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ON THE HORIZONTAL COHOMOLOGY WITH GENERAL COEFFICIENTS

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This paper is a continuation of the author's paper [5], where the Vinogradov category [9],[12] of nonlinear partial differential equations was shown to be comonadic. This means that it belongs to a class of categories well known to the category theorists and exhaustively studied during the last 30 years in connection with categorical algebra and categorical homology theory (cf. [3],[4], our general references for all categorical concepts).

In this paper we profit from the results achieved. Namely, we show, that the Van Osdol [8] bicohomology theory, originally developed for a better understanding of certain facts occurring in sheaf theory, fits our situation as well. This gives rise to a new cohomology theory for differential equations, naturally generalizing the horizontal cohomology theory of [10],[11].

Throughout the paper it will be

∞ .... $\aleph_0$,
M .... a finite-dimensional paracompact smooth manifold,
m .... its dimension,

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\mathcal{M}_M$ ... any category of smooth $\le\infty$-dimensional fibered manifolds over M with smooth maps over M as morphisms, with Whitney sums as finite products, which admits:

j^r ... the r-jet prolongation functor $\mathcal{M}_M \longrightarrow \mathcal{M}_M$, $r\le\infty$, i.e. an assignment to a manifold $Y \in \mathcal{M}_M$ of the manifold $j^r Y$ of all r-jets $j^r_x\gamma$ of local sections γ of Y, $x\in M$.
The reader should check his favorite category of ∞-dimensional manifolds for these properties.

$\mathcal{J}$ .... j^∞: $\mathcal{M}_M \longrightarrow \mathcal{M}_M$.

$\mathbb{J}$ .... the comonad $(\mathcal{J}, \pi, \iota)$ in $\mathcal{M}_M$, with the counit π defined by πY: $j^\infty Y \longrightarrow \mathrm{Id}$, $j^\infty_x\gamma \longmapsto \gamma(x)$, and the comultiplication ι defined by ιY: $j^\infty Y \longrightarrow j^\infty j^\infty Y$, $j^\infty_x\gamma \longmapsto j^\infty_x j^\infty\gamma$, where $j^\infty\gamma$: $x \longmapsto j^\infty_x\gamma$.

$\mathcal{DE}$ ... the Vinogradov [9],[11],[12] category of infinitely prolonged systems of nonlinear partial differential equations (henceforth simply equations) and solution preserving differential operators between them.

$\mathcal{DE}_M$ .. the subcategory of $\mathcal{DE}$ of equations with the base manifold of independent variables M, and independent variables preserving differential operators between them.

$\mathcal{M}^{\mathbb{J}}_M$ ... the Eilenberg-Moore category of $\mathbb{J}$-coalgebras, in [5] identified with $\mathcal{DE}_M$.
In what follows, **$\mathbb{J}$-coalgebras and equations are synonyma.**

$\mathcal{J}$ .... $\mathcal{M}_M \longrightarrow \mathcal{M}^{\mathbb{J}}_M$ - the cofree coalgebra = "empty equation" functor $\mathcal{M} \longrightarrow \mathcal{M}^{\mathbb{J}}_M$, $Y \longmapsto (j^\infty Y, \iota Y)$ = the right adjoint to the forgetful functor $\mathcal{M}^{\mathbb{J}}_M \longrightarrow \mathcal{M}_M$, $(X,\xi) \longmapsto X$.

We also make an agreement that $[\ ,\ ]_M$ denotes hom-sets in $\mathcal{M}^{\mathbb{J}}_M$ to distinguish them from hom-sets $(\ ,\ )_M$ in $\mathcal{M}_M$.

As the functor j^∞ preserves Whitney sums in $\mathcal{M}_M$, so does the functor $\mathcal{J}$: $\mathcal{M}_M \longrightarrow \mathcal{M}^{\mathbb{J}}_M$, so that all requirements of Van Osdol [8] to construct the bicohomology theory relative to functors $\mathcal{J}$ and Id: $\mathcal{M}^{\mathbb{J}}_M \longrightarrow \mathcal{M}^{\mathbb{J}}_M$ are fulfilled.

