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INVARIANCE PROPERTIES OF THE LAPLACE OPERATOR

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1. Introduction

This paper arose from some work in gauge theory on noncompact manifolds. Let (M^n,g) be open complete, G a compact Lie group with Lie algebra $\mathfrak{g}$, P(M,G) a G-principal fibre bundle, $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$, $\mathcal{C}_P$ the set of G-connections and $\mathcal{G}_P$ the gauge group. Assuming that there are already well defined Sobolev completions $\bar{\mathcal{G}}^k_P$, $\bar{\mathcal{C}}^k_P$ (which is a rather delicate problem but solved by ourselves), there arises the question about the structure of the configuration space $\bar{\mathcal{C}}^k_P / \bar{\mathcal{G}}^{k+1}_P$. One would like to obtain closed orbits as submanifolds of $\bar{\mathcal{C}}^k_P$ and a stratification of $\bar{\mathcal{C}}^k_P / \bar{\mathcal{G}}^{k+1}_P$ as in [3]. Consider the question of closed orbits as submanifolds of $\bar{\mathcal{C}}^k_P$. To do this, one has to consider how for $\omega \in \bar{\mathcal{C}}^k_P$ the differential of the map $\Phi_\omega : \bar{\mathcal{G}}^{k+1}_P \longrightarrow \bar{\mathcal{C}}^k_P$, $f \longrightarrow f^*\omega$, acts. A necessary condition for the submanifold property of the orbits is the closedness of $\operatorname{im} T_e\Phi_\omega$. There holds $T_e\Phi_\omega \eta = -\nabla^\omega \eta$, $\eta \in \Omega^0(\mathfrak{g}_P)$ (=0-forms with values in $\mathfrak{g}_P$). But on closed manifolds $\operatorname{im} \nabla^\omega$ is closed since ∇^ω has an injective symbol. On open manifolds this is far from being true. Nevertheless, also on open manifolds it can occur that $\operatorname{im} \nabla^\omega$ is closed. $\bar{\mathcal{C}}^k_P$ splits into in general uncountable many components, each of them being an affine space. To establish a reasonable theory, one would ask if $\operatorname{im} \nabla^{\omega'}$ is closed for all ω' of a component comp(ω) in $\bar{\mathcal{C}}^k_P$. This leads to questions of spectral invariance for the Laplace operator Δ^ω with respect to the connection, as we point out below. In fact, we prove that the closedness of $\operatorname{im} \nabla^\omega$ is a property of the whole component of ω in $\bar{\mathcal{C}}^k_P$. But we show more, namely the invariance of the essential spectrum which implies the result for $\operatorname{im} \nabla^\omega$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

This is the main result of the paper.
In §2 we recall some simple facts from Hilbert space theory. In §3 we clarify the topology in the connection space. Finally, §4 is devoted to the main results 4.9 - 4.12, which are of selfconsistent interest for the spectral theory of open manifolds.

2. Hilbert space preliminaries

Let X be a Hilbert space; $A:D_A \longrightarrow X$ a self-adjoint nonnegative unbounded operator. Then the spectrum $\sigma(A)$ splits into the purely discrete point spectrum $\sigma_{pd}(A)$ and the essential spectrum $\sigma_e(A)$,

$$\sigma(A) = \sigma_{pd}(A) \cup \sigma_e(A), \quad \sigma_{pd}(A) \cap \sigma_e(A) = \emptyset.$$

The essential spectrum is characterized by the existence of Weyl sequences, i.e.

$$\sigma_e(A) = \{\lambda \in \sigma(A) \mid \text{There exists a Weyl sequence for } \lambda\} .$$

A Weyl sequence for λ is a bounded non-precompact sequence $(x_i)_i$ in D_A such that

$$\lim_{i \to \infty} Ax_i - \lambda x_i = 0.$$

Without loss of generality one can assume that a Weyl sequence consists of orthonormal elements .
The Hilbert space X decomposes as an orthogonal direct sum

$$X = \overline{\text{im } A} \oplus \ker A.$$

We would like to give a spectral theoretic description for the closedness of im A. For this we need another non-disjoint decomposition of the spectrum. Set

$$\sigma_{pf}(A) = \{\lambda \in \sigma(A) \mid \lambda \text{ is an eigenvalue of finite multiplicity}\}.$$

Then $\sigma(A) = \sigma_{pf}(A) \cup \sigma_e(A)$ and $\sigma_{pf}(A) \cap \sigma_e(A)$ = set of all eigenvalues of finite multiplicity that are embedded into the essential spectrum. We reall a simple fact from [5] , p. 223.
<u>Proposition 2.1.</u> Let A be self-adjoint, $\lambda \in \mathbb{C}$. Then the following cases are possible.
a. im $(A - \lambda E) = X$. Then λ belongs to the resolvent set.

b. im $(A-\lambda E)$ is a proper subspaces of $\overline{\mathrm{im}\,(A-\lambda E)}$, $\overline{\mathrm{im}\,(A-\lambda E)} = X$. Then $\lambda \in \sigma_e(A)$, $\lambda \notin \sigma_{pf}(A)$.

c. $\mathrm{im}\,(A-\lambda E) = \overline{\mathrm{im}\,(A-\lambda E)}$, $\overline{\mathrm{im}\,(A-\lambda E)}$ is a proper subspace of finite codimension. Then $\lambda \in \sigma_{pf}(A)$, $\lambda \notin \sigma_e(A)$.

d. im $(A-\lambda E)$ is a proper subspace of $\overline{\mathrm{im}\,(A-\lambda E)}$ and $\overline{\mathrm{im}\,(A-\lambda E)}$ is a proper subspace of X of finite codimension. Then $\lambda \in \sigma_{pf}(A) \cap \sigma_e(A)$.

e. codim $\mathrm{im}(A-\lambda E) = \infty$. Then $\lambda \in \sigma_e(A)$, $\lambda \notin \sigma_{pf}(A)$.

