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MANIFOLDS OF MAPPINGS BETWEEN OPEN MANIFOLDS

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1. Introduction

More then twenty years it is well known that the space of all smooth mappings $f:M^n \to N^r$ between closed smooth manifolds of finite dimensions forms after suitable completion a Hilbert resp. Banach manifold $H^s(M,N)$ resp. $C^k(M,N)$ if one completes with respect to a certain Sobolev space resp. Banach space norm. The smooth structure of $H^s(M,N)$ or $C^k(M,N)$, respectively, is constructed by use of Riemannian metrics on M and N but it does not depend on the metrics. The reason for this is that on a closed manifold all Riemannian metrics are quasi-isometric and the Sobolev spaces of order s are equivalent. The latter follows from the fact that for two elliptic operators P,P' of order m on M there exist constants C_1, C_2 such that

$$C_1(\|P'f\| + \|f\|) \leq \|Pf\| + \|f\| \leq C_2(\|P'f\| + \|f\|),$$

$f \in C^\infty(M)$, $\| \; \| = L_2$ norm. The latter is a consequence of the existence of a parametrix for an elliptic operator on a closed manifold. The same holds for the Banach space norm = supremum norm of covariant derivatives. A further essential step in defining $H^s(M,N)$ is the Sobolev embedding theorem $H^s(M) \hookrightarrow C^k(M)$ continuously for $s > n/2 + k$. All this breaks completely down for noncompact manifolds. Arbitrary Riemannian metrics on an open manifold are far from beiing quasi-isometric, there do not exist parametrices, different metrics and connections (in vector bundles) give in general non-equivalent Sobolev spaces, these spaces depend on the differential operators generating them, and versions of the Sobolev embedding theorem in general do not hold. Therefore, one has to work on open manifolds much more carefully. On the other hand, as in the

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case of closed manifolds, mapping spaces between open manifolds are very important in mathematical physics on open manifolds, for example in gauge theory for the description of the configuration space $\mathcal{C}^k_P/\mathcal{G}^{k+1}_P$, and for the study of Diff(M) in general relativity.

As a conclusion, one has to find out those cases in which one can work and establish a reasonable theory. This paper is devoted to this problem. More concretely, studying the gauge group $\mathcal{G}^{k+1}_P$ acting on a principal fibre bundle $P(M,G) \longrightarrow M$, M closed and G compact, and establishing that $\mathcal{G}^{k+1}_P$ is a Hilbert Lie group, one needs the fact that the space $H^{k+1}(\tilde{P})$ of H^{k+1}-sections of $\tilde{P} = \mathcal{C}^\infty(P \times_{Ad} G) \cong \mathcal{G}_P$ is a Hilbert manifold (cf. [8]). The same fact is needed for $\mathrm{Diff}^s(P)$. Since our final aim is a description of $\mathcal{C}^s_P$, $\mathcal{G}^{s+1}_P$ and a stratification for $\mathcal{C}^s_P/\mathcal{G}^{s+1}_P$ for M^n open, and we succeeded considerably in this direction, we have to establish the needed properties for some version of $H^s(\tilde{P})$, $\mathrm{Diff}^s(P)$ on open manifolds. Moreover, the study of the mapping spaces between open manifolds is of its own interest. Until now, no approach to this question is known to the author.

The paper is organized as follows. In section 2 we recall Sobolev spaces and those facts from their theory on open manifolds which are needed later on. Section 3 is devoted to manifolds of C^s-bounded geometry. In section 4 we show that for complete manifolds (M^n,g), (N^r,h) with C^s-bounded geometry of sufficiently high order and sufficiently high bounded mappings we get in fact Banach or Hilbert manifolds ${}^b\Omega^s(M,N)$, ${}^p\Omega^s(M,N)$, ${}^2\Omega^s(M,N)$, respectively. Section 5 is devoted to the diffeomorphism group which is in general not an open subspace, but the restriction to diffeormorphisms whose differential is bounded from below and above, has this property. Counterexamples show that the restriction to $\inf_x |df|_x > 0$ is in fact necessary. The bounded diffeomorphism group will be the main subject of forthcoming papers.

2. Sobolev spaces

Assume (M^n,g) being complete, $(E,h) \longrightarrow M$ a Riemannian vector bundle over M with metric connection ${}^h\nabla = \nabla^E$. Then the Levi-Civita connection ${}^g\nabla$ and the connection ${}^h\nabla$ define metric connections ∇ in all tensor bundles $T^q_r \otimes E$ and, in particular, in

$\Lambda^q T^* M \otimes E$ where $\Lambda^q T^* \otimes E \subset T^q_0 \otimes E$. By $\Omega^q(E)$ or $\Omega^0(T^q_r \otimes E)$ we denote the space of smooth q-forms or tensor fields of type (q,r) with values in E, respectively. $\Omega^q_0(E)$ or $\Omega^0_0(T^q_r \otimes E)$ shall denote the subspace of forms or tensors with compact support. Then we define for $p \in \mathbb{R}$, $1 \le p < \infty$ and k a nonnegative integer

$$^p\Omega^q_k(E) = \{ \varphi \in \Omega^q(E) \mid {}^p\| \varphi \|_k := \sum_{i=0}^{k} (\int | \nabla^i \varphi |^p dvol)^{1/p} < \infty \}$$

and

$^p\bar{\Omega}^{q,k}(E)$ = completion of $^p\Omega^q_k(E)$ with respect to $^p\| \ \|_k$,
$^p\mathring{\Omega}^{q,k}(E)$ = completion of $\Omega^q_0(E)$ with respect to $^p\| \ \|_k$ and
$^p\Omega^{q,k}(E) = \{ \varphi \mid \varphi$ measurable regular distribution with $^p\| \varphi \|_k < \infty \}$.
In the same manner we define $^p\Omega^0_k(T^q_r \otimes E)$, $^p\bar{\Omega}^{0,k}(T^q_r \otimes E)$, $^p\mathring{\Omega}^{0,k}(T^q_r \otimes E)$, $^p\Omega^{0,k}(T^q_r \otimes E)$. Clearly, $^p\bar{\Omega}^{q,k}(E) \subset {}^p\bar{\Omega}^{0,k}(T^q_0 \otimes E)$ as a closed subspace with the induced norm, the same holds for the other cases, but we treat $\Omega^q(E)$ and $\Omega^0(T^q_r \otimes E)$ separately since during working with the Laplace operator as derivatives we have to do with two different operators, $\Delta = \nabla^* \nabla + \rho$ on $\Omega^q(E)$, $\Delta = \nabla^* \nabla$ on $\Omega^0(T^q_r \otimes E)$. Furthermore, we define

