Manuel de León; Paulo R. Rodrigues

Higher order almost tangent geometry and non-autonomous Lagrangian dynamics

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 16. pp. [157]--171.

Persistent URL: http://dml.cz/dmlcz/701419

HIGHER ORDER ALMOST TANGENT GEOMETRY AND
NON-AUTONOMOUS LAGRANGIAN DYNAMICS

Manuel de León Paulo R. Rodrigues

Abstract.- This paper is a sequel of a previous article |DLR1| . We generalize the intrinsical formulation of non-autonomous Lagrangian dynamics for Lagrangians depending on higher order derivatives with respect to the time. The study is developed from the almost tangent geometry point of view. Some geometric structures are examined. The higher order Poincaré-Cartan theory is presented in terms of the almost tangent structures.

Key words: Almost tangent geometry, Lagrangian dynamics,non-autonomous.
Mathematical A.M.S. Classification: 58F/70H.

1.- Introduction.

In a previous paper (de León & Rodrigues (see |DLR1|)) we have examined the intrinsical description of the non-autonomous (or time dependent) Lagrangian formalism in the framework of the almost-tangent geometry (see Clark & Bruckheimer |CB|). We have seen there, for instance, that the theory of connections proposed by Grifone (|G1|, |G2|) is more simpler than the theory for the autonomous (or time-independent) situation. Also, the intrinsical version of the Poincaré -Cartan form in terms of the almost-tangent structure was investigated (see also the paper of Crampin, Prince & Thompson |CPT|).

The purpose of the present article is the extension of our study to the formalism of non-autonomous Lagrangians higher-order derivatives. The study of higher order theories, from some different point of views, has been object of a certain number of authors: for example, Aldaya & Azcárraga |AA1|, |AA2|, de León & Rodrigues |DLR2|, Francaviglia & Krupka |FK|, García & Muñoz |GM|, Hórak & Kolár |HK|, Kolár |K|, Krupka |KR|, Shadwick |S|, Tulczyjew |T1|,|T2|. In the León & Rodrigues |DLR2|, for example, we have clarified how the autonomous higher order situation is formulated in terms of the almost tangent geometry machinery (we suggest to the reader the paper by Crampin, Sarlet & Cantrijn |CSC|, where a different aproach is presented, gi

ven emphasis on the role of the higher order differential equations). A non-autonomous (resp. autonomous) Lagrangian formalism for a higher order particle Mechanics is given by a real smooth (C^{∞}) function L defined on the jet bundle $J^k(R,M)$ of all smooth functions from R to M (resp. on the bundle $J^k_0(R,M)$ of all smooth functions from R to M with the source at the origin $0 \in R$). Here, k is the highest order of derivation involved in the variables from which L is dependent and M is the configuration manifold. As these bundles may be identified with $R \times T^kM$ and T^kM, respectively, where T^kM is the tangent bundle of order k of M, we may transport the geometrical structures intrinsically defined on T^kM to $J^k(R,M)$. We use this fact to give the corresponding intrinsical formulation on $J^k(R,M)$.
The present paper is organized as follows. In section 2, we give some basic definitions and results necessary for the development of the theory. In section 3, we characterize the semisprays of higher order by means of the higher order almost-tangent geometry. In section 4, we introduce a kind of connections (called dynamical connections) on the fibration $J^k(R,M) \longrightarrow J^{k-1}(R,M)$. Section 5 and 6 are devoted to study the relationship between senisprays and dynamical connection. Finally, in section 7, we show that the Poincaré-Cartan 2-form may be constructed by using the almost tangent structure of higher order and prove that there exists a dynamical connection whose paths are solutions of the generalized Lagrange equations.

1.- Notations, definitions and some results.

Troughout this paper it is assumed that all differential structures are of C^{∞}- class (smooth). Let R be the field of real numbers. M a m-dimensional manifold and $(R \times M, p, R)$ the corresponding (trivial) fibred manifold. By Sec $(R\times M)$ we denote the set of all sections of $(R\times M, p, R)$. Locally $(R\times M)$ is characterized by coordinates (t, y^A), $1 \leq A \leq m$. The manifold of k-jets of sections $s \in$ Sec $(R\times M)$, denoted by $J^k(R,M)$ is locally given by coordinates of type $(t, y^A, y^A_1, \ldots, y^A_k)$, $1 \leq k < \infty$ (when k=0, the $J^0(R,M) = R\times M$). If $s \in$ Sec $(R\times M)$ then $s^k(t)$ or $j^k_t s$ denotes the corresponding k-jet of s at $t \in R$. By $\alpha^k : J^k(R,M) \longrightarrow R$, $\beta^k : J^k(R,M) \longrightarrow M$ and $\pi^k : J^k(R,M) \longrightarrow R\times M$, we denote, respectively, the canonical projections $s^k(t) \longrightarrow t$, $s^k(t) \longrightarrow s(t)$ and $s^k(t) \longrightarrow (t, s(t))$. The map $s^k : R \longrightarrow J^k(R,M)$, $t \longrightarrow s^k(t)$, such that $s^k(t) \in$ Sec $(J^k(R\times M), \alpha^k, R)$ is called k-jet prolongation (or extension) of $s \in$ Sec $(R\times M)$. If $s \in$ Sec $(R\times M)$ and $s^k(t)$ is the corresponding k-jet of s at $t \in R$ then locally we have:

$$y^A = s^A(t), \quad y^A_i = \frac{1}{i!} \frac{d^i}{dt^i} s^A(t), \quad 1 \leq i \leq k.$$