<u>Corollary 2.2.</u> im A is closed if and only if $0 \notin \sigma_e(A|_{(\ker A)^\perp})$.

<u>Proof.</u> We denote $A|_{(\ker A)^\perp}$ by A'. Since im A = im A' the proof reduces to A' and to the Hilbert space $X' = \overline{\mathrm{im}\,A}$. We apply 2.1 to X', A', $\lambda=0$. If im A' is closed then we have to apply 2.1.a. and 0 belongs to the resolvent set, in particular $0 \notin \sigma_e(A')$. If $0 \notin \sigma_e(A')$, then only the cases 2.1.a. and c. are possible, in fact only the case a., i.e. in particular im A' is closed. □

<u>Corollary 2.3.</u> If inf $\sigma_e(A) > 0$, then im A is closed. □

3. The topology of the connection space

We assume that $\mathfrak{g}$ is endowed with a G-invariant positive definite scalar product (,). This is in particular in canonical manner possible if G is semisimple. (,) on $\mathfrak{g}$ induces a fibrewise scalar product $(\ ,\ )_x$ in $\mathfrak{g}_P$ and a global scalar product

$$\langle s,t\rangle = \int_M (s,t)_x \mathrm{dvol}_x, \quad s,t \in C_0^\infty(\mathfrak{g}_P) \equiv \Omega_0^0(\mathfrak{g}_P).$$

More general, let $\Omega^0(T_r^q \otimes \mathfrak{g}_P)$ resp. $\Omega^q(\mathfrak{g}_P)$ be the space of smooth tensor fields resp. q-forms with values in $\mathfrak{g}_P$. Then the pointwise norm $|\alpha \otimes s|_x = |\alpha|_x \cdot |s|_x$ for $\alpha \in \Omega^0(T_r^q)$ resp. $\alpha \in \Omega^q$ is well-defined, can be extended to $\varphi \in \Omega^0(T_r^q \otimes \mathfrak{g}_P)$ resp. $\varphi \in \Omega^q(\mathfrak{g}_P)$ and defines a global scalar product

$$\langle \varphi, \psi\rangle = \int (\varphi,\psi)_x \,\mathrm{dvol},$$

φ, ψ with compact support. If $\omega \in \mathcal{C}_P$ then ω induces a unique metric connection $\nabla^\omega : \Omega^0(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)$, which extends by tensoring with the Levi-Civita connection ${}^g\nabla$ of M to connections in all tensor bundles $T_r^q \otimes \mathfrak{g}_P$ and in $\Lambda^q T^* M \otimes \mathfrak{g}_P$.

Let $S = S(\omega)$ be a finite set of polynomials in ∇^ω, $(\nabla^\omega)^*$ with constant coefficients. Then we set for $1 \le p < \infty$

$${}^p\Omega^q_S(\mathfrak{g}_P,\omega) = \{\varphi \in \Omega^q(\mathfrak{g}_P) \mid {}^p\|\varphi\| < \infty \;,\; {}^p\|D\varphi\| < \infty \text{ for all } D \in S\}$$

and ${}^p\bar{\Omega}^{q,S}(\mathfrak{g}_P,\omega)$ = completion of ${}^p\Omega^q_S(\mathfrak{g}_P,\omega)$ with respect to

$${}^p\|\varphi\|_S := {}^p\|\varphi\| + \sum_{D\in S} {}^p\|D\varphi\| .$$

Here as usual ${}^p\|\varphi\| = \left(\int |\varphi|^p_x \, d\mathrm{vol}_x\right)^{1/p}$.

Cases of particular interest are

$${}^p\bar{\Omega}^{q,k,l}(\mathfrak{g}_P,\omega) = {}^p\bar{\Omega}^{q,\{\nabla^0,\ldots,\nabla^k,\nabla^*,\ldots,(\nabla^*)^l\}}(\mathfrak{g}_P,\omega),$$

${}^p\bar{\Omega}^{q,k}(\mathfrak{g}_P,\omega) \equiv {}^p\bar{\Omega}^{q,k,0}(\mathfrak{g}_P,\omega)$.
In the same manner one defines ${}^p\bar{\Omega}^{0,S}(T^q_r \otimes \mathfrak{g}_P,\omega)$.
For $\varphi \in \Omega^q(\mathfrak{g}_P)$ we set

$${}^b|\varphi|_S := \sup_{x\in M} \left(\{|\varphi|_x\} \cup \{|D\varphi|_x \mid D \in S\}\right),$$

$${}^b\Omega^q_S(\mathfrak{g}_P,\omega) = \{\varphi \in \Omega^q(\mathfrak{g}_P) \mid {}^b|\varphi|_S < \infty\}$$

and
${}^b\bar{\Omega}^{q,S}(\mathfrak{g}_P,\omega)$ = completion of ${}^b\Omega^q_S(\mathfrak{g}_P,\omega)$ with respect to ${}^b|\;|_S$. Analogously one defines ${}^b\bar{\Omega}^{0,S}(T^q_r \otimes \mathfrak{g}_P,\omega)$.
Finally we set

$${}^{b,p}U_{\varepsilon,S}(\omega) = \{\omega' \in \mathcal{C}_P \mid \omega' - \omega \in {}^b\Omega^1_S(\mathfrak{g}_P,\omega) \cap {}^p\Omega^1_S(\mathfrak{g}_P,\omega) \text{ and }$$

$${}^{p,b}\|\omega'-\omega\|_{S(\omega)} := {}^b|\omega'-\omega|_S + {}^p\|\omega'-\omega\|_S < \varepsilon\} \quad .$$

For $S(\omega) = \{\nabla^0,\ldots,\nabla^k,\nabla^*,\ldots,(\nabla^*)^m\}$ we set ${}^{p,b}\|\;\|_{S(\omega)} = {}^{p,b}\|\;\|_{k,m}$ and ${}^{p,b}U_{\varepsilon,S(\omega)} = {}^{p,b}U_{\varepsilon,k,m}$.