$$^b\Omega^{q,k}(E) = \{ \varphi \mid \varphi \ C^k\text{-form and } {}^b|\varphi|_k = \sup_{\substack{x \in M \\ 0 \le i \le k}} |\nabla^i \varphi|_x < \infty \}$$

and
$^b\mathring{\Omega}^{q,k}(E)$ = completion of $\Omega^q_0(E)$ with respect to $^b| \ |_k$.
In the same manner we define $^b\Omega^{0,k}(T^q_r \otimes E)$, $^b\mathring{\Omega}^{0,k}(T^q_r \otimes E)$.
<u>Proposition 2.1.</u> All defined spaces $^p\mathring{\Omega}^{q,k}(E)$, $^p\bar{\Omega}^{q,k}(E), \dots$, $^b\mathring{\Omega}^{0,k}(T^q_r \otimes E)$, $^b\Omega^{0,k}(T^q_r \otimes E)$ are Banach spaces and there are inclusions

$$^p\mathring{\Omega}^{q,k}(E) \subseteq {}^p\bar{\Omega}^{q,k}(E) \subseteq {}^p\Omega^{q,k}(E),$$

$$^p\mathring{\Omega}^{0,k}(T^q_r \otimes E) \subseteq {}^p\bar{\Omega}^{0,k}(T^q_r \otimes E) \subseteq {}^p\Omega^{0,k}(T^q_r \otimes E),$$

$$^b\mathring{\Omega}^{q,k}(E) \subsetneqq {}^b\Omega^{q,k}(E), \quad {}^b\mathring{\Omega}^{0,k}(T^q_r \otimes E) \subsetneqq {}^b\Omega^{0,k}(T^q_r \otimes E).$$

For the proof we refer to [3] . □
If p=2 then $^2\mathring{\Omega}^{q,k}(E), \dots, {}^2\Omega^{0,k}(T^q_r \otimes E)$ are Hilbert spaces with respect to the usual scalar product $\langle \varphi, \psi \rangle_k = \sum_{i=0}^{k} \langle \nabla^i \varphi, \nabla^i \psi \rangle = \sum_{i=0}^{k} \int | \nabla^i \varphi |^2 dvol$.

In the case k=2m we have a second canonical variant of Sobolev spaces replacing $\nabla^0,\dots,\nabla^{2m}$ by $(1+\Delta)^m$, where $\Delta = d\delta + \delta d = \nabla^*\nabla + \rho$ or $\Delta = \nabla^*\nabla$ is the Laplace operator acting in $\Omega^q(E)$ or $\Omega^0(T^q_r \otimes E)$, respectively, i.e. we replace ${}^p\|\cdot\|_k$ by ${}^p\|\cdot\|'_k = {}^p\|(1+\Delta)^m\cdot\|$. Then we get corresponding spaces ${}^p\mathring{\Omega}'^{,q,k}(E)$, ${}^p\bar{\Omega}'^{,q,k}(E)$, ${}^p\mathring{\Omega}'^{,0,k}(T^q_r \otimes E),\dots$. There arise several natural questions concerning the coincidence between ${}^p\mathring{\Omega}^{q,k}(E)$, ${}^p\bar{\Omega}^{q,k}(E)$, ${}^p\Omega^{q,k}(E)$, the coincidence between ${}^p\mathring{\Omega}'^{,q,k}(E)$, ${}^p\bar{\Omega}'^{,q,k}(E)$, ${}^p\Omega'^{,q,k}(E)$, the coincidence between ${}^p\mathring{\Omega}^{q,k}(E)$ and ${}^p\mathring{\Omega}'^{,q,k}(E)$, ${}^p\bar{\Omega}^{q,k}(E)$ and ${}^p\bar{\Omega}'^{,q,k}(E)$, ${}^p\Omega^{q,k}(E)$ and ${}^p\Omega'^{,q,k}(E)$. Finally, one has to put the same questions for the corresponding subspaces of ${}^p\Omega^0(T^q_r \otimes E) = {}^p\Omega^{0,0}(T^q_r \otimes E)$.

To clear up the situation we consider the following conditions (I), $(B_m(M))$, $(B_m(E))$, $(B_m(M,E))$.

(I). The injectivity radius of M has a positive lower bound,

$$r_{inj}(M) = \sup_{x\in M} r_{inj}(x) = a > 0.$$

$(B_m(M))$. $$\sup_{\substack{x\in M\\ 0\le i\le m}} |\nabla^i R^M|_x < \infty .$$

$(B_m(E))$. $$\sup_{\substack{x\in M\\ 0=i=m}} |\nabla^i R^E|_x < \infty .$$

$(B_m(M,E))$. $$\sup_{\substack{x\in M\\ 0\le i\le m}} |\nabla^i \rho|_x < \infty .$$

Here R^M resp. R^E resp. ρ denotes the curvature tensor of M resp. of E resp. the curvature endomorphism in the Weitzenboeck formula $\Delta = \nabla^*\nabla + \rho$. By definition of ρ imply two of the (B_m)-conditions the third.

We say that M resp. E resp. M and E have bounded geometry up to order m if M satisfies (I) and M satisfies $(B_m(M))$ resp. E satisfies $(B_m(E))$ resp. M and E satisfy $(B_m(M,E))$.

Theorem 2.2. Suppose (M^n,g) being open, complete and of bounded geometry up to order m. Then there holds

$$ {}^p\mathring{\Omega}^{q,k}(E) = {}^p\bar{\Omega}^{q,k}(E) = {}^p\Omega^{q,k}(E), \tag{2.1}$$

$$ {}^p\mathring{\Omega}^{0,k}(T^q_r \otimes E) = {}^p\bar{\Omega}^{0,k}(T^q_r \otimes E) = {}^p\Omega^{0,k}(T^q_r \otimes E) \tag{2.2}$$

for $0\le k\le m+2$.