The factor $1/i!$ appears only for technical reasons. We may adopt the following coordinate system for $J^k(R,M)$: $(t,q^A,q_1^A,\dots,q_k^A)$, where $q^A=s^A(t)$, $q_i^A=(d^i/dt^i)s^A(t)$, $1\leqslant i\leqslant k$. Clearly, we have:

$$q^A = (i!)\ y_i^A,\ 0\leqslant i\leqslant k,\ 1\leqslant A\leqslant m.$$

As $(R\times M, p, R)$ is a trivial bundle we may identify maps from R to M with sections of $(R\times M,p,R)$ as well as their k-jets. Thus we put $J^k(R\times M) = J^k(R,M)$ (= the k-jet manifold of all maps from R to M). Furthermore, we notice that $J^k(R,M)$ can be identify with $R\times T^kM$ in a natural way by the map $s^k(t) \longrightarrow (t,(d/dt)(s(t)),\dots,(d^k/dt^k)(s(t))$, where T^kM is the tangent bundle of order k of M, that is, $T^kM = J_0^k(R,M)$ is the k-jet bundle of all maps from R to M with source at the origin $0\in R$.

Let $g : J^k(R,M)\longrightarrow R$ be a smooth function. Thus d_T is the Tulczjew's operator which maps g on a function d_Tg on $J^{k+1}(R,M)$ locally expressed by

$$d_Tg(t,y^A,\dots,y_k^A) = \frac{\partial g}{\partial t} + \sum_{i=0}^{k} (i+1)\ y_{i+1}^A \frac{\partial g}{\partial y_i^A} \qquad (y_0^A = y^A) \qquad (2.1).$$

(for an intrinsical definition of d_T see |DLR2|,p.80).

Definition (2.1). Let N be a $(k+1)m$-dimensional manifold. An endomorphism $S : TN\longrightarrow TN$ such that rank $S = km$ and $S^{k+1} = 0$ is called almost tangent structure (of order k). The couple (N,S) is said almost tangent manifold (of order k).

A first interesting result says that for all manifold M its tangent bundle TM is endowed with an almost tangent structure (see Godbillon |G|). Furthermore, for any integer k, there exists a family of endomorphisms $J_r: T(T^kM)\longrightarrow T(T^kM)$, $1\leqslant r\leqslant k$, such that $J_1 : T(T^kM)\longrightarrow T(T^kM)$ is an almost tangent structure (of order k) on T^kM. For $1\leqslant r\leqslant k$, one has

$$J_r = (J_1)^r.$$

(see de León & Rodrigues |DLR2|, p.24-31). For a local coordinate system $(y^A,y_1^A,\dots,y_k^A)$ the endomorphism J_r has the following expression:

$$J_r = \sum_{i=0}^{k-r} \frac{\partial}{\partial y_{r+i}^A} \otimes dy_i^A \qquad (2.2).$$

Also, there exists on T^kM a family of vector fields C_r, $1\leqslant r\leqslant k$, locally given by

$$C_r = \sum_{i=0}^{k-r} (i+1)y_{i+1}^A \frac{\partial}{\partial y_{r+i}^A} \qquad (2.3).$$

When r = 1, then C_1 is called the (generalized) Liouville vector field. One has

$$C_r = J_1 C_{r-1} \quad (\text{or } C_r = J_{r-1} C_1), \ r \geq 2 \ .$$

Definition (2.2). Let ξ be a vector field on T^kM. We say that ξ is a semispray (or a (k+1)th order different equation) if $J_1\xi = C_1$. A curve $s : R \longrightarrow M$ is called a path of ξ if s^k is an integral curve of ξ, that is,

$$(d/dt)s^k = \xi \circ s^k \ .$$

Therefore, s is a path of ξ if and only if verifies the following system of differential equations:

$$\frac{1}{k!} \ \frac{d^{k+1}}{dt^{k+1}} \, s^A = \xi^A(s^A, (d/dt)s^A, \ldots, (d^k/dt^k)s^A) \quad ,$$

where the semispray ξ has the local expression

$$\xi = \sum_{i=0}^{k-1} (i+1)y^A_{i+1} \frac{\partial}{\partial y^A_i} + \xi^A \frac{\partial}{\partial y^A_{(k)}} \qquad (2.4).$$

(for further details, see |DLR2|, p