Proposition 3.1. The ${}^{p,b}U_{\varepsilon,k,m}(\omega)$, $\varepsilon > 0$, $\omega \in \mathcal{C}_P$, form a bas of neighbourhood filters for a locally metrizable topology on $\mathcal{C}_P$

Proof. For m=0 the (rather delicate) proof is that of propositon 7.3 of [1] . For m>0 it follows using equation (5.3) of [1] . □

Taking completions, we obtain spaces ${}^{p,b}\bar{\mathcal{C}}^{k,m}_P$. By comp($\omega$) we denote the component of $\omega \in \mathcal{C}_P$ in ${}^{p,b}\bar{\mathcal{C}}^{k,m}_P$.

Proposition 3.2. ${}^{p,b}\bar{\mathcal{C}}^{k,m}_P$ is a locally affine space, where comp(ω) is affine with ${}^{p,b}\bar{\Omega}^{1,k,m}(\mathfrak{g}_P,\omega)$ as vector space.
For the proof we refer to that of 7.5 of [1] . □
If m=0, we write simply ${}^{p,b}\bar{\mathcal{C}}^k_P$ and ${}^{p,b}U_{\varepsilon,k}$.

4. The invariance of the essential spectrum

Consider $\omega \in \mathcal{C}_P$, $\omega' \in \mathrm{comp}(\omega) \subset {}^{p,b}\overline{\mathcal{C}}_P^k$. Then $\omega' = \omega + \eta$ and η is only $\in C^k$, i.e. $\nabla^{\omega'}$ has non-smooth coefficients. But since the coefficients are $\in C^k$ the iteration $(\nabla^{\omega'})^i$, $0 \leq i \leq k+1$, is still well-defined and the same holds for the Sobolev spaces ${}^p\overline{\Omega}^{q,s}(\mathfrak{g}_P, \omega)$, $0 \leq s \leq k+1$. We wish to compare ${}^p\overline{\Omega}^{q,s}(\mathfrak{g}_P, \omega)$ and ${}^p\overline{\Omega}^{q,s}(\mathfrak{g}_P, \omega')$. For this we recall some simple equations between ∇^{ω} and $\nabla^{\omega'}$. According to §4 of [1]

$$(\nabla^{\omega'} - \nabla^{\omega})\varphi = [\omega' - \omega, \varphi] \quad , \tag{4.1}$$

$$\nabla[\varphi, \psi] \underset{\pi}{=} [\nabla\varphi, \psi] + [\psi, \nabla\varphi] \quad , \tag{4.2}$$

where $[\omega' - \omega, \varphi](s) = (\omega' - \omega)(\varphi(s)) - \varphi((\omega' - \omega)(s))$ and $\underset{\pi}{=}$ denotes the validity of the equation up to a permutation in the tensor products.

Then (4.1),(4.2), $|s \otimes t|_x = |t|_x \cdot |s|_x$ imply for the pointwise norm

$$|(\nabla^{\omega'} - \nabla^{\omega})\varphi| = |[\omega' - \omega, \varphi]| \leq C \cdot |\omega' - \omega| \cdot |\varphi| \ , \tag{4.3}$$

$$|\nabla[\xi, \zeta]| \leq |[\nabla\xi, \zeta]| + |[\xi, \nabla\zeta]| \leq$$

$$\leq C' \cdot (|\nabla\xi| \cdot |\zeta| + |\xi| \cdot |\nabla\zeta|). \tag{4.4}$$

By means of (4.1) - (4.4) we proved in [1]

<u>Proposition 4.1.</u> Assume $\omega \in \mathcal{C}_P$, $\omega' \in \mathrm{comp}(\omega) \subset {}^{p,b}\overline{\mathcal{C}}_P^k$. Then

$${}^p\overline{\Omega}^{q,s}(\mathfrak{g}_P, \omega) = {}^p\overline{\Omega}^{q,s}(\mathfrak{g}_P, \omega') \tag{4.5}$$

as equivalent Banach spaces, $0 \leq s \leq k+1$. □

We now specialize to p=2.

On compact manifolds and for a differential operator A of order r with smooth coefficients and injective symbol there holds

$${}^2\Omega^s = \mathrm{im}\, A|_{s+r} \oplus \ker A \qquad (A|_{s+r} = A|_{{}^2\Omega^{s+r}}) \tag{4.6}$$

as orthogonal direct sum of closed subspaces.

On open manifolds one has to replace the image by its closure in the corresponding space. If one is working with non-smooth coefficients there arise additional troubles. But in our case

$A = \nabla^{\omega} + \eta$, ω smooth and $\eta \in {}^{2,b}\bar{\Omega}^{1,k}(\mathcal{G}_P, \omega)$ this troubles can be overcome (cf. [4,] and [3] , p.34-36), and we conclude to

Proposition 4.2. For $\omega \in \mathcal{C}_P$, $\omega' \in \mathrm{comp}(\omega) \in {}^{2,b}\bar{\mathcal{C}}^k$

$$
{}^2\bar{\Omega}^{1,s}(\mathcal{G}_P, \omega) = {}^2\bar{\Omega}^{1,s}(\mathcal{G}_P, \omega') = \overline{\mathrm{im}\,\nabla^{\omega}|_{s+1}} \oplus \ker(\nabla^{\omega})^* =
$$

$$
= \overline{\mathrm{im}(\nabla^{\omega} + \eta)|_{s+1}} \oplus \ker(\nabla^{\omega} + \eta)^*, \ s \leq k. \ \square
$$

At this stage we do not need the assumption $k > n/2$, since by our assumption $\eta \in {}^{2,b}\bar{\Omega}^{1,k}(\mathcal{G}_P, \omega)$ η is automatically $\in C^k$ and bounded.