A complete proof is contained in [3], [4] . Therefore we here indicate only the general line. From (I) and $(B_m(M))$ one gets the existence of exhaustion functions $h_j \in C_0^{m+2}(M)$, $h_j \xrightarrow[j\to\infty]{} 1$, $|\nabla^i h_j|_x = C$ for all x and $0 \le i \le m+2$. Then one approximates $\varphi \in {}^p\Omega^{q,k}(E)$ or $\varphi \in {}^p\Omega^{0,k}(T^q_r \otimes E)$ by $\varphi_j = h_j \cdot \varphi$. The φ_j are not necessarily smooth and have to be regularized, $\tilde{\varphi}_{ij} \xrightarrow[i\to\infty]{} \varphi_j$. Diagonal choice gives $\tilde{\varphi}_{j,j} \to \varphi$ which proves (2.1),(2.2).

For p=2 we have the same result for the Ω'-spaces which are built up by means of the powers of the Laplace operator, but without the assumption of bounded geometry.

Theorem 2.3. Suppose (M^n,g) being open, complete, k=2m. Then

$$ {}^2\mathring{\Omega}'^{q,k}(E) = {}^2\bar{\Omega}'^{q,k}(E) = {}^2\Omega'^{q,k}(E) \tag{2.3} $$

and

$$ {}^2\mathring{\Omega}'^{0,k}(T^q_r \otimes E) = {}^2\bar{\Omega}'^{0,k}(T^q_r \otimes E) = {}^2\Omega'^{0,k}(T^q_r \otimes E). \tag{2.4} $$

Proof. (2.3) is just the essential self-adjointness of $(1+\Delta)^m \big|_{\Omega_0^q(E)}$ in ${}^2\Omega^q(E)$ for Δ acting on q-forms. But (2.3) implies (2.4) setting q=0 and replacing the vector bundle E by $T^q_r \otimes E$. □

A connection between the Ω-spaces and the Ω'-spaces is established until now only for the case p=2.

Theorem 2.4. Suppose (M^n,g) being open, complete, (M^n,g) and $(E,h) \longrightarrow M$ satisfying $(B_{2m}(M))$ and $(B_{2m}(E))$. Then with $\tilde{\Omega} = \mathring{\Omega}, \bar{\Omega}, \Omega$

$$ {}^2\tilde{\Omega}^{q,2m+2}(E) = {}^2\tilde{\Omega}'^{q,2m+2}(E) \tag{2.5} $$

and

$$ {}^2\tilde{\Omega}^{0,2m+2}(T^q_r \otimes E) = {}^2\tilde{\Omega}'^{0,2m+2}(T^q_r \otimes E) \tag{2.6} $$

with equivalent norms.

The proof of (2.5) is contained in [3] and the proof of (2.6) is still easier since in this case one has to work with $\Delta = \nabla^* \nabla$ instead of $\Delta = \nabla^* \nabla + \rho$. □

Corollary 2.5. Suppose the hypotheses of 2.4. Then

$$ \begin{aligned} {}^2\mathring{\Omega}^{q,2m+2}(E) &= {}^2\bar{\Omega}^{q,2m+2}(E) = {}^2\Omega^{q,2m+2}(E) = \\ = {}^2\mathring{\Omega}'^{q,2m+2}(E) &= {}^2\bar{\Omega}'^{q,2m+2}(E) = {}^2\Omega'^{2m+2}(E) \end{aligned} \tag{2.7} $$

and the analogous assertion holds for tensor fields with values in E

Remark. Corollary 2.5 shows that for p=2 the assertions of theorem 2.2 are valid without assuming the condition (I).

As in the case of compact manifolds, the Sobolev embedding theorems play a fundamental role for the definition of a manifold structure on mapping spaces between open manifolds.

Theorem 2.6. Assume (M^n,g) being open, complete and of bounded geometry up to order 0, i.e. satisfying (I) and $(B_0(M))$. If $s > \frac{n}{p} + k$, then there are continuous embeddings

$$^p\mathring{\Omega}^{q,s}(E) \hookrightarrow {}^b\mathring{\Omega}^{q,k}(E), \quad ^p\mathring{\Omega}^{0,s}(T^q_r \otimes E) \hookrightarrow {}^b\mathring{\Omega}^{0,k}(T^q_r \otimes E), \tag{2.8}$$

$$^p\bar{\Omega}^{q,s}(E) \hookrightarrow {}^b\Omega^{q,k}(E), \quad ^p\bar{\Omega}^{0,s}(T^q_r \otimes E) \hookrightarrow {}^b\Omega^{0,k}(T^q_r \otimes E). \tag{2.9}$$

(2.8) was already proved in [2] , and the proof carries over to that of (2.9) which is indicated in [3]. □

Corollary 2.7. If (M^n,g) additionally satisfies $(B_{s-2}(M))$, then there are continuous embeddings

$$^p\Omega^{q,s}(E) \hookrightarrow {}^b\Omega^{q,k}(E), \quad ^p\Omega^{0,s}(T^q_r \otimes E) \hookrightarrow {}^b\Omega^{0,k}(T^q_r \otimes E). \tag{2.10}$$

Corollary 2.8. Suppose the hypotheses of 2.7 with s even, $(B_{s-2}(E))$ and p=2. Then there are continuous embeddings

$$^2\Omega'^{q,s}(E) \hookrightarrow {}^b\Omega^{q,k}(E), \quad ^2\Omega'^{0,s}(T^q_r \otimes E) \hookrightarrow {}^b\Omega^{0,k}(T^q_r \otimes E). \ \square \tag{2.11}$$

3. C^s-bounded geometry and bounded mappings

The main purpose of this section consists in an explanation of C^s-bounded geometry and of implications if this kind of geometry is assumed to be given. The notion of C^s-bounded geometry can always be defined for classes of coordinates which map a neighborhood $U(x_0)$ onto an open set $\subset T_{x_0}M$. Although we in the next section work with exponential coordinates, we start as an example with almost linear coordinates. At first we recall some facts on almost linear coordinates. Let $B_\varrho(x_0) \subset M^n$ be a ball which is disjoint to the cut locus of x_0 and suppose curvature bounds $-K_1^2 \le K \le K_2^2$, $|K| \le C$ for the sectional curvature K and $\varrho < \pi/2K_2$. This implies the geodesic convexity of $B_\varrho(x_0)$. If $u \in T_{x_0}M$ is a unit vector then we extend u to a vector field u(x) on B_ϱ by radial parallel translation. We set $r(x) := d(x_0,x)$, $p(x) := \exp_{x_0} r(x)u$, $q(x) :=$ $= \exp_{x_0} -r(x)u$ and

$$\lambda(x) := (d(x,q(x))^2 - d(x,p(x))^2)/4r^2.$$

From [6] we recall

Proposition 3.1. There holds

$$|\nabla^2\lambda|_x \doteq (8\,\kappa_2 \frac{\sinh(2Cr)}{\sin(2\kappa_2 r)}\,\kappa_1 r\cdot \mathrm{ctgh}(\kappa_1 r))r(x). \qquad (3.1)$$

