PONTRYAGIN ALGEBRA OF A TRANSITIVE LIE ALGEBROID

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INTRODUCTION

The Chern-Weil homomorphism h^P of a principal fibre bundle (pfb) P has been known for some forty years [Ch]. On the other hand, in analogy to the theory of Lie groups and Lie algebras, each pfb P has its algebraic equivalent: a transitive Lie algebroid (tLa) $A(P)$ - constructed on the basis of the right-invariant vector fields on P. $A(P)$ is simply a vector bundle equipped with some structures (of an algebraic nature) like a structure of a Lie algebra in the module of sections.

It turns out that the Chern-Weil homomorphism of P is a notion of the Lie algebroid of this pfb! This means that, knowing only the Lie algebroid $A(P)$ of $P=P(M,G)$, one can uniquely reproduce the ring of invariant polynomials $(V\mathfrak{g}^*)_I$ and the Chern-Weil homomorphism $h^P:(V\mathfrak{g}^*)_I\longrightarrow H(M)$ ($\mathfrak{g}$ denotes the Lie algebra of G).

We pay our attention to the fact that this holds although in the Lie algebroid $A(P)$ there is no direct information about the structural Lie group of P!

This paper is in final form and no version of it will be submitted for publication elsewere.

In addition, we must point out two things:

1) A tLa is - in some sense - a simpler structure than a pfb. Namely, nonisomorphic pfb's can possess isomorphic Lie algebroids. For example, there exists a nontrivial pfb for which the Lie algebroid is trivial (the nontrivial *Spin(3)*-structure of the trivial pfb $\mathbb{R}P(5)\times SO(3)$ [Kub]$_2$).

2) There exist "nonintegrable" tLa's, ie tLa's which cannot be realized as the Lie algebroids of pfb's. First examples were constructed by R. Almeida and P. Molino [Al-Mo]$_{1-2}$ (see also [Mo]) basing themselves on transwersally complete foliations. The tLa of the foliation of a compact simply connected Lie group by the left cosets of a connected and nonclosed subgroup is an example of a nonintegrable tLa [Mo].

In connection with the above, it seems important to construct the Chern-Weil homomorphism of a tLa *A* in such a way that it will agree with the Chern-Weil homomorphism of any pfb *P* for which *A* is its Lie algebroid. In addition, this homomorphism will probably be useful to investigate some nonintegrable tLa's.

Originally, the notion of a Lie algebroid was invented in connection with the study of differential groupoids [J. Pradines in [Pra]$_{1-2}$ introduced the so-called *Lie functor* which assigns a Lie algebroid to any differential groupoid]. Since each pfb *P* determines a differential groupoid (the so-called *Lie groupoid* PP^{-1} *of Ehresmann* [Ehr]), therefore each pfb *P* defines - in an indirect manner - a tLa $A(P)$. P. Libermann noticed [Lib] that the vector bundle of this tLa $A(P)$, $P=P(M,G)$, is canonically isomorphic to the vector bundle TP/G (investigated earlier by M. Atiyah in the context of the problem of the existence of a complex connection in a complex pfb [At]). The construction of the Lie functor for pfb's with the omission of the indirect step of differential groupoids was made independently by K. Mackenzie [Mac] and by the author [Kub]$_1$.

LIE FUNCTOR FOR PFB'S

We begin with giving the fundamental (for our considerations) definition of a tLa and with a construction of the Lie functor. We assume that all the manifolds considered are C^{∞} and Hausdorff, and that M - the base of tLa's - is connected.

Definition. [Pra]$_{1-2}$. By a *transitive Lie algebroid* (tLa) on a manifold M we shall mean a system

$$A = (A, [\![\cdot,\cdot]\!], \gamma) \tag{1}$$

consisting of a vector bundle A over M and mappings

$$[\![\cdot,\cdot]\!]: SecA\times SecA \longrightarrow SecA \ , \quad \gamma: A\longrightarrow TM,$$

such that

(a) $(SecA, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

(b) γ is an epimorphism of vector bundles,

(c) $Sec\gamma: SecA\longrightarrow \mathfrak{X}(M)$ is a homomorphism of Lie algebras,

(d) $[\![\xi, f\eta]\!] = f\cdot[\![\xi,\eta]\!]+(\gamma\circ\xi)(f)\cdot\eta$ for $f\in C^{\infty}(M)$, $\xi,\eta\in SecA$.