To relate the closedness of $\mathrm{im}\,\nabla^{\omega}$ with spectral theory, we must still introduce the Sobolev spaces associated to the powers of Δ^{ω} and relate them to the spaces ${}^2\bar{\Omega}^{q,k}(\mathcal{G}_P, \omega)$. We define

$$
{}^2\Omega^{q}_{2m}(\mathcal{G}_P, \omega) = {}^2\Omega^{q}_{\{\Delta^0, \dots, \Delta^m\}}(\mathcal{G}_P, \omega) =
$$

$$
= \{\varphi \in \Omega^q(\mathcal{G}_P) \mid {}^2\|\varphi\|'_{2m} := \sum_{i=0}^{m} {}^2\|\Delta^i \varphi\| < \infty\}
$$

and

${}^2\bar{\Omega}'^{q,2m}(\mathcal{G}_P, \omega)$ = completion of ${}^2\Omega^{q}_{2m}(\mathcal{G}_P, \omega)$ with respect to ${}^2\|\ \|'_{2m}$. In particular, $D_{\overline{\Delta^{\omega}}} = {}^2\bar{\Omega}'^{q,2}(\mathcal{G}_P, \omega)$, the closure of $\overline{\Delta^{\omega}}$.

We consider the following two conditions $(B_k(M))$ for the metric g and $(B_k(\mathcal{G}_P))$ for the connection ω.

$(B_k(M))$. The curvature of M is bounded up to order k, i.e. there exist constants C_i such that

$$
|({}^g\nabla)^i R^M| \leq C_i, \ 0 \leq i \leq k.
$$

$(B_k(\mathcal{G}_P))$. The curvature of $\mathcal{G}_P$ is bounded up to order k, i.e. there exist constants D_i, such that

$$
|(\nabla^{\omega})^i R^{\omega}| \leq D_i, \ 0 \leq i \leq k,
$$

where R^M resp. R^{ω} denotes the curvature (M^n, g) resp. of $(\mathcal{G}_P, \nabla^{\omega})$.

From [2] we use

Proposition 4.3. if (M^n, g) and $(\mathcal{G}_P, \omega)$ satisfy the conditions $(B_{2m}(M))$ and $(B_{2m}(\mathcal{G}_P))$ then

$$
{}^2\bar{\Omega}^{q,2m+2}(\mathcal{G}_P, \omega) = {}^2\bar{\Omega}'^{q,2m+2}(\mathcal{G}_P, \omega).
$$

Denote by $\mathcal{C}_{P,b,r} \subset \mathcal{C}_P$ the subset of all connections satisfying $(B_r(\mathfrak{g}_P))$ and ${}^{p,b}\overline{\mathcal{C}}^k_{P,b,r} \subset {}^{p,b}\overline{\mathcal{C}}^k_P$ the corresponding completion.

<u>Proposition 4.4.</u> If (M^n,g) satisfies $(B_r(M))$, $\omega \in \mathcal{C}_{P,b,r}$ and $r \le k$, then $\mathrm{comp}(\omega) \subset {}^{p,b}\overline{\mathcal{C}}^k_P$ belongs to ${}^{p,b}\overline{\mathcal{C}}^k_{P,b,r}$, i.e. with $\omega \in \mathcal{C}_{P,b,r}$ belongs its whole component in ${}^{p,b}\overline{\mathcal{C}}^k_P$ to ${}^{p,b}\overline{\mathcal{C}}^k_{P,b,r}$. In particular, ${}^{p,b}\overline{\mathcal{C}}^k_{P,b,r}$ is open in ${}^{p,b}\overline{\mathcal{C}}^k_P$.

<u>Proof.</u> According to 3.2 there remains to show that $\omega \in \mathcal{C}_{P,b,r}$ and $\eta \in {}^{p,b}\overline{\Omega}^{1,k}(\mathfrak{g}_P,\omega)$ implies

$$|(\nabla^{\omega+\eta})^i R^{\omega+\eta}| \le C_i, \quad 0 \le i \le r. \tag{4.7}$$

This follows immediately from

$$R^{\omega+\eta} = R^\omega + d^\omega\eta + \tfrac{1}{2}[\eta,\eta],$$

iterated covariant differentiation, the assumption $\eta \in {}^{b}\overline{\Omega}^{1,k}(\mathfrak{g}_P,\omega)$ and $(B_{r-1}(M))$. We remark that $\nabla^\omega\eta$ bounded implies $d^\omega\eta$ bounded. The assumption $(B_r(M))$ is needed since in $(\nabla^{\omega+\eta})^i$ appear the (i-1)-th partial derivatives of the Christoffel symbols which are bounded since we have $(B_r(M))$, which implies in normal coordinates the boundedness of all partial derivatives of the Γ^k_{ij} of order $\le r-1$. □

<u>Corollary 4.5.</u> If M satisfies $(B_{2m}(M))$, $\omega \in \mathcal{C}_{P,b,<m}$ and $\omega' \in \mathrm{comp}(\omega)$ then there holds for $s \le m$

$${}^{2}\overline{\Omega}^{0,2s}(\mathfrak{g}_P,\omega') = \overline{\mathrm{im}\,\Delta^{\omega'}|_{2(s+1)}} \oplus \ker\Delta^{\omega'} \quad . \tag{4.8}$$

<u>Proof.</u> This equation holds for the spaces ${}^{2}\overline{\Omega}^{0,2s}(\mathfrak{g}_P,\omega)$. For ${}^{2}\overline{\Omega}^{0,2s}(\mathfrak{g}_P,\omega')$ it follows from 4.3,4.4. □

<u>Lemma 4.6.</u> Under the assumption 4.5 $\mathrm{im}\,(\nabla^{\omega'})^*|_{2s+1}$ is closed if and only if $\mathrm{im}\,\Delta^{\omega'}|_{2(s+1)}$ is closed.