This class of functions can be used to define the so called almost linear coordinates. Choose $u_1,\dots,u_n \in T_{x_0}M$ orthonormal and define by means of the corresponding almost linear functions $\lambda^1,\dots,\lambda^n$ the map

$$L: B_\varrho(x_0) \longrightarrow T_{x_0}M,\ L(x) := \sum_{i=1}^{n} \lambda^i(x)u_i(x_0). \qquad (3.2)$$

From 3.1 we immediately obtain

$$|\nabla^2 L|_x \leq 8\sqrt{n}\,\kappa_2 \frac{\sinh(2Cr)}{\sin(2\kappa_2 r)}\,\kappa_1 r\cdot \mathrm{ctgh}(\kappa_1 r)\cdot r(x). \qquad (3.3)$$

It is a simple matter of fact that the Christoffel symbols are given by the second covariant derivative of the coordinate functions. Then (3.1)-(3.3) imply

Proposition 3.2. In almost linear coordinates there holds for the Christoffel symbols Γ_{ij}^k

$$|\Gamma_{ij}^k(x)|_x \leq \text{const.}(\text{curvature})\cdot d(x_0,x). \qquad (3.4)$$

From [7] we cite

Proposition 3.3. In normal coordinates there holds for the Christoffel symbols

$$|\Gamma_{ij}^k(x)|_x \leq \text{const.}(\text{curvature},\nabla(\text{curvature})). \qquad (3.5)$$

In this sense almost linear coordinates are "better" than normal coordinates. On the other hand, normal coordinates are more geometrical and quite natural.

Assume that $(M^n,g),(N^r,h)$ are open, complete manifolds of bounded geometry up to order 0. This implies, in particular, the existence of numbers δ_M, δ_N, $0 < \delta_M < r_{inj}(M)$, $0 < \delta_N < r_{inj}(N)$, and uniformly locally finite coverings $\mathfrak{U}_M = \{U_{\delta_M}(x_i)\}_i$, $\mathfrak{U}_N = \{U'_{\delta_N}(y_j)\}_j$ by almost linear coordinate neighborhoods $(U_{\delta_M}(x_i),x^1,\dots,x^n)$, $U'_{\delta_N}(y_j),y^1,\dots,y^r)$ such that the Christoffel symbols ${}^g\Gamma_{ab}^c$ in

$U_{\delta_M}(x_i)$ and the ${}^h\Gamma^m_{ik}$ in $U'_{\delta_N}(y_j)$ are bounded. The same is valid for normal coordinates if we assume bounded geometry up to order 1. Consider now $f \in C^\infty(M,N)$. f induces the connection $f^{*\,h}\nabla$ in the induced bundle f^*TN which is locally given by

$$\Gamma^i_{aj} = {}^{f^*h\nabla}\Gamma^i_{aj} = \partial_a f^k(x)\,{}^h\Gamma^i_{kj}(f(x)). \qquad (3.6)$$

A coordinate free description is given by

$$(f^{*h}\nabla)_X(Y_{f(x)},x) = ({}^h\nabla_{f_*X}Y_{f(x)},x). \qquad (3.7)$$

Next we consider the condition

$$\frac{\partial^\alpha}{\partial x^\alpha} f^k \quad \text{bounded for } |\alpha| \le m+1, \qquad (3.8)$$

which makes sense if we refer to the coverings $\mathfrak{U}_M$, $\mathfrak{U}_N$ of almost linear or normal coordinates. Then, assuming (3.4) resp. (3.5) in almost linear resp. normal **coordinates** for ${}^h\nabla$, (3.8) for m=0 implies the boundedness of the ${}^{f^*h\nabla}\Gamma^k_{aj}(x)$. A coordinate free description of (3.8) for m 1 **can be assured if we have** the boundedness of the partial derivatives of the Christoffel symbols. Using the Jacobi field techniques of [6], [7], we can in fact show that bounded geometry up to order m implies the boundedness of the partial derivatives of the Christoffel symbols up to order m in almost linear coordinates. The same holds for normal coordinates if we assume bounded geometry up to order m+1. For the reasons of place we can't present here the proofand refer to the forthcoming paper [1] . Therefore we have to work here with another notion of bounded geometry which immediately implies the boundedness of the partial derivatives of the Christoffel symbols. We say (M^n,g) has C^k-bounded geometry with respect to a class of coordinates $\Phi: U(x_0) \longrightarrow T_{x_0}M$, provided it satisfies the conditions (I) and

(C^k). There exists a radius δ_M, $0 < \delta_M < r_{i_1}(M)$, for the chosen class of coordinates such that with respect to this class for every $x_0 \in M$ the metric tensor g_{ij} on $B_{\delta_M}(x_0)$ pulled back to $B_{\delta_M}(0) \subset T_{x_0}M$ is bounded in the C^k-topology, in particular the matrices (g_{ij}), (g^{ij}) are bounded in the sup norm. We include the case $k=\infty$ with homogeneous Riemannian spaces as a class of examples. Then it is clear that (I) and (C^k) imply (B_{k-2}). On the other hand, (I) and (B_0) are for almost linear coordinates really stron-

ger than (I) and (C^0), they in fact imply (I) and (C^1) (cf. [6]). If we work with normal coordinates $\exp_{x_0} : B_\delta(0) \longrightarrow B_\delta(x_0)$, this is not true but (I) and (B_1) imply (I) and (C^1). For $k = \infty$, assuming the hypothesis (I), the conditions (B_∞) and (C^∞) are equivalent, both for almost linear and for normal coordinates. To guarantee the boundedness of $\frac{\partial^\alpha}{\partial x^\alpha} \Gamma^k_{ij}$ we assume in the sequel C^k-bounded geometry of sufficiently high order.
For $f \in C^\infty(M,N)$ induce ${}^g\nabla$ and $f^{*h}\nabla$ connections ∇ in all tensor bundles $T^r_s(M) \otimes f^* T^u_v(N)$. The differential $df = f_*$ can be considered as a section of $T^*M \otimes f^* TN$. Therefore is $\nabla^m df$ well defined.
If (M^n,g), (N^r,h) have bounded geometry up to order 0 then the condition df bounded, in local coordinates

$$|df|^2 = {}^g\mathrm{tr}(f^*h) = g^{ab} h_{ij} \partial_a f^i \partial_b f^j \qquad (3.9)$$

bounded, and $\partial_a f^i$ bounded are equivalent (since (B_0),(I) imply the boundedness of the g_{ab}, h_{ij},(cf. [5]). But (3.9) can be understood as a coordinate free description.
The assertions 3.4 - 3.6 are valid as well for almost linear as for exponential coordinates.