Namely, for any abelian group object $\mathcal{A} = (A, \alpha, +, -, 0)$ in $\mathcal{M}^{\mathbb{J}}_M$, we have abelian groups $\mathcal{J}\mathcal{A}$, $\mathcal{J}^2\mathcal{A} = \mathcal{J}\mathcal{J}\mathcal{A}$, $\mathcal{J}^3\mathcal{A} = \mathcal{J}\mathcal{J}\mathcal{J}\mathcal{A}$, etc., and abelian group homomorphisms

$$\chi^n_n\mathcal{A}\colon \mathcal{J}^n\mathcal{A} \xrightarrow{\mathcal{J}^n\alpha} \mathcal{J}^{n+1}\mathcal{A},$$

$$\chi^n_i\mathcal{A}\colon \mathcal{J}^n\mathcal{A} \xrightarrow{\mathcal{J}^i\iota\mathcal{J}^{n-i-1}\mathcal{A}} \mathcal{J}^{n+1}\mathcal{A}, \quad i=0,\dots,n-1.$$

This allows us to construct a complex of abelian groups

$$(1) \qquad 0 \longrightarrow [\mathcal{X}, \mathcal{J}\mathcal{A}]_M \xrightarrow{\partial_1} [\mathcal{X}, \mathcal{J}^2\mathcal{A}]_M \xrightarrow{\partial_2} [\mathcal{X}, \mathcal{J}^3\mathcal{A}]_M \xrightarrow{\partial_3} \dots$$

for any coalgebra $\mathcal{X} = (X, \xi)$, where

$$[\mathcal{X}, \mathcal{J}^n\mathcal{A}]_M \ni \varphi \xmapsto{\partial_n} \sum_{i=0}^{n} (-1)^i \ \chi^n_i\mathcal{A}\circ\varphi \in [\mathcal{X}, \mathcal{J}^{n+1}\mathcal{A}]_M$$

The condition $\partial_{n+1}\circ\partial_n = 0$ then follows immediately from the definitions. The group

$$H^n_{\mathbb{J}}(\mathcal{X}, \mathcal{A}) := \frac{\operatorname{Ker} \partial_{n+1}}{\operatorname{Im} \partial_n}$$

is called the n-th *$\mathbb{J}$-cohomology group* of the equation $\mathcal{X}$ with coefficients in the group $\mathcal{A}$.

Because of the adjointness isomorphism $\#$: $(X, A)_M \cong [\mathcal{X}, \mathcal{J}\mathcal{A}]_M$, the complex (1) is isomorphic to

$$(1') \qquad 0 \longrightarrow (X, A)_M \xrightarrow{\partial'_1} (X, \mathcal{J}A)_M \xrightarrow{\partial'_2} (X, \mathcal{J}^2A)_M \xrightarrow{\partial'_3} \dots$$

where $\partial'_1\colon f \longmapsto \mathcal{J}f\circ\xi - \alpha\circ f$, $\partial'_2\colon f \longmapsto \mathcal{J}f\circ\xi - \iota A\circ f + \mathcal{J}\alpha\circ f$ etc. From the first assignment it immediately follows, that $\partial'_1 f = 0$ if and only if f is a $\mathbb{J}$-homomorphism $\mathcal{X} \longrightarrow \mathcal{A}$. Hence

$$(2) \qquad H^0_{\mathbb{J}}(\mathcal{X}, \mathcal{A}) \cong [\mathcal{X}, \mathcal{A}]_M \cong \mathcal{DE}_M(\mathcal{X}, \mathcal{A})$$

The expression for ∂'_2 then serves as the basis for the identification of the elements of $H^1_{\mathbb{J}}(\mathcal{X}, \mathcal{A})$ with isomorphism classes of principal bundles over $\mathcal{X}$ with the structure group $\mathcal{A}$ in [8], Th. 7. We skip the identification here, but remark that according to Theorem 1 below this reveals the categorical background of Khorkova [1] work on $\bar{H}^1\mathcal{X}$ and might result in a generalization of [1] to a wider class of coverings $\tilde{\mathcal{X}} \longrightarrow \mathcal{X}$ in the sense of [2], [9].