<u>Proof.</u> Clearly, $\mathrm{im}\,\Delta^{\omega'}|_{2(s+1)} \subseteq \mathrm{im}\,(\nabla^{\omega'})^*|_{2s+1}$. Moreover, $\mathrm{im}\,(\nabla^{\omega'})^*|_{2s+1}$ is orthogonal to $\ker\nabla^{\omega'}|_{2s+1}$ with respect to the L_2-scalar product, and $\ker\Delta^{\omega'} = \ker\nabla^{\omega'}$. Since by 4.5 ${}^{2}\overline{\Omega}^{0,2s}(\mathfrak{g}_P,\omega') = \overline{\mathrm{im}\,\Delta^{\omega'}|_{2(s+1)}} \oplus \ker\Delta^{\omega'}$, $\ker\Delta^{\omega'} = \ker(\Delta^{\omega'})^*$ and $\mathrm{im}(\nabla^{\omega'})^*|_{2s+1}$ orthogonal to $\ker\,(\Delta^{\omega'})^*$, we obtain

$$\mathrm{im}\,(\nabla^{\omega'})^*|_{2s+1} \subseteq \overline{\mathrm{im}\,\Delta^{\omega'}|_{2(s+1)}}.$$

Thus we obtain

$$\operatorname{im} \Delta^{\omega'}|_{2(s+1)} \subseteq \operatorname{im}(\nabla^{\omega'})^*|_{2s+1} \subseteq \overline{\operatorname{im} \Delta^{\omega'}|_{2(s+1)}}. \quad (4.9)$$

From these inclusions we conclude that the closedness of $\operatorname{im} \Delta^{\omega'}|_{2(s+1)}$ implies the closedness of $\operatorname{im}(\nabla^{\omega'})^*|_{2s+1}$. There remains to prove the inverse implication. Assume $\operatorname{im}(\nabla^{\omega'})^*|_{2s+1}$ being closed and $\operatorname{im}\Delta^{\omega'}|_{2(s+1)}$ not. According to (4.9), then there exists a $\varphi \in \operatorname{im}(\nabla^{\omega'})^*|_{2s+1}$ such that $\varphi \notin \operatorname{im} \Delta^{\omega'}|_{2(s+1)}$. Then $\varphi = (\nabla^{\omega'})^* \psi$. We can assume $\psi \in \operatorname{im}\nabla^{\omega'}|_{2(s+1)}$. Namely, if $\psi = \psi_1 + \psi_2$, $\psi_1 \in (\operatorname{im} \nabla^{\omega'}|_{2(s+1)})$, $\psi_2 \in (\operatorname{im} \nabla^{\omega'}|_{2(s+1)})^{\perp}$, then $0 = \langle \psi_2, \nabla^{\omega'} \chi \rangle$ for all $\chi \in {}^2\bar{\Omega}^{0,2(s+1)}(\mathfrak{g}_P, \omega')$, i.e. $(\nabla^{\omega'})^*\psi_2 = 0$. But $\operatorname{im}(\nabla^{\omega'})^*|_{2s+1}$ closed is equivalent to $\operatorname{im} \nabla^{\omega'}|_{2(s+1)}$ closed (cf. [6], p. 205). Therefore $\psi = \nabla^{\omega'} \chi$, $\chi \in {}^2\bar{\Omega}^{0,2(s+1)}(\mathfrak{g}_P, \omega')$, which implies $\varphi = (\nabla^{\omega'})^*\psi = (\nabla^{\omega'})^* \nabla^{\omega'} \chi = \Delta^{\omega'} \chi$, i.e. a contradiction. $\operatorname{im} \Delta^{\omega'}|_{2(s+1)}$ has to be closed. ◻

<u>Proposition 4.7.</u> Under the assumptions of 4.5 the following assertions a. and b. are equivalent.

a. $\operatorname{im} \Delta^{\omega'}|_2$ is closed.

b. $\operatorname{im} \Delta^{\omega'}|_i$ is closed, $i=2,4,\dots,2(s+1)$.

<u>Proof.</u> Clearly, b. implies a.. Assume $\operatorname{im} \Delta^{\omega'}|_2$ being closed and $\varphi \in \operatorname{im} \Delta^{\omega'}|_{2j}$, $j \le s+1$. Then there exists a sequence $\psi_\nu \in {}^2\bar{\Omega}^{0,2j}(\mathfrak{g}_P, \omega')$, $\psi_\nu \in (\ker \Delta^{\omega'})^{\perp}$ such that $\Delta \psi_\nu \longrightarrow \varphi$ in ${}^2\bar{\Omega}^{0,2(j-1)}(\mathfrak{g}_P, \omega')$. Since $\operatorname{im} \Delta^{\omega'}|_{2j} \subseteq \operatorname{im}\Delta^{\omega'}|_2 = \overline{\operatorname{im} \Delta^{\omega'}|_2}$ and $\Delta^{\omega'}$ is one to one outside $\ker \Delta^{\omega'}$ there exists a unique $\psi \in {}^2\bar{\Omega}^{0,2}(\mathfrak{g}_P, \omega')$ such that $\Delta^{\omega'} \psi = \varphi$. It follows $\Delta \psi_\nu \longrightarrow \Delta \psi$ in ${}^2\bar{\Omega}^{0,2(j-1)}$, $\Delta \psi \in {}^2\bar{\Omega}^{0,2(j-1)}(\mathfrak{g}_P, \omega')$ i.e. $\varphi \in \operatorname{im}\Delta^{\omega'}|_{2j}$. ◻