Let (1) and $(A', [\![\cdot,\cdot]\!]', \gamma')$ be two Lie algebroids on the same manifold M. By a *homomorphism* between them we mean a strong homomorphism $H: A\longrightarrow A'$ of vector bundles, such that

(a) $\gamma'\circ H=\gamma$,

(b) $SecH: SecA\longrightarrow SecA'$ is a homomorphism of Lie algebras.

With each tLa (1) we associate a short exact sequence

$$0\longrightarrow \mathbf{g} \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow 0 \tag{2}$$

($\mathbf{g}=Ker\gamma$) called the *Atiyah sequence* of (1). $\mathbf{g}$ is a Lie algebra bundle if in each fibre $\mathbf{g}_{|x}$ the Lie algebra structure is defined by: $[v,w]:=[\![\xi,\eta]\!](x)$, $\xi,\eta\in SecA$, $\xi(x)=v$, $\eta(x)=w$ (see [Al-Mo]$_1$, [Mac] and [Kub]$_2$).

For the construction of the *Lie functor*, we take any pfb $P=P(M,G)$ with the projection $\pi:P\longrightarrow M$ and the action $R:P\times G\longrightarrow P$, and define another pfb $TP(TM,TG)$ with the projection $\pi_*:TP\longrightarrow TM$ and the action $R_*:TP\times TG\longrightarrow TP$. We can treat G as a closed subgroup of TG $(G\cong\{\vartheta_a;a\in G\}$, ϑ_a being the null tangent vector at a). The restriction of R_* to G is then equal to $R^T:TP\times G\longrightarrow TP$, $(v,a)\longmapsto(R_a)_{*v}$, R_a being the action of a on P. We put

$$A(P)=TP/G$$

- the space of all orbits of R^T, and denote by $\pi^A:TP\longrightarrow A(P)$, $v\longmapsto[v]$, the natural projection. By [Ko-No], we see that the structure of a Hausdorff C^∞-manifold, such that π^A is a submersion, exists in $A(P)$ .In the end, we define the projection $p:A(P)\longrightarrow M$, $[v]\longmapsto\pi z$, if $v\in T_zP$. For each point $x\in M$, in the fibre $p^{-1}(x)$ there exists exactly one $\mathbb{R}$-vector space structure such that $[v]+[w]=[v+w]$ if $\pi_P(v)=\pi_P(w)$, $\pi_P:TP\longrightarrow P$ being the projection. The system $(A(P),p,M)$ is a vector bundle [Mac], $[\mathrm{Kub}]_1$. Let $\mathfrak{X}^R(P)$ denote the $C^\infty(M)$-module of all C^∞ global right-invariant vector fields on P .

Proposition. [Mac],$[\mathrm{Kub}]_1$. For each cross-section $\eta\in SecA(P)$, there exists exactly one C^∞ right-invariant vector field $\eta'\in\mathfrak{X}^R(P)$ such that $[\eta'(z)]=\eta(\pi z)$. The mapping

$$SecA(P)\longrightarrow\mathfrak{X}^R(P),\quad \eta\longmapsto\eta', \tag{3}$$

is an isomorphism of $C^\infty(M)$-modules.■

Now, we define some $\mathbb{R}$-Lie algebra structure $[\![\cdot,\cdot]\!]$ in the $\mathbb{R}$-vector space $SecA(P)$ by demanding that (3) be an isomorphism of $\mathbb{R}$-Lie algebras. We also take the mapping $\gamma:A(P)\longrightarrow TM$, $[v]\longmapsto\pi_*v$.

Theorem. [Mac], [Kub]$_1$. The object

$$A(P)=(A(P),[\![\cdot,\cdot]\!],\gamma)$$

is a transitive Lie algebroid. A homomorphism $F=(F,\mu):P(M,G)\longrightarrow P'(M,G')$ of pfb's [$\mu:G\longrightarrow G'$ - a homomorphism of Lie groups, $F(za)=F(z)\cdot\mu(a)$] determines a mapping $dF:A(P)\longrightarrow A(P')$, $[v]\longmapsto[F_* v]$, which is a homomorphism of Lie algebroids. The correspondence $P\longmapsto A(P)$, $F\longmapsto dF$, is a covariant functor (called the *Lie functor for pfb's*).■

An interpretation of sections of the Lie algebroid of the Lie groupoid GL(f)