Lemma 1. *For any equation* $\mathcal{X} \in \mathfrak{M}_M^{\mathbb{J}}$ *and any vector bundle* $B \in \mathfrak{M}_M$ *the groups* $\mathrm{H}^n_{\mathbb{J}}(\mathcal{X}, \mathfrak{F}B)$ *are zero for any* $n = 1,2,3,\dots$

Proof (cf. [4], exercise 3.1.22(b)): According to (2) it is to be verified the exactness of the sequence

$$0 \longrightarrow [\mathcal{X},\mathfrak{F}\mathcal{A}]_M \xrightarrow{\ker \partial_1} [\mathcal{X},\mathfrak{F}^2\mathcal{A}]_M \xrightarrow{\partial_1} [\mathcal{X},\mathfrak{F}^3\mathcal{A}]_M \xrightarrow{\partial_2} \dots$$

where now $\partial_n\varphi = \Sigma_{i=0}^n (-1)^i \chi_i^n \mathfrak{F}\mathcal{A}\circ\varphi = \Sigma_{i=0}^n (-1)^i \chi_i^{n+1}\mathcal{A}\circ\varphi$. The map

$$s_{n+1} = (-1)^n . \mathfrak{F}^{n+1}\pi\mathcal{A}\colon \mathfrak{F}^{n+2}\mathcal{A} \longrightarrow \mathfrak{F}^{n+1}\mathcal{A}$$

induces a contracting homotopy

$$[\mathcal{X}, s_{n+1}]_M\colon [\mathcal{X},\mathfrak{F}^{n+2}\mathcal{A}]_M \longrightarrow [\mathcal{X},\mathfrak{F}^{n+1}\mathcal{A}]_M.$$

Indeed, $s_{n+1}\circ\chi_i^{n+1}\mathcal{A} + \chi_i^n\mathcal{A}\circ s_n = 0$ for $i = 0,1,\dots,n-1$, whence $s_{n+1}\circ\partial_n + \partial_{n-1}\circ s_n = (-1)^n\ s_{n+1}\circ\chi_n^{n+1} = \mathrm{id}$ for $n>0$.

In what follows we restrict our choice of abelian group objects in $\mathfrak{M}_M^{\mathbb{J}}$ to linear equations. For a linear equation, say $\mathcal{A} = (A,\alpha,+,-,0) \in \mathfrak{M}_M^{\mathbb{J}}$, A is a $\le\infty$-dimensional vector bundle over M. We define a homomorphism of linear equations as a $\mathbb{J}$-homomorphism, which is simultaneously a linear map of the underlying vector bundles. We call a sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ of homomorphisms of linear equations exact, if $\operatorname{Ker} g$ and $\operatorname{Im} f$ exist as vector bundles and are equal.

Lemma 2. *Let* $A \hookrightarrow B \twoheadrightarrow C$ *be a short exact sequence of vector bundles over* M. *Then the induced sequences* $\mathfrak{F}A \hookrightarrow \mathfrak{F}B \twoheadrightarrow \mathfrak{F}C$ *and* $(X,A)_M \hookrightarrow (X,B)_M \twoheadrightarrow (X,C)_M$, *are exact for any* $X \in \mathfrak{M}_M$ *as well.*

Proof: Since M is paracompact, any short exact sequence of vector bundles over M splits, whence any product preserving functor is exact, particularly $\mathfrak{F}$ and $(X,-)_M$.