<u>Corollary 4.8.</u> Under the hypotheses 4.5 the following assertions are equivalent.

a. $\operatorname{im} \Delta^{\omega'}|_2 \equiv \operatorname{im} \overline{\Delta^{\omega'}}$ is closed.

b. $\operatorname{im} \Delta^{\omega'}|_i$ is closed, $i=2,4,\dots,2(s+1)$.

c. $\operatorname{im} (\nabla^{\omega'})^*|_i$ is closed, $i=1,3,\dots,2s+1$.

d. $\operatorname{im} \nabla^{\omega'}|_i$ is closed, $i=1,3,\dots,2s+1$.

e. $\operatorname{im} T_e \Phi_{\omega'}$ is closed in ${}^2\bar{\Omega}^{1,i}(\mathfrak{g}_P, \omega')$, $i=1,3,\dots,2s+1$. ◻

According to 2.2 $\operatorname{im} \overline{\Delta^{\omega'}}$ is closed if and only if $0 \notin \sigma_e(\Delta^{\omega'}|_{(\ker \Delta^{\omega'})^{\perp}})$.

Now we put the main question of our paper. Suppose $(B_{2m}(M))$, $\omega \in \mathcal{C}_{P,b,2m}$ and $\operatorname{im} \nabla^{\omega} \subset {}^2\bar{\Omega}^{1,2m+1}(\mathfrak{g}_P, \omega)$ being closed. Does the same hold for all $\omega' \in \operatorname{comp}(\omega)^{2,b}\overline{\mathcal{C}}^{k}_{P,b,2m}$, $2m \le k$?

Equivalent question. Suppose (M^n, g) and ω as above and

$0 \notin \sigma_e(\overline{\Delta^\omega}|(\ker \overline{\Delta^\omega})^\perp)$. Does the same hold for all $\omega' \in \mathrm{comp}(\omega)$? The affirmative answer follows from the following more general result.

Theorem 4.9. Assume $(B_0(M))$, $\omega \in \mathcal{C}_{P,b,0}$ and $\omega' \in \mathrm{comp}(\omega) \subset {}^{2,b}\overline{\mathcal{C}}^2_{P,b,0}$. Then $\sigma_e(\overline{\Delta^\omega}) = \sigma_e(\overline{\Delta^{\omega'}})$, where the closure of Δ is taken in ${}^2\overline{\Omega}^0(\mathfrak{g}_P)$.

Proof. We write $\Delta^\omega = \Delta$, $\Delta^{\omega'} = \Delta'$. $\omega' \in \mathrm{comp}(\omega)$ implies

$$\Delta = \nabla'^* \nabla \underset{\pi}{=} [\omega'-\omega, \;]^* \circ [\omega'-\omega, \;] + [\omega'-\omega, \;] \circ \nabla + \tag{4.10}$$
$$+ \nabla^* \circ [\omega'-\omega, \;] + \nabla^* \nabla = \nabla^* \nabla + \mathcal{K} ,$$

i.e. for $\varphi \in {}^2\Omega^0_{\{\Delta\}}(\mathfrak{g}_P, \omega)$

$$\Delta' \varphi = \nabla'^* \nabla' \varphi \underset{\text{II}}{=} [\omega'-\omega, [\omega'-\omega, \varphi]]_c + [\omega'-\omega, \nabla \varphi]_c + \tag{4.11}$$
$$+ \nabla^*([\omega'-\omega, \varphi]) + \Delta \varphi ,$$

where $[\omega'-\omega, \psi]_c$ means $[\omega'-\omega, \psi]$ as usual followed by contraction with the 1-form $\omega'-\omega$. On complete manifolds there holds ${}^2\Omega^0_{\{\Delta\}}(\mathfrak{g}_P, \omega) \subseteq {}^2\Omega^0_{\{\nabla\}}(\mathfrak{g}_P, \omega)$. There are constants C_1', C_2' such that for the pointwise norm

$$[\omega'-\omega, [\omega'-\omega, \varphi]]_c \le C_1' \cdot |\omega'-\omega|^2 \cdot |\varphi| , \tag{4.12}$$

$$|[\omega'-\omega, \nabla \varphi]_c| \le C_2' \cdot |\omega'-\omega| \cdot |\nabla \varphi| . \tag{4.13}$$

The estimation of $\nabla^*[\omega'-\omega, \varphi]$ amounts to the estimation of the g^{kl}, $\nabla(\omega'-\omega)$, $\omega'-\omega$, $\nabla \varphi$. Since we assumed $(B_0(M))$ the g^{kl} in normal coordinates are bounded and there exists a constant C_3' süch that

$$|\nabla^*[\omega'-\omega, \varphi]| \le C_3' \left(|\nabla(\omega'-\omega)| \cdot |\varphi| + |\omega'-\omega| \cdot |\nabla \varphi|\right) \tag{4.14}$$

where we used a version of (5.3) of [1]). Furthermore, there exists a constant C_4' such that

$$|\omega'-\omega| , |\nabla(\omega'-\omega)| \le C_4'. \tag{4.15}$$

(4.12) - (4.14) are used now to estimate $\mathcal{K}\varphi$.