Let $\mathfrak{f}$ be any vector bundle over M and $GL(\mathfrak{f})$ - the Lie groupoid of all linear isomorphisms between fibres of $\mathfrak{f}$. Ngo-Van-Que [NVQ] discovered an operator interpretation of sections of the Lie algebroids $A(GL(\mathfrak{f}))$ (see also [Kum] and [Mac]). We describe it in a little different manner. For a section $\sigma\in Sec\,\mathfrak{f}$ and for $x\in M$, we put $\tilde{\sigma}_x:GL(\mathfrak{f})_x\longrightarrow\mathfrak{f}_{|x}$, $h\longmapsto h^{-1}(\sigma(\beta h))$. Then we have the following

Proposition. Let $\xi\in Sec\,A(GL(\mathfrak{f}))$. Then the mapping

$$\mathfrak{L}_\xi(\sigma):M\longrightarrow\mathfrak{f},\quad x\longmapsto\xi_x(\tilde{\sigma}_x)$$

is a C^∞-section of $\mathfrak{f}$, and $\mathfrak{L}_\xi:Sec\,\mathfrak{f}\longrightarrow Sec\,\mathfrak{f}$, $\sigma\longmapsto\mathfrak{L}_\xi(\sigma)$, is a differential operator of order ≤ 1 such that

$$(4)\qquad \mathfrak{L}_\xi(f\cdot\sigma)=f\cdot\mathfrak{L}_\xi(\sigma)+X(f)\cdot\sigma,\quad f\in C^\infty(M),\ \sigma\in Sec\,\mathfrak{f},\ \text{where } X=\gamma\circ\xi.$$

Conversely, for any differential operator $\mathfrak{L}$ of order ≤ 1 in the vector bundle $\mathfrak{f}$, such that (4) holds for some $X\in\mathfrak{X}(M)$, there exists exactly one section $\xi\in Sec\,A(GL(\mathfrak{f}))$ such that $\mathfrak{L}=\mathfrak{L}_\xi$ and $X=\gamma\circ\xi$. ■

Representations of Lie groupoids and Lie algebroids in vector bundles

By a *representation of a transitive Lie groupoid Φ in a vector bundle $\mathfrak{f}$* (both over M) we mean a strong homomorphism $T:\Phi\longrightarrow GL(\mathfrak{f})$ of Lie groupoids; whereas by a *representation of a transitive Lie algebroid A in a vector bundle $\mathfrak{f}$* we mean a strong homomorphism $T':A\longrightarrow A(GL(\mathfrak{f}))$ of tLa's. Of course, for a representation $T:\Phi\longrightarrow GL(\mathfrak{f})$, the differential $dT:A(\Phi)\longrightarrow A(GL(\mathfrak{f}))$ is a representation of the Lie algebroid $A(\Phi)$ in $\mathfrak{f}$.

Definition. (a). For a representation $T:\Phi\longrightarrow GL(\mathfrak{f})$ of a Lie groupoid Φ in $\mathfrak{f}$, we define the vector space of *invariant sections* of $\mathfrak{f}$ in the following way

$$(Sec\,\mathfrak{f})_I = \left\{\sigma\in Sec\,\mathfrak{f}:\ \bigwedge_{h\in\Phi}\left[T(h)(\sigma_{\alpha h})=\sigma_{\beta h}\right]\right\}$$

(b). For a representation $T':A\longrightarrow A(GL(\mathfrak{f}))$ of a transitive Lie algebroid A in $\mathfrak{f}$, we define analogously

$$(Sec\,\mathfrak{f})_{I^o} = \left\{\sigma\in Sec\,\mathfrak{f}:\ \bigwedge_{\xi\in Sec\,A}\left[\mathfrak{L}_{T'\circ\xi}(\sigma)=0\right]\right\}.$$

The following facts play the fundamental role in our theory:

Theorem 1. Let $T:\Phi\longrightarrow GL(\mathfrak{f})$ be any representation of a Lie groupoid Φ in $\mathfrak{f}$, and $dT:A(\Phi)\longrightarrow A(GL(\mathfrak{f}))$ its differential. Then

(a) $(Sec\,\mathfrak{f})_I\subset(Sec\,\mathfrak{f})_{I^o}$,

(b) if Φ is connected, then $(Sec\,\mathfrak{f})_I=(Sec\,\mathfrak{f})_{I^o}$. ■

For a representation $T:\Phi\longrightarrow GL(\mathfrak{f})$ and for $x\in M$ we take the induced representation $T_x:G_x\longrightarrow GL(\mathfrak{f}_{|x})$ of the isotropy Lie group G_x in the vector space $\mathfrak{f}_{|x}$. By $(\mathfrak{f}_{|x})_I$ we denote the space of T_x-invariant vectors. Then we have