Lemma 3. *Assigned to any short exact sequence of linear equations* $\mathcal{A} \xhookrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ *and any equation* $\mathcal{X} \in \mathfrak{M}_M^{\mathbb{J}}$ *there is an exact sequence of abelian groups*

$$(3)\quad \begin{array}{l} 0 \longrightarrow [\mathcal{X},\mathcal{A}]_M \longrightarrow [\mathcal{X},\mathcal{B}]_M \longrightarrow [\mathcal{X},\mathcal{C}]_M \longrightarrow H^1_{\mathbb{J}}(\mathcal{X},\mathcal{A}) \longrightarrow \dots \\ \dots \longrightarrow H^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}) \longrightarrow H^n_{\mathbb{J}}(\mathcal{X},\mathcal{B}) \longrightarrow H^n_{\mathbb{J}}(\mathcal{X},\mathcal{C}) \longrightarrow H^{n+1}_{\mathbb{J}}(\mathcal{X},\mathcal{A}) \rightarrow \dots \end{array}$$

Proof: From the naturality of the homomorphisms ∂ it follows the existence of a short sequence of complexes

$$(1')\quad \begin{array}{ccccccccc} 0 & \longrightarrow & (X,A)_M & \xrightarrow{\partial_1'} & (X,\mathfrak{J}A)_M & \xrightarrow{\partial_2'} & (X,\mathfrak{J}^2A)_M & \xrightarrow{\partial_3'} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (X,B)_M & \xrightarrow{\partial_1'} & (X,\mathfrak{J}B)_M & \xrightarrow{\partial_2'} & (X,\mathfrak{J}^2B)_M & \xrightarrow{\partial_3'} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (X,C)_M & \xrightarrow{\partial_1'} & (X,\mathfrak{J}C)_M & \xrightarrow{\partial_2'} & (X,\mathfrak{J}^2C)_M & \xrightarrow{\partial_3'} & \dots \end{array}$$

which is exact due to the preceding lemma and induces the exact sequence of the assertion. •

To compute the $\mathbb{J}$-cohomology we use the standard method of resolutions. We define a resolution of a linear equation $\mathcal{A}$ as an exact sequence $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 \longrightarrow \dots$ for which it is $\mathcal{A} \cong \mathrm{Ker}\ (\mathcal{A}_0 \longrightarrow \mathcal{A}_1)$. Let us call a resolution $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \dots$ acyclic, if $H^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}_i) = 0$ for every $\mathcal{X}\in\mathcal{M}^{\mathbb{J}}_M$ and every $n>0$, $i\geq 0$. Let us call the resolution $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \dots$ cofree, if all the equations $\mathcal{A}_i$ are cofree, i.e. are of the form $\mathcal{A}_i = \mathfrak{J}B$, $B\in\mathcal{M}_M$. By Lemma 1, all cofree resolutions are acyclic.

Definition: Let $\mathcal{X}\in\mathcal{M}^{\mathbb{J}}_M$ be an equation, let $\mathcal{A}\in\mathcal{M}^{\mathbb{J}}_M$ be a linear equation and let $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \dots$ be a resolution of the latter. Define a *horizontal complex* of the equation $\mathcal{X}$, corresponding to this equation, as the complex

$$(4)\qquad 0 \longrightarrow [\mathcal{X},\mathcal{A}_0]_M \longrightarrow [\mathcal{X},\mathcal{A}_1]_M \longrightarrow [\mathcal{X},\mathcal{A}_2]_M \longrightarrow \dots$$

Denote by $\bar{H}^n(\mathcal{X},\mathcal{A})$ the factor

$$\frac{\mathrm{Ker}\ ([\mathcal{X},\mathcal{A}_n]_M \longrightarrow [\mathcal{X},\mathcal{A}_{n+1}]_M)}{\mathrm{Im}\ ([\mathcal{X},\mathcal{A}_{n-1}]_M \longrightarrow [\mathcal{X},\mathcal{A}_n]_M)}.$$

Theorem 1. *Let* $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \dots$ *be an acyclic resolution of an equation* $\mathcal{A}$. *Then for every equation* $\mathcal{X}$ *and each natural number* n *there is a natural isomorphism*

$$\bar{\mathrm{H}}^n(\mathcal{X},\mathcal{A}) \cong \mathrm{H}^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}).$$