Let be $\lambda \in \sigma_e(\Delta^\omega)$. We have to show $\lambda \in \sigma_e(\Delta^{\omega'})$ and proceed as follows. If $(u_i)_i$ is a Weyl sequence for λ, then we construct starting from $(u_i)_i$ a Weyl sequence $(\varphi_\nu)_\nu$ such that

$$\|\Delta^{\omega'}\varphi_\nu - \lambda\varphi_\nu\| \xrightarrow[\nu\to\infty]{} 0.$$

There exists a compact submanifold K_1^n such that

$$\|\omega' - \omega\|_{M\setminus K_1}, \quad \|(\omega' - \omega)\|_{M\setminus K_1} < 1/4^1,$$

where $\|\psi\|_{M\setminus K_1} = (\int_{M\setminus K_1} |\psi|^2 dvol)^{1/2}$. Let $(u_i)_i$ be an orthonormal Weyl sequence for $\lambda \in \sigma_e(\Delta^\omega)$,

$$\|\Delta^\omega u_i - \lambda u_i\| \xrightarrow[i\to\infty]{} 0,$$

$K_1 \subset U_1' \subset U_1$, $\bar{U}_1$ compact, $\phi = \phi_1 \in C_0^\infty(U_1)$ such that $\phi = 1$ on U_1'. Then $(\phi u_i) \in {}^2\bar{\Omega}^{0,2}(\mathcal{G}_P|_{U_1'}, \omega)$ and by the Rellich lemma $(\phi u_i)_i$ contains a convergent subsequence in ${}^2\bar{\Omega}^{0,1}(\mathcal{G}_P|_{U_1'}, \omega)$ which we again denote by $(\phi u_i)_i$. Now we consider $v_i = u_{2i+1} - u_{2i}$. Then $((1-\phi)v_i)_i$ is a Weyl sequence: $((1-\phi)v_i)_i$ is bounded, non-precompact and

$$\Delta(1-\phi)v_i - \lambda(1-\phi)v_i = \Delta v_i - \lambda v_i - \Delta(\phi v_i) - \lambda(\phi v_i) \xrightarrow[i\to\infty]{} 0$$

since $\Delta v_i - \lambda v_i \longrightarrow 0$ by assumption and $\Delta(\phi v_i) - \lambda(\phi v_i) \longrightarrow 0$ by construction of the subsequence ($(\Delta(\phi u_{2i+1})_i, \Delta(\phi u_{2i})_i$ have the same limit, $(\phi u_{2i+1})_i, (\phi u_{2i})_i$ have the same limit). $\Delta v_i - \lambda v_i \longrightarrow 0$ implies $\|\nabla v_i\| \longrightarrow \sqrt{2\lambda}$ since by completeness of (M^n, g) $\langle\Delta v_i, v_i\rangle - \lambda\langle v_i, v_i\rangle = \|\nabla v_i\|^2 - \lambda\|v_i\|^2 = \|\nabla v_i\|^2 - 2\lambda \to 0$. Moreover, we get from $\|v_i\| = \sqrt{2}$, $\|\phi v_i\|, \|\nabla(\phi v_i)\| \longrightarrow 0$, that for all sufficiently high indices

$$\frac{\|\nabla(1-\phi)v_i\|}{\|(1-\phi)v_i\|} \le \frac{\|\nabla v_i\|}{\|(1-\phi)v_i\|} + \frac{\|\nabla(\phi v_i)\|}{\|(1-\phi)v_i\|} \le \frac{(\sqrt{2}+1/20)\sqrt{\lambda}}{\|(1-\phi)v_i\|}$$
$$+ \frac{\|(\phi v_i)\|}{\|(1-\phi)v_i\|} \le \sqrt{\lambda}(\sqrt{2}+1/20) + \sqrt{\lambda}/20 = (\sqrt{2}+1/10)\sqrt{\lambda} \tag{4.16}$$

if $\lambda > 0$ and the left hand side of (4.16) $\le 1/10$ if $\lambda = 0$. We start with the first index such that for all higher indices (4.16) is valid and denote $(u_i^{(1)})_i = ((1-\phi)v_i/\|(1-\phi)v_i\|)_i$, in particular $\|u_i^{(1)}\| = 1$ and $\|\nabla u_i^{(1)}\| \le \sqrt{\lambda}(\sqrt{2}+1/10)$ resp. $\le 1/10$. Then we obtain from (4.12) - (4.16) using Schwarz inequality

$$\begin{aligned}(\int |[\omega'-\omega,[\omega'-\omega,u_i^{(1)}]]_c|^2 dvol)^{1/2} \le \\ \le C_1 (\int_{M-K_1} |\omega'-\omega|^2 dvol)^{1/2} \le C_1/4^1,\end{aligned} \tag{4.17}$$

$$\begin{aligned}(\int |[\omega'-\omega,\nabla u_i^{(1)}]_c|^2 dvol)^{1/2} \le C_2 (\int_{M\setminus K_1} |\omega'-\omega|^2 dvol)^{1/2} \cdot \\ \cdot (\int_{M\setminus K_1} |\nabla u_i^{(1)}|^2 dvol)^{1/2} \le C_2 \cdot \frac{1}{4^1}\sqrt{\lambda}(\sqrt{2}+1/10) \text{ resp. } \le \frac{C_2'}{4^1}\cdot\frac{1}{10}.\end{aligned} \tag{4.18}$$

$$\begin{aligned}(\int |\nabla^*[\omega'-\omega,u_i^{(1)}]|^2 dvol)^{1/2} \le \frac{C_3}{4^1}(1+\sqrt{\lambda}(\sqrt{2}+1/10)), \\ \text{resp. } \le \frac{C_3}{4^1}\cdot\frac{11}{10}\end{aligned} \tag{4.19}$$