Theorem 2. For an arbitrary $\upsilon\in(\mathfrak{f}_{|x})_I$, the section

$$\sigma_\upsilon : M\longrightarrow\mathfrak{f},\quad y\longmapsto T(h)(\upsilon),\quad \text{where } h\in\Phi \text{ and } \alpha h=x,\ \beta h=y,$$

is a correctly defined smooth invariant section, and the mapping

$$(\mathfrak{f}_{|x})_I\longrightarrow(Sec\,\mathfrak{f})_I,\quad \upsilon\longmapsto\sigma_\upsilon,$$

is an isomorphism of vector spaces.■

In addition, we have the following fact: For an arbitrary representation $T':A\longrightarrow A(GL(\mathfrak{f}))$ of a tLa A in $\mathfrak{f}$, each T'-invariant section $\sigma\in(Sec\,\mathfrak{f})_I{}^{o}$ is uniquely determined by its value at any point.

Now, for a tLa A , having (2) as its Atiyah sequence, we define the <u>*adjoint representation*</u>

$$ad: A\longrightarrow A(GL\mathfrak{g})$$

in such a way that $\mathfrak{L}_{ad\circ\xi}(\sigma)=[\![\xi,\sigma]\!]$, $\sigma\in Sec\,\mathfrak{g}$. ad induces the representation $ad^{\vee}:A\longrightarrow A(GL\bigvee^k\mathfrak{g}^*)$ by the formula

$$\langle\mathfrak{L}_{ad^{\vee}\circ\xi}\Gamma,\sigma_1\vee\ldots\vee\sigma_k\rangle=(\gamma\circ\xi)\langle\Gamma,\sigma_1\vee\ldots\vee\sigma_k\rangle$$
$$-\textstyle\sum_{i=1}^{k}\langle\Gamma,\sigma_1\vee\ldots\vee[\![\xi,\sigma_i]\!]\vee\ldots\vee\sigma_k\rangle.$$

In particular, we have

$$(Sec\bigvee^k\mathfrak{g}^*)_I{}^{o}=\Big\{\Gamma\in Sec\bigvee^k\mathfrak{g}^*:\ \bigwedge_{\xi\in Sec A}\ \bigwedge_{\sigma_1,\ldots,\sigma_k\in Sec\,\mathfrak{f}}\Big[(\gamma\circ\xi)\langle\Gamma,\sigma_1\vee\ldots\vee\sigma_k\rangle$$
$$=\textstyle\sum_{i=1}^{k}\langle\Gamma,\sigma_1\vee\ldots\vee[\![\xi,\sigma_i]\!]\vee\ldots\vee\sigma_k\rangle\Big]\Big\}$$

In addition, if $\Gamma^s\in(Sec\bigvee^{k_s}\mathfrak{g}^*)_I{}^{o}$, $s=1,2,$, then

$$\Gamma^1\vee\Gamma^2\in(Sec\bigvee^{k_1+k_2}\mathfrak{g}^*)_I{}^{o}$$

so $\bigoplus_k (Sec \vee^k g^*)_{I^o}$ is a subalgebra of $\bigoplus_k (Sec \vee^k g^*)$.

K. Mackenzie [Mac] proved that if $A=A(\Phi)$, then ad is a differential of the adjoint representation $Ad:\Phi \longrightarrow GL(g)$ defined by: $Ad(h)=(\tau_h)_{*u_x}$, $\tau_h: G_x \longrightarrow G_y$, $a \longmapsto hah^{-1}$, $x=\alpha h$, $y=\beta h$.

The Chern-Weil homomorphism of a transitive Lie algebroid

By a *connection* in a tLa $A=(A,[\![\cdot,\cdot]\!],\gamma)$ we mean a splitting of the Atiyah sequence $O \longrightarrow g \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow O$ of A, ie a mapping $\lambda: TM \longrightarrow A$ such that $\gamma \circ \lambda = id_{TM}$. For a connection λ in A, the uniquely determined morphism of vector bundles $\omega: A \longrightarrow g$ fulfilling $\omega | g = id$ and $Ker\omega = Im\lambda$ is called a *connection form* of λ. By a *curvature base-form* (or a *curvature tensor*) of a connection λ we shall mean the 2-form Ω_M on M with values in the vector bundle g, defined by the formula

$$\Omega_M(X,Y)=-\omega([\![\lambda\circ X,\lambda\circ Y]\!]) \left(=\lambda\circ[X,Y]-[\![\lambda\circ X,\lambda\circ Y]\!]\right).$$