Proof: Denote by $\mathcal{B}_i$ the vector bundle $\operatorname{Ker}(A_i \longrightarrow A_{i+1}) \cong \operatorname{Im}(A_{i-1} \longrightarrow A_i)$, equipped with the $\mathbb{J}$-coalgebra structure induced from $\mathcal{A}_i$. Then for any of the short exact sequences

$$0 \longrightarrow \mathcal{A} \hookrightarrow \mathcal{A}_0 \twoheadrightarrow \mathcal{B}_1 \longrightarrow 0, \quad \dots ,$$

$$0 \longrightarrow \mathcal{B}_i \hookrightarrow \mathcal{A}_i \twoheadrightarrow \mathcal{B}_{i+1} \to 0, \quad \dots ,$$

the corresponding exact sequences (3) decompose into

$$0 \longrightarrow [\mathcal{X},\mathcal{A}]_M \longrightarrow [\mathcal{X},\mathcal{A}_0]_M \longrightarrow [\mathcal{X},\mathcal{B}_1]_M \longrightarrow \mathrm{H}^1_{\mathbb{J}}(\mathcal{X},\mathcal{A}) \longrightarrow 0, \quad \dots$$

$$\dots\dots\dots\dots\dots\dots, \quad 0 \longrightarrow \mathrm{H}^n_{\mathbb{J}}(\mathcal{X},\mathcal{B}_1) \cong \mathrm{H}^{n+1}_{\mathbb{J}}(\mathcal{X},\mathcal{A}) \longrightarrow 0, \quad \dots$$

$$0 \longrightarrow [\mathcal{X},\mathcal{B}_i]_M \longrightarrow [\mathcal{X},\mathcal{A}_i]_M \longrightarrow [\mathcal{X},\mathcal{B}_{i+1}]_M \longrightarrow \mathrm{H}^1_{\mathbb{J}}(\mathcal{X},\mathcal{B}_i) \longrightarrow 0, \quad \dots$$

$$\dots\dots\dots\dots\dots\dots, \quad 0 \longrightarrow \mathrm{H}^n_{\mathbb{J}}(\mathcal{X},\mathcal{B}_{i+1}) \cong \mathrm{H}^{n+1}_{\mathbb{J}}(\mathcal{X},\mathcal{B}_i) \longrightarrow 0, \dots$$

Therefore, in the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & & & & \mathrm{H}^1_{\mathbb{J}}(\mathcal{X},\mathcal{A}) & & & & \mathrm{H}^1_{\mathbb{J}}(\mathcal{X},\mathcal{B}_2) \cong \mathrm{H}^3_{\mathbb{J}}(\mathcal{X},\mathcal{A}) & & \\
 & & & {[\mathcal{X},\mathcal{B}_1]_M} \arrow[ur, two heads] \arrow[dr, hook] & & & & {[\mathcal{X},\mathcal{B}_3]_M} \arrow[ur, two heads] \arrow[dr, hook] & & & \\
0 \arrow[rr] & & {[\mathcal{X},\mathcal{A}_0]_M} \arrow[rr] \arrow[ur] & & {[\mathcal{X},\mathcal{A}_1]_M} \arrow[rr] \arrow[dr] & & {[\mathcal{X},\mathcal{A}_2]_M} \arrow[rr] \arrow[ur] & & {[\mathcal{X},\mathcal{A}_3]_M} \arrow[rr] \arrow[dr] & & \dots\dots \\
 & {[\mathcal{X},\mathcal{A}]_M} \arrow[ur, hook] & & & & {[\mathcal{X},\mathcal{B}_2]_M} \arrow[ur, hook] \arrow[dr, two heads] & & & & {[\mathcal{X},\mathcal{B}_4]_M} \arrow[ur, hook] \arrow[dr, two heads] & \\
 & & & & & & \mathrm{H}^1_{\mathbb{J}}(\mathcal{X},\mathcal{B}_1) \cong \mathrm{H}^2_{\mathbb{J}}(\mathcal{X},\mathcal{A}) & & & & \dots\dots
\end{tikzcd}$$