(4.17) - (4.19) imply

$$\|\Delta' u_i^{(1)} - \lambda u_i^{(1)}\| \le \|\Delta u_i^{(1)} - \lambda u_i^{(1)}\| + \|\mathcal{K} u_i^{(1)}\| \tag{4.20}$$

with

$$\|\Delta u_i^{(1)} - \lambda u_i^{(1)}\| \xrightarrow[i\to\infty]{} 0 \tag{4.21}$$

and

$$\|\mathcal{K} u_i^{(1)}\| \le \frac{C}{4^1}. \tag{4.22}$$

There exists a compact submanifold $K_2 \supset K_1$ such that $\|\omega'-\omega\|_{M\setminus K_2}$, $\|\nabla(\omega'-\omega)\|_{M\setminus K_2} < 1/4^2$. Let $K_2 \subset U_2' \subset U_2$, $\overline{U}_2$ compact and $\phi = \phi_2$ such that $\phi \in C_0^\infty(U_2)$, $\phi = 1$ on U_1'. Now proceed with $\phi = \phi_2$, K_2, $(u_i^{(1)})_i$ as with $\phi_1, K_1, (u_i)_i$, getting by this procedure a sequence $(u_i^{(2)})_i$, $\|u_i^{(2)}\| = 1$, non-precompact and satisfying

$$\|\Delta u_i^{(2)} - \lambda u_i^{(2)}\|_i \xrightarrow[i\to\infty]{} 0 \tag{4.23}$$

and

$$\|\mathcal{K} u_i^{(2)}\| \le \frac{C}{4^2}. \tag{4.24}$$

Iterating this procedure, we obtain for each j a sequence $(u_i^{(j)})_i$, $\|u_i^{(j)}\| = 1$, non -precompact and satisfying

$$\|\Delta u_i^{(j)} - \lambda u_i^{(j)}\|_{i\to\infty} \longrightarrow 0 \tag{4.25}$$

and
$$\|\mathcal{K} u_i^{(j)}\| < \frac{c}{4^j} . \tag{4.26}$$

Set $\varepsilon_j = \frac{1}{j}$. Then there exists an i_j such that

$$\|\Delta u_i^{(j)} - \lambda u_i^{(j)}\| < \frac{1}{j}$$

for all $i \geq i_j$. Finally we set $\varphi_j = u_{i_j}^{(j)}$. Then $\|\varphi_j\| = 1$, $(\varphi_j)_j$ is non-precompact (since $(u_i)_i$ is non-precompact) and

$$\|\Delta^{\omega'} \varphi_j - \lambda \varphi_j\| \leq \|\Delta^{\omega} \varphi_j - \lambda \varphi_j\| + \frac{c}{4^j} \xrightarrow[j \to \infty]{} 0,$$

i.e. $\lambda \in \sigma_e(\Delta^{\omega'})$, $\sigma_e(\Delta^{\omega}) \subseteq \sigma_e(\Delta^{\omega'})$. Exchanging ω', ω and repeating verbatim gives $\sigma_e(\Delta^{\omega'}) \subseteq \sigma_e(\Delta^{\omega})$. □

<u>Corollary 4.10.</u> Assume $(B_0(M))$, $\mathrm{vol}(M) = \infty$, $\omega \in \mathcal{C}_{P,b,0}$ and $\omega' \in \mathrm{comp}(\omega) \in {}^{2,b}\overline{\mathcal{C}}^{2}_{P,b,0}$. Then

$$\sigma_e(\overline{\Delta^{\omega}}|_{(\ker \Delta^{\omega})^{\perp}}) = \sigma_e(\overline{\Delta^{\omega'}}|_{(\ker \Delta^{\omega'})^{\perp}}).$$

In particular, im Δ^{ω} is closed if and only if im $\Delta^{\omega'}$ is closed.

<u>Proof.</u> $\mathrm{vol}(M) = \infty$ implies $\ker \Delta^{\omega} = \ker \Delta^{\omega'} = \{0\}$ ($\Delta = \nabla^* \nabla$ implies that L_2-harmonic sections of $\Omega(\mathfrak{g}_P)$ are parallel, i.e. $= 0$, since $\mathrm{vol}(M) = \infty$). □

<u>Remark.</u> If we replace the assumption $\mathrm{vol}(M) = \infty$ by $r_{inj}(M) > 0$, then 4.10 remains still valid since $(B_0(M))$ and $r_{inj}(M) > 0$ imply $\mathrm{vol}(M) = \infty$.

<u>Corollary 4.11.</u> Assume (M^n, g) being open, complete, satisfying $(B_{2m}(M))$, $\mathrm{vol}(M) = \infty$, $\omega \in {}^{2,b}\mathcal{C}_{P,b,2m}$ and $\omega' \in \mathrm{comp}(\omega) \subset {}^{2,b}\overline{\mathcal{C}}^{2m}_{P,b,2m}$. If $\mathrm{im}(\nabla^{\omega} : {}^{2}\overline{\Omega}^{0,2s+2}(\mathfrak{g}_P, \omega) \longrightarrow {}^{2}\overline{\Omega}^{1,2s+1}(\mathfrak{g}_P, \omega))$ is closed, $0 \leq s \leq m$, then the same holds for ω'. □

<u>Corollary 4.12.</u> Assume (M^n, g), ω, ω' as in 4.11. If im $T_e \Phi_{\omega}$ is closed in ${}^{2}\overline{\Omega}^{1,2s+1}(\mathfrak{g}_P, \omega)$ then the same holds for $T_e \Phi_{\omega'}$. □

This has good and far reaching consequences for the study of the configuration space in gauge theory on open manifolds.

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