<u>Proposition 3.4.</u> Assume $(M^n,g),(N^r,h)$ being open, complete, of C^{m+2}-bounded geometry, $f \in C^\infty(M,N)$, ${}^b|df| < \infty$. Then the following conditions are equivalent.

a. All $\frac{\partial^\alpha}{\partial x^\alpha} f^k$ are bounded, $|\alpha| \le m+1$, $k=1,\dots,r$, (3.10)

where the derivatives are taken with respect to some uniformly locally finite atlas for M resp. N.

b. All $\frac{\partial^\alpha}{\partial x^\alpha} \Gamma^i_{aj}(x)$ are bounded, $|\alpha| \le m$, $a=1,\dots,n$, $i,j=1,\dots,r$. (3.11)

c. All $\nabla^\mu df$ are bounded, $0 \le \mu \le m$. (3.12)

<u>Proof.</u> a. implies b.: We perform induction. For m=0 the assertion coincides with (3.6). If we assume the assertion for $\mu - 1 \le m$ then it follows for μ by the assumption a. and the Leibniz rule.
b. implies c.: ∇ here denotes the product connection of ${}^g\nabla$ and $f^{*h}\nabla$. Then in local coordinates

$$\nabla_a df = \nabla_a \partial_b f^k = \partial_a \partial_b f^k - {}^g\Gamma^c_{ab} \partial_c f^k + {}^h\Gamma^k_{ij} \partial_a f^i \partial_b f^j . \qquad (.13)$$

For $\mu = 0$ the assertion is just the assumption ${}^b|df| < \infty$. For $\mu = 1$ the assertion follows from the equation (3.13) and the assumptions. If we assume the assertion for $\mu-1 \leq m$ then the validity for μ follows from this, the $\mu-1$ times derived equation (3.13) and the boundedness of the Christoffel symbols of ${}^g\nabla$, $f^{*h}\nabla$ and their partial derivatives (for $f^{*h}\nabla$ this is just the assumption b.). c implies a.: For m=0 there is nothing to show. For m=1 (3.13), the boundedness of the ${}^g\Gamma^c_{ab}$, ${}^h\Gamma^k_{ij}$, $\partial_a f^i$, $\partial_b f^j$ and of ∇df implies the boundedness of the $\partial_\alpha f^k$ for $|\alpha| \leq 2$. From the validity of the assertion for $\mu-1 \leq m$, ${}^b|\nabla^\mu df| < \infty$, the boundedness of $\frac{\partial^\alpha}{\partial x^\alpha}\Gamma^c_{ab}$, $\frac{\partial^\alpha}{\partial x^\alpha}(\partial_a f^k \Gamma^i_{jk}(f(x))$, $|\alpha| \leq \mu$ and the $\mu-1$ times derived equation (3.13) we obtain the assertion for $\mu \leq m$. □

Corollary 3.5. Assume $1 \leq p < \infty$, (M^n,g), (N^r,h) being open, complete of C^{m+2}-bounded geometry, $s > \frac{n}{p} + m$ and $df \in {}^p\Omega^{0,s}(T^*M \otimes f^*TN)$. Then f satisfies a.,b.,c. of 3.4. □

Proof. From the assumptions and theorem 2.6 we obtain $\nabla^\mu df$ bounded, $0 \leq \mu \leq m$. □

Corollary 3.6. Assume (M^n,g), (N^r,h) being open, complete, of C^s-bounded geometry, $s > \frac{n}{2} + m$ and $df \in {}^2\bar{\Omega}^{10,s}(T^* M \otimes f^* TN)$. Then f satisfies a.,b.,c. of 3.4. □

4. Manifolds of mappings between open manifolds of C^s-bounded geometry

For complete manifolds (M^n,g), (N^r,h) of C^s-bounded geometry with respect to normal coordinates we denote by $C^{\infty,s}(M,N)$ the set of all $f \in C^\infty(M,N)$ which satisfy

$$\sup_{x \in M} |\nabla^\mu df|_x < \infty, \quad 0 \leq \mu \leq s, \tag{4.1}$$

where ∇ equals to the tensor product of ${}^g\nabla$ and $f^{*h}\nabla$. Suppose now $0 < \delta \leq \delta_N < r_{inj}(N)$ as in section 3 and $Y \in C^\infty(f^*TN) \equiv \Omega^0(f^*TN)$ with ${}^{b,f^*h}|Y| = \sup_{x \in M} {}^h|Y_{f(x)}| < \delta$. Then the mapping $x \xrightarrow{g_Y} \exp_{f(x)} Y_{f(x)}$, i.e. $g_Y = \exp_f Y \circ f$, defines an element of $C^\infty(M,N)$.

More general, if $\sup_{\substack{x \in M \\ 0 \leq \mu \leq s}} |\nabla^\mu Y| = {}^b|Y|_s < \delta$, then $g_Y \in C^{\infty,s}(M,N)$.

This follows from the following facts:

1. the Leibniz product rule for $g_Y = \exp_f Y \circ f$,
2. for $\mu = 1$ there holds in B_δ

$$|d\exp - P_r|_V \le \left(\frac{\sinh(\kappa_1|V|)}{|V|} - 1\right),$$

where $-\kappa_1^2 \le K \le \kappa_2^2$ and P_r denotes the radial parallel translation (cf. [6], [7]),

3. $|\nabla^2 \exp| = |\nabla\, d \exp| = |\Gamma| \le \mathrm{const}(R, \nabla R)$,

4. the higher covariant derivatives of exp are bounded since the partial derivatives of the Christoffel symbols are bounded by the assumption of C^s-bounded geometry.