Theorem. The mapping

$$\begin{aligned} \gamma^M: \bigoplus_k (Sec \vee^k g^*) &\longrightarrow \Omega(M) \\ Sec \vee^k g^* \ni \Gamma &\longmapsto \frac{1}{k!}\langle \Gamma, \underbrace{\Omega_M \vee \ldots \vee \Omega_M}_{k\text{-times}} \rangle \end{aligned}$$

is a homomorphism of algebras such that the form $\gamma_M(\Gamma)$ is closed when Γ is invariant. ■

The superposition

$$h^A: \bigoplus_k (Sec \vee^k g^*)_{I^o} \xrightarrow{\gamma} Z(M) \longrightarrow H(M)$$

is called the *Chern-Weil homomorphism* of A. Its image Imh^A is a subalgebra of $H(M)$ called the *Pontryagin algebra* of A.

Theorem. The Chern-Weil homomorphism h^A of a tLa A is independent of the choice of a connection.■

Now, take any pfb $P(M,G)$ and let $A=A(P)$ be its Lie algebroid. Then, for the Lie groupoid of Ehresmann $\Phi=PP^{-1}$ and for the adjoint representation $Ad:\Phi\longrightarrow GL(g)$, we have, by Theorems 1 and 2, the commuting diagram:

$$\begin{array}{ccc} \bigoplus_k (Sec \bigvee^k g^*)_{I^0} & \xrightarrow{\quad h^A \quad} & \\ \cong\uparrow & & \\ (\bigvee(g_{|x})^*)_I & \xrightarrow{\quad h^{(\Phi_x)} \quad} & H(M) \\ \cong\uparrow & & \\ (\bigvee g^*)_I & \xrightarrow{\quad h^P \quad} & \end{array}$$

from which we obtain that the Chern-Weil homomorphism h^P of a pfb P is an invariant of the Lie algebroid of P .

Remarks. 1/. It is possible to construct the characteristic homomorphism h^A of those nontransitive Lie algebroids A for which γ is of the constant rank (such Lie algebroids are called *regular*).

2/. There exists a characteristic homomorphism of flat (and of partially flat) regular Lie algebroids - an object analogous to that for flat (and for foliated) pfb's.

3/. The proofs of the above-mentioned theorems will appear in the next work by the author.

REFERENCES

[Al-Mo]$_1$- R. ALMEIDA, P. MOLINO "Suites d'Atiyah, feuilletages et quantification geometrique", Estrait du Seminaire du Geometrie differentielle, 1984-85, Montpellier.

[Al-Mo]$_2$-R. ALMEIDA, P. MOLINO "Suites d'Atiyah et feuilletages transversallement complets", C.R.A.S. Paris t.300, 13-15, 1985

[Ch] - S.S. CHERN "Differential Geometry of fiber bundles", Proc Intern. Congress Math (1950) VolII, 397-411.

[Ko-No]-S.KOBAYASHI, K.NOMIZU "Foundations of differential Geometry" VolI, New York-London, 1963.

[Kub]$_1$-J.KUBARSKI, "Lie algebroid of a principal fibre bundle - three equivalent definitions" Zeszyty Naukowe Politechniki Szczecińskiej (in printing) Konferencja z Geometrii Różniczkowej, Szczecin 1987.

[Kub]$_2$- J.KUBARSKI "Lie algebroid of a principal fibre bundle" Preprint No7, Institute of Mathematics, Technical University of Łódź, January 1988.

[Kum]- A.KUMPERA, "An Introduction to Lie groupoids", Nucleo de Estudos e Pasquisas Cientificas, Rio de Janeiro, 1971.

[Lib]- P.LIBERMANN,"Sur les prolongements des fibres principaux et les groupoides differentiables" Seminaire Analyse Global, Montreal 1969.

[Mac]- K.MACKENZIE, "Lie groupoids and Lie algebroids in differential Geometry" Cambridge, 1987.

[Mo]- P.MOLINO, "Riemannian Foliations", Birkhäuser 1988.

[NVQ]- NGO-VAN-QUE,"Nonabelian Spencer cohomology and deformation theory", Journal of Differential Geometry, Vol3, No2, 1969, pp165-211.

[Pra]$_1$- J.PRADINES, "Theorie de Lie pour les groupoides differentiables dans la categorie des groupoides", Calcul differential dans la categorie des groupoides infinitesimaux", C.R.A.S.Paris, t.264, 265-248 1967.

[Pra]$_2$- J.PRADINES, "Theorie de Lie pour les groupoides differerentiables", Atti Conv Intern Geoom Diff. Bologna, 1967, Bologna-Amsterdam.

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