all the $\nearrow$ and $\searrow$ sequences are exact, whence

$$\begin{aligned}
\bar{\mathrm{H}}^0(\mathcal{X},\mathcal{A}) &= \operatorname{Ker}([\mathcal{X},\mathcal{A}_0] \longrightarrow [\mathcal{X},\mathcal{A}_1]) \cong \\
&\cong \operatorname{Ker}([\mathcal{X},\mathcal{A}_0] \longrightarrow [\mathcal{X},\mathcal{B}_1]) \cong \\
&\cong [\mathcal{X},\mathcal{A}]_M,
\end{aligned}$$

and

$$\bar{H}^n(\mathcal{X},\mathcal{A}) \cong \frac{\mathrm{Ker}\ ([\mathcal{X},\mathcal{A}_n]_M \longrightarrow [\mathcal{X},\mathcal{A}_{n+1}]_M)}{\mathrm{Im}\ ([\mathcal{X},\mathcal{A}_{n-1}]_M \longrightarrow [\mathcal{X},\mathcal{A}_n]_M)} \cong$$

$$\cong \frac{\mathrm{Ker}\ ([\mathcal{X},\mathcal{A}_n]_M \longrightarrow [\mathcal{X},\mathcal{B}_{n+1}]_M)}{\mathrm{Im}\ ([\mathcal{X},\mathcal{A}_{n-1}]_M \longrightarrow [\mathcal{X},\mathcal{A}_n]_M)} \cong$$

$$\cong \frac{[\mathcal{X},\mathcal{B}_n]_M}{\mathrm{Im}\ ([\mathcal{X},\mathcal{A}_{n-1}]M \longrightarrow [\mathcal{X},\mathcal{B}_n]\ )} \cong$$

$$\cong H^1_{\mathbb{J}}(\mathcal{X},\mathcal{B}_{n-1}) \cong$$

$$\cong H^2_{\mathbb{J}}(\mathcal{X},\mathcal{B}_{n-2}) \cong$$

$$\cdots\cdots\cdots\cdots$$

$$\cong H^{n-1}_{\mathbb{J}}(\mathcal{X},\mathcal{B}_1) \cong$$

$$\cong H^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}).$$

Thus the groups $\bar{H}^n(\mathcal{X},\mathcal{A})$ do not depend on the choice of the resolution $\mathcal{A}_0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \ldots,$ if only it is acyclic. Let us call the group $\bar{H}^n(\mathcal{X},\mathcal{A})$ the n-th *horizontal cohomology group of the equation* $\mathcal{X}$ *with coefficients in the linear equation* $\mathcal{A}$.

There is a wide class of linear equations possessing a cofree resolution of a finite length. See [7], Theorem 5.5 for the following statement:

For any involutive linear equation $\mathcal{A}\in\mathcal{M}^{\mathbb{J}}_M$, $\dim M = m$ *there exists a cofree resolution of the form*

$$(5) \qquad \mathcal{J}B_0 \xrightarrow{\Phi_1} \mathcal{J}B_1 \xrightarrow{\Phi_2} \ldots \xrightarrow{\Phi_m} \mathcal{J}B_m \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

In what follows it is called the *Janet resolution* and the corresponding complex of differential operators

$$(6) \qquad 0 \longrightarrow B_0 \xrightarrow{\varphi_1} B_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_m} B_m \longrightarrow 0 \longrightarrow \ldots,$$

$\Phi_i = \varphi_i^{H}$, is called the *Janet sequence*.