Remark. $|\Gamma| \le$ const. already holds for C^1-bounded geometry.

We define

$${}^bU_{\delta,s}(f) = \{g \in C^{\infty,s}(M,N) \mid \text{There exists an } Y \in {}^b\Omega^{0,s}(f^* TN) \text{ such that } g = g_Y = \exp_f Y \circ f \text{ and } {}^b|Y|_s < \delta$$

and set $\quad {}^b|f-g|_s := {}^b|Y|_s.$

Lemma 4.1. The system of all ${}^bU_{\delta,s}(f)$, $0 < \delta \le \delta_N < r_{inj}(M)$, $f \in C^{\infty,s}(M,N)$ forms a base of neighborhood filters for a locally metrizable topology on $C^{\infty,s}(M,N)$.

Proof. The only nontrivial fact that remains to show is the following: For each ${}^bU_{\varepsilon,s}(f)$ there exists an ${}^bU_{\tau,s}(f)$ such that ${}^bU_{\varepsilon,s}(f)$ is a neighborhood for each $g \in {}^bU_{\tau,s}(f)$, i.e. there exists a $\tau' = \tau'(g) < \varepsilon$ such that ${}^bU_{\tau',s}(g) \subset {}^bU_{\varepsilon,s}(f)$. Suppose $g = g_Y \in {}^bU_{\varepsilon,s}(f)$. f^*TN and g^*TN are for ε sufficiently small canonically isomorphic since $f = f_0$ and $g = f_1$ are canonically homotopic by the smooth and and up to order s+1 bounded homotopy f_t, $f_t(x) = \exp_{f(x)} t\cdot Y$, i.e. f_t satisfies (4.1). Therefore there exist positive constants C_1, C_2 such that

$$C_1 \sup_{0\le\mu\le s} |\nabla^\mu Y'|_x^f \le \sup_{0=\mu=s} |\nabla^\mu Y'|_x^g \le C_2 \sup_{0=\mu=s} |\nabla^\mu Y'|_x^f, \quad (4.2)$$

where $|\ \ |_x^f$ indicates that into ∇ and the pointwise norm enter $f^{*h}\nabla$, f^*h. The same holds for $|\ \ |_x^g$.

Now we set $\tau = \frac{\varepsilon}{4}$. Then ${}^bU_{\varepsilon/4C_1,s}(g) \subset {}^bU_{\varepsilon,s}(f)$:

If $g' = g_{Y'} \in {}^bU_{\varepsilon/4C_1,s}(g)$, then

$${}^b|f - g'|_s^f \le {}^b|f - g|_s^f + {}^b|g - g'|_s^f < \frac{\varepsilon}{4} + {}^b|Y'|_s^f <$$

$$\frac{\varepsilon}{4} + C_1 \cdot \frac{\varepsilon}{4C_1} = \frac{\varepsilon}{2} < \varepsilon \ . \ \square$$

Let ${}^b\bar{\Omega}^s(M,N)$ be the completion of $C^{\infty,s}(M,N)$. The neighborhoods in ${}^b\bar{\Omega}^s(M,N)$ shall be denoted by ${}^bU_\delta^s(f)$.

Proposition 4.2. Suppose (M^n,g), (N^r,h) being open, complete and of C^s-bounded geometry, $s\geq 1$. Then ${}^b\bar{\Omega}^s(M,N)$ is a Banach manifold.

Proof. Set $T_f{}^b\bar{\Omega}^s(M,N) := {}^b\Omega^{0,s}(f^*TN)$. Then $Y \to g_Y$, $g_Y(x) = \exp_{f(x)} Y_{f(x)} = (\exp_f Y \circ f)(x)$ is for $0 < \delta \leq \delta_N$ a homeomorphism between $B_\delta(0) \subset {}^b\Omega^{0,s}(f^*TN)$ and ${}^bU^s_\delta(f)$, i.e. a chart. That the transition mappings are C^∞ (in the Banach category) can be established in the same manner as for M,N closed. □

In similar manner we can work with ${}^b\mathring{\Omega}^{0,s}(f^*TN)$ and obtain a Banach manifold ${}^b\mathring{\Omega}^s(M,N)$.

The next case is given by $1\leq p < \infty$. We start again with $C^{\infty,s}(M,N)$, M and N of C^s-bounded geometry, $s\geq 1$. We set for $f \in C^{\infty,s}(M,N)$, $\varepsilon < \delta_N$,

$$ {}^pU_{\varepsilon,s}(f) = \{ g = g_Y = \exp_f Y \circ f \mid Y \in {}^p\mathring{\Omega}^0_{s+[\frac{n}{p}]+2}(f^*TN) \text{ and } $$

$$ {}^p|Y|_{s+[\frac{n}{p}]+2} < \varepsilon \}, $$

where ${}^p|Y|_k = \sum_{\mu=0}^{k} (\int |\nabla^\mu Y|^p dvol)^{1/p}$, $\nabla = \nabla^1 = f^{*h}\nabla$, for $\mu\geq 2$ = composition of $f^{*h}\nabla$ and of the tensor product of ${}^g\nabla$ and $f^{*h}\nabla$.

Lemma 4.3. The system of all ${}^pU_{\varepsilon,s}(f)$, $0<\varepsilon<\delta_N$, $f\in C^{\infty,s}(M,N)$, forms a base of neighborhood filters for a locally metrizable topology on $C^{\infty,s}(M,N)$.

Proof. Assume at first $s\geq 2$. C^s-bounded geometry for $s\geq 2$ implies (I) and (B_0) for M, the Sobolev embedding theorem 2.6 is applicable and $Y \in {}^p\Omega^0_{s+[\frac{n}{p}]+2}(f^*TN)$ implies $Y \in {}^b\Omega^{0,s}(f^*TN)$. Therefore ${}^pU_{\varepsilon,s}(f) \subset C^{\infty,s}(M,N)$. The compatibility condition with respect to ${}^p|\ |_s$ again follows from (4.2). If $s\geq 1$ then (B_0) is not guaranted. But C^1-bounded geometry still implies the boundedness of the Jacobian dexp, and this fact, which is part of the Rauch comparison theorem, is needed during the proof of 2.6 (cf. [2]). □

Let ${}^p\bar{\Omega}^s(M,N)$ be the completion of $C^{\infty,s}(M,N)$ and ${}^pU^s_\varepsilon(f)$ the completed neighborhoods.

