{\epsilon,\epsilon'}$, $A > 0$ and δ $(0 < \delta < 1)$ such that

$$|MF(w) - \sum_{n=0}^{N} F(n) w^n| < C_{\epsilon,\epsilon'} A^N N! |w|^{N+\delta}.$$

For details of the strong asymptotic expansion, we refer the reader to Nevanlinna [6], [7] and Reed and Simon [9] and Sokal [10].

The following proposition shows the importance of strong asymptotic expansion.

Proposition 2. ([6],[7],[9] and [10]).

Let $f(w)$ be holomorphic in the sector S_k and have strong asymptotic expansion there. If the coefficients in expansion are all zero and k is less than $\frac{\pi}{2}$, then $f(w)$ vanishes identically there.

3. Integral representation of $F(z)$ by $MF(w)$.

In this section we give integral representation of $F(z)$ by $MF(w)$. Namely the following integral formula is valid.

Proposition 3.

Let $F(z)$ be a holomorphic function defined on the right half plane with proximate order $1 + \log(\log r)/\log r$. Then the following integral representation is valid.

$$F(z) = (2\pi i)^{-1} \int_\Gamma MF(w) w^{-z-1} dw,$$

where Γ is a contour shown in Figure 1.

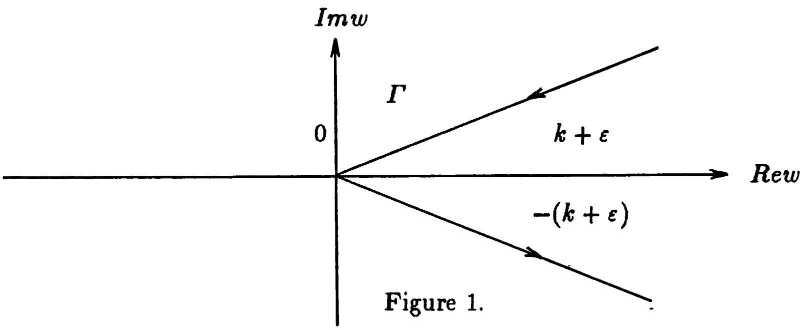

Figure 1.

Proof.

We calculate the right hand side of the formula by making use of residue theorem. First we insert the definition of $MF(w)$ and exchange the order of integrations. Then we obtain

$$(2\pi i)^{-1}(2i)^{-1} \int_{c-i\infty}^{c+i\infty} dt F(t)(\sin(\pi t))^{-1} \int_\Gamma w^{-z-1}(-w)^t dw.$$

The integral

$$(2i)^{-1} \int_\Gamma w^{-z-1}(-w)^t dw$$

is equal to $(t-z)^{-1}\sin(\pi z - (k+\varepsilon)(t-z))\varepsilon^{t-z}$. By the residue theorem we have

$$-(2\pi i)^{-1} \int_{Ret=c'} F(t)(t-z)^{-1}\sin(\pi z - (k+\varepsilon)(t-z))\sin(\pi z)^{-1}\varepsilon^{t-z} dt + F(z).$$

As t varies on the vertical line $Rez = c$, $Re(t-z)$ is positive. If ε tends to zero then the integral above goes to zero. Hence we obtain the desired result.

4. Applications.

In this section we show some applications concerning the holomorphic functions with proximate order $1 + \log(\log r)/\log r$. We begin with the Carlson type theorem.

Theorem 1.

Let $F(z)$ be holomorphic in the right half plane $\{Rez > 0\}$ and satisfy the estimate $(*)$ with $k < \pi/2$. If $F(n)$ vanishes for all natural numbers n, then $F(z)$ vanishes identically.

Proof.

We consider the Mellin transform of $F(z)$. From Proposition 1 and the assumption, $MF(w)$ has the strong asymptotic expansion with 0 coefficient. So $MF(w)$ vanishes identically. By virtue of the integral representation, $F(z)$ vanishes identically.

Remark.

The assumption $k < \pi/2$ is crucial in Theorem 1. Let $F(z)$ be $1/\Gamma(z)$ (Γ denotes Euler Gamma function). Then $F(z)$ satisfies the estimate $(*)$ with $k = \pi/2$ and all assumptions in Theorem 1, but $F(z)$ does not vanish identically.