For a coalgebra $\mathcal{X} = (X,\xi) \in \mathfrak{M}_M^{\mathbb{J}}$ the corresponding complex (4),

$$0 \longrightarrow [\mathcal{X},\mathfrak{J}B_0]_M \xrightarrow{[\mathcal{X},\Phi_1]_M} [\mathcal{X},\mathfrak{J}B_1]_M \xrightarrow{[\mathcal{X},\Phi_2]_M} [\mathcal{X},\mathfrak{J}B_2]_M \longrightarrow \dots$$

is isomorphic to the complex

$$(7) \qquad 0 \longrightarrow (\mathcal{X},B_0)_M \xrightarrow{(\mathcal{X},\varphi_1)_M} (\mathcal{X},B_1)_M \xrightarrow{(\mathcal{X},\varphi_2)_M} (\mathcal{X},B_2)_M \longrightarrow \dots$$

which we shall call the *horizontal Janet complex*.

Corollary: *$H^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}) = 0$ for $n>m$, for any equation $\mathcal{X} \in \mathfrak{M}_M^{\mathbb{J}}$ and any involutive linear equation $\mathcal{A} \in \mathfrak{M}_M^{\mathbb{J}}$.*

Moreover, for non-overdetermined equations we have $B_2 = B_3 = \dots = B_m = 0$ (see [7], Theorem 6.8), so that both Janet sequence and Janet complex have exactly two terms.

Corollary: *$H^n_{\mathbb{J}}(\mathcal{X},\mathcal{A}) = 0$ for $n>2$, for any equation $\mathcal{X} \in \mathfrak{M}_M^{\mathbb{J}}$ and any non-overdetermined linear equation $\mathcal{A} \in \mathfrak{M}_M^{\mathbb{J}}$.*

Example: The common de Rham complex

$$\mathcal{F}M \xrightarrow{\mathrm{d}} \Lambda M \xrightarrow{\mathrm{d}} \Lambda^2 M \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Lambda^m M \longrightarrow 0$$

and the corresponding Spencer sequence

$$\mathfrak{J}\mathcal{F}M \xrightarrow{\mathrm{S}} \mathfrak{J}\Lambda M \xrightarrow{\mathrm{S}} \mathfrak{J}\Lambda^2 M \xrightarrow{\mathrm{S}} \dots \xrightarrow{\mathrm{S}} \mathfrak{J}\Lambda^3 M \longrightarrow 0$$

serve us as the Janet complex and Janet sequence of the "equation of constants" $\partial y/\partial x^i = 0$, $i = 1,\dots,m$, correspondingly. The horizontal Janet complex then coincides with the *horizontal de Rham complex*

$$\mathcal{F}X \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda} X \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^2 X \xrightarrow{\bar{\mathrm{d}}} \dots \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^m X \longrightarrow 0$$

studied in Vinogradov [10],[11] by means of the so called *$\mathcal{C}$-spectral sequence*, associated with the restriction on $\mathcal{X}$ of the famous "*variational bicomplex*" $\Lambda^{p,q}$.

By similar methods we are able to prove the following:

Theorem 3. *Associated with an equation* $\mathcal{X} \in \mathcal{DE}_M$ *and an involutive linear equation* $\mathcal{A} \in \mathcal{DE}_M$ *possessing a Janet resolution* (1), *there is a bicomplex* $B^{p,q}\mathcal{X}$ *such that*

I. *Its first spectral sequence* $E_r^{p,q}(\mathcal{X})$ *locally reduces to the Janet cohomology of the equation* $\mathcal{A}$,

II. *Its second spectral sequence* $Ш_r^{p,q}(\mathcal{X})$ *satisfies* $Ш_1^{0,q}(\mathcal{X}) = H_{\mathbb{J}}^q(\mathcal{X}, \mathcal{A})$

and both converge.

Finally, Vinogradov [10],[11] methods allow us to compute the terms $Ш_r^{p,q}$ necessary to find $H_{\mathbb{J}}^q(\mathcal{X}, \mathcal{A})$. Essentially the same picture is observed: Generalized Spencer complexes occur and the two-line theorem is valid. The details should appear in [6]. This enlarges the class [1] of coverings $\tilde{\mathcal{X}} \longrightarrow \mathcal{X}$ computable by means of a spectral sequence.

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