Proposition 4.4. Assume $(M^n,g),(N^r,h)$ being open, complete and of C^s-bounded geometry, $s\geq 1$. Then ${}^p\bar{\Omega}^s(M,N)$ is a Banach manifold.

Proof. If $f\in {}^p\bar{\Omega}^s(M.N)$ then by construction $f \in {}^b\Omega^s(M,N)$. Set $T_f{}^p\bar{\Omega}^s(M,N) := {}^p\bar{\Omega}^{0,s+[n/p]+2}(f^*TN) \subset {}^b\Omega^{0,s}(f^*TN)$ continuously. The map $Y \to g_Y = \exp_f Y \circ f$ defines a homeomorphism of some open ball $B_\varepsilon(0) \subset {}^p\bar{\Omega}^{0,s+[n/p]+2}(f^*TN) = T_f$ onto some ${}^pU^s_\varepsilon(f)$. The smoothness of the transition maps follows as in the case M,N closed, since according to our assumption the Sobolev embedding theorem is valid. □

Working with ${}^{p}\mathring{\Omega}^{0,s+[n/2]+2}(f^* TN)$ instead of ${}^{p}\bar{\Omega}^{0,s+[n/2]+2}(f^*TN)$, we obtain Banach manifolds ${}^{p}\mathring{\Omega}^{s}(M,N) \subseteq {}^{p}\bar{\Omega}^{s}(M,N)$. Theorem 2.2 then immediately implies

Corollary 4.5. If $(M^n,g),(N^r,h)$ have $C^{s+[n/2]+2}$-bounded geometry then ${}^{p}\bar{\Omega}^{s}(M,N) = {}^{p}\mathring{\Omega}^{s}(M,N)$.

Proof. The assumption implies (I) and $(B_{s+[n/2]})$, ${}^{p}\bar{\Omega}^{0,s+[n/2]+2}(f^*TN) = {}^{p}\mathring{\Omega}^{0,s+[n/2]+2}(f^*TN)$. ◻

Corollary 4.6. Assume p=2 and the hypotheses of 4.4. Then ${}^{2}\bar{\Omega}^{s}(M,N)$ is a Hilbert manifold. If we additionally assume the hypothesis of 4.5, then ${}^{2}\bar{\Omega}^{s}(M,N) = {}^{2}\mathring{\Omega}^{s}(M,N)$. ◻

Working with the powers of $1+\Delta$ instead with the powers of ∇, we obtain Hilbert manifolds ${}^{2}\bar{\Omega}'^{s}(M,N)$ and ${}^{2}\mathring{\Omega}'^{s}(M,N)$.

Theorem 4.7. Suppose all hypotheses of 4.6. Then

$${}^{2}\mathring{\Omega}'^{s}(M,N) = {}^{2}\bar{\Omega}'^{s}(M,N) = {}^{2}\bar{\Omega}^{s}(M,N) = {}^{2}\mathring{\Omega}^{s}(M,N).$$

Proof. Corollary 2.5. ◻

Instead of starting with $C^{\infty,s}(M,N)$, we could start with

$${}^{p}C^{1,s+[n/2]+2}(M,N) = \{f \in C^1(M,N) \mid df \in {}^{p}\Omega^{0,s+[n/2]+2}(TM \otimes f^*TN)\}.$$

If $(M^n,g),(N^r,h)$ have C^s-bounded geometry then $f \in C^{\infty,s}(M,N)$. The same procedure as above then yields manifolds ${}^{p,p}\Omega^{s}(M,N)$, i.e. we introduce manifolds of p-integrable mappings (i.e. the differentials and their derivatives are p-integrable) which are completed with respect to the p-norm.

All the concepts above are applicable to sections of fibre bundles $E \xrightarrow{F} M$ if we assume that M,E and the vertical bundle T_vE have C^s-bounded geometry and the fibres are totally geodesic. The case of the product bundle $M \times F \longrightarrow M$ just corresponds to the case above: A section can be identified with a map $(f,id):M \longrightarrow M \times F$. For an arbitrary fibre bundle which satisfies the above geometric assumptions we consider the set $C^{\infty,s}(E)$ of smooth bounded sections, define metrizable neighborhoods by composition with the exponential map in the vertical bundle and the fibres and perform completion as above. Thus we get Banach manifolds ${}^{b}\bar{\Omega}^{s}(E), {}^{b}\mathring{\Omega}^{s}(E), {}^{p}\bar{\Omega}^{s}(E)$, ${}^{p}\mathring{\Omega}^{s}(E)$, in particular Hilbert manifolds ${}^{2}\bar{\Omega}^{s}(E) \supseteq {}^{2}\mathring{\Omega}^{s}(E)$. We conclude this section with the assertion which was the main motivation for the whole paper.

Proposition 4.8. Assume (M^n,g) being open, complete, $P(M,G) \longrightarrow M$ a principal fibre bundle, $\tilde{P} = P \times_{Ad} G$. If M,P,T_vP have C^s-bounded geometry and the fibres of P are totally geodesic, then ${}^{2}\bar{\Omega}^{s}(\tilde{P})$ is a Hilbert manifold.

Remark. Keep in mind that ${}^{2}\bar{\Omega}^{s}(\tilde{P})$ by definition above consists of mappings of Sobolev class $s+[n/2]+2$. Furthermore, for f homotopic g in ${}^{p}\bar{\Omega}^{s}(M,N)$ there holds $f^{*}TN\cong g^{*}TN$ and $T_f\cong T_g$.

5. Groups of diffeomorphisms

As we have seen until now, the assumption of sufficiently high bounded geometry allowed to establish the manifold property for the space of smooth bounded mappings between open, complete manifolds $(M^n,g),(N^r,h)$. If we restrict to bounded diffeomorphisms, there arise additional difficulties. This subset is not longer an open subset. But, if we restrict to diffeomorphisms which are bounded from below too, then we again get an open subset and a manifold structure.