Next we will prove the following Liouville type theorem.

Theorem 2.

Let $F(z)$ be a holomorphic function defined in the right half plane $\{z \in C; Rez > 0\}$ satisfy the estimate $(*)$ with $k < \pi/2$. If $\limsup_{n\to\infty} |F(n)|^{1/n} = A$, then $F(z)$ is a holomorphic function of exponential type.

Proof.

From Proposition 1, $MF(w)$ has the following strong asymptotic expansion in the sector S_ε

$$MF(w) \sim \sum_{n=o}^{\infty} F(n)w^n.$$

By the assumption on $F(n)$, the formal series in the above converges in the circle with radius $1/A$ and center 0. So the series define a holomorphic function in this circle. Hence we obtain the following equality:

$$MF(w) = \sum_{n=0}^{\infty} F(n)w^n \qquad (|w| < 1/A).$$

Hence $MF(w)$ is holomorphic in the shaded region shown in Figure 2. By virtue of the integral representation of $F(z)$, $F(z)$ is a holomorphic function of exponential type defined in the right half plane.

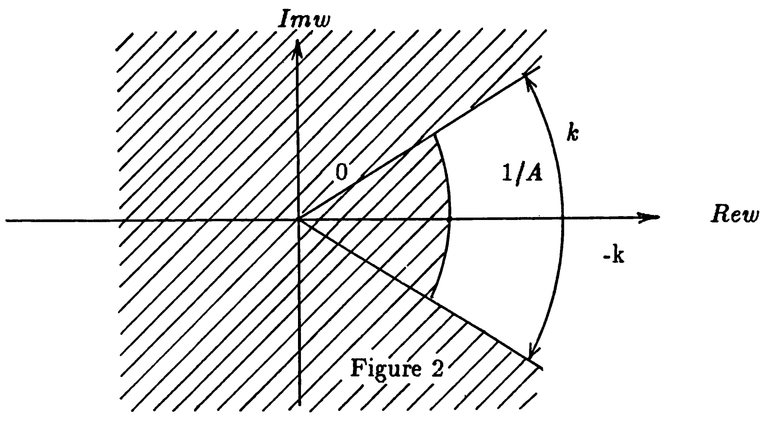

Figure 2

Corollary 1. (Carlson type theorem)

Let $F(z)$ be holomorphic function with the estimate $(*)$ defined on the right half plane. If $F(n) = 0$ is valid for all natural number n, then $F(z)$ vanishes identically.

Proof.

By virtue of Theorem 2, $F(z)$ is holomorphic function of exponential type. So Carlson's theorem yields our desired result (see [1] and [5]).

Corollary 2. (Cartright type theorem)

Let $F(z)$ be holomorphic in the right half plane and satisfy the estimate $(*)$. If $|F(n)| \leq M$ is valid for all natural number n, then $F(z)$ is bounded on the real axis.

Proof

From the assumption and Theorem 1, $F(z)$ is exponential type function in the right half plane. So we obtain our desired result from the usual Cartright theorem (see [1]).

Theorem 3. (Phragmen-Lindelof type theorem)

Let $F(z)$ be an entire function with estimate $(*)$ and satisfy $\limsup_{n\to\infty} |F(n)|^{1/n} = A$ and $\limsup_{n\to\infty} |F(-n)|^{1/n} = B$, then $F(z)$ is an entire function of exponential type.

Proof.

This is a consequence of Theorem 2.

Theorem 4. (Liouville type theorem, see Yoshino [12])

Let $F(z)$ be entire function with estimate $(*)$ with $k = 0$ and furthermore $F(n) = O(|n|^p)$ for all integer n . Then $F(z)$ is a polynomial with degree at most p.

Proof.

From same argument in proof of Theorem 1, we conclude that $MF(w)$ is holomorphic except the origin. The origin is pole (degree at most p) of $MF(w)$. The integral representation of $F(z)$ and residue theorem yield our desired result.

Note.

All results in this paper can be generalized to n-dimensional case without any difficulties.

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