Suppose (M^n,g) being open, oriented, complete, of C^s-bounded geometry, $s > n/p+2$. Then we set

$$ {}^{p}\mathcal{D}^{s}(M) = \{ f \in {}^{p}\bar{\Omega}^{s}(M,M) \mid f \text{ is one-one, orientation preserving and } f^{-1} \in {}^{p}\bar{\Omega}^{s}(M,M) \} $$

and

$$ {}^{p}\mathcal{D}^{s}_{bb}(M) = \{ f \in {}^{p}\mathcal{D}^{s}(M) \mid \inf_{x\in M} |df|_x > 0 \}. $$

"bb" means the norm of the differential is positively bounded from below. The situation will be cleared up by

Theorem 5.1. If $s > n/p+2$, then ${}^{p}\mathcal{D}^{s}_{bb}$ is open in ${}^{p}\bar{\Omega}^{s}(M,M)$.

Corollary 5.2. ${}^{p}\mathcal{D}^{s}_{bb}(M)$ is a manifold. □

The proof of 5.1 will be prepared by

Lemma 5.3. Suppose (M^n,g) being open, complete, connected, of C^s--bounded geometry, $s > n/p+2$, $f: M \to M$ a C^1-diffeomorphism and $g: M \to M$ a local C^1-diffeomorphism which can be connected with f by an arc in ${}^{p}\bar{\Omega}^{s}(M,M)$ of local C^1-diffeomorphisms. Then $g(M) = M$.

Proof. Fix some point $z \in M$ and consider the open metric balls $B_k = B_k(z) = \{ x \in M \mid d(x,z) < k \}$. Then $B_1 \subset B_2 \subset \cdots$ and $\bigcup B_k = M$. Moreover, $f(B_1) \subset f(B_2) \subset \cdots$ and

$$ \bigcup_k f(B_k) = M \tag{5.1} $$

since f is a diffeomorphism. Consider an arc $\{g_t\}_{0=t=1}$ of local C^1-diffeomorphisms between f and g, $f=g_0$, $g=g_1$. By the Sobolev embedding theorem we have ${}^{p}\bar{\Omega}^{0,s}(TM) \hookrightarrow {}^{b}\Omega^{0,1}(TM)$ continuously, ${}^{b}|Y|_1 = C\cdot {}^{p}\|Y\|_s$. Fix $0 < \delta_0 < \delta_M < r_{inj}(M)$ and after that $0 < \delta_0' < \delta_0$, such that $C\cdot\delta_0' \le \delta_0$. The arc $\{g_t\}_t$ can be covered by a finite number of neighborhoods ${}^p\mathcal{U}^s_{\delta_0}(g_{t_i})$, $i=0,\ldots,r$, $t_0=0$, $t_r = 1$. We set $g_{t_i} = g_i$. According to the definition of local neighborhoods, this implies the existence of C^1-vector fields Y_i, $i=0,\ldots,r-1$, ${}^b|Y_i|_1 < \delta_0$, such that the mapping $x \longrightarrow g(x)$ is given by

$$x \longrightarrow f(x)=g_0(x) \longrightarrow \exp_{g_0(x)}Y_0 \longrightarrow \exp_{g_1(x)}Y_1 \longrightarrow \cdots \longrightarrow \exp_{g_{r-1}(x)}Y_{r-1}, \quad (5.2)$$

where $g_\varrho(x) = \exp_{g_{\varrho-1}} Y_{\varrho-1}$, $\varrho=1,\ldots,r$. Suppose now $y_0 \in M \setminus g(M)$, $d(y_0,z) = \varepsilon$. Then we choose k such that $k-\varepsilon > 2r\delta_0$ and $m > k$ such that $f(B_m) \supset B_k$. (5.2) implies

$$d(f(x),g(x)) < r\cdot\delta_0. \quad (5.3)$$

All $g_t(B_m)$ are open manifolds. (5.3) now yields

$$g_1(B_m)=g(B_m) \supset B_{k-r\delta_0} \supset B_{\varepsilon+r\delta_0}$$

which contradicts $y_0 \notin g(M)$.

Proof of theorem 5.1. Since $s > n/p+2$ we can represent all $g \in {}^p\Omega^s(M,M)$ by bounded C^1-maps. Suppose $f \in {}^p\mathcal{D}^s_{bb}(M)$, $\inf_{x\in M} |df|_x > 0$. According to the continuity of $\inf_{x\in M} |df|_x$ as a function of f, there is a contractible neighborhood $U(f) \subset {}^p\Omega^s(M,M)$ such that $\inf_{x\in M} |dg|_x > 0$. According to the inverse function theorem, consists U of local diffeomorpisms. Lemma 5.3 now yields that each $g \in U$ is a surjective local C^1-diffeomorphism $g: M \longrightarrow M$, i.e. a covering map. f has the leave number 1, by continuity the same holds for g, g has to be a diffeomorphism. ▫

Remarks. 1. Lemma 5.3 becomes false if we do not observe the special topology in ${}^p\Omega^s(M,M)$. Consider, for example, $M = \mathbb{R}$, $f = \mathrm{id}: \mathbb{R} \longrightarrow \mathbb{R}$ and the "arc" $\{g_t\}_{0\le t\le 1}$, $g_t(x) = t\cdot \mathrm{arctg}x + (1-t)x$. $f=g_0=\mathrm{id}_{\mathbb{R}}$ is surjective, each g_t is a local diffeomorphism, but $g_1(x) = \mathrm{arctg}x \in \left]-\pi/2, \pi/2\right[$ is not surjective. The reason is evident. The above "arc" is not an arc in our topology, since during a t-interval of length δ a point x moves through an interval of length $\ge |\delta\cdot x| - \pi/2$ which becomes arbitrary large for $|x|$ sufficiently large. In our topology, a point x moves during a δ-interval along a way of bounded length, independently of x.

2. Theorem 5.1 becomes definitely false if one replaces ${}^p\mathcal{D}^s_{bb}(M)$ by ${}^p\mathcal{D}^s(M)$ (for closed manifolds they coincide). Let $\gamma(t)$, $0 \le t < \infty$, be a geodesic ray in M, $T_2 \subset T_1$ tubes of radius $e^{-2t} \subset e^{-t}$ around γ and f a C^1-diffeomorphism which is outside some neigh-

borhood of T_1 the identity and maps T_1 onto T_2. Then every neighborhood of f contains a C^1-map which maps T_1 onto T_2 and additionally T_2 onto γ for $t \geq a$ sufficiently large. One has to choose the latter map in a C^1-manner, i.e. by some radial contraction and along some vector field which touches $\gamma(t)$ at least of first order. We omit the simple and more or less standard details.
Many further questions concerning manifolds of mappings between manifolds are still open and under investigation. We devote them forthcoming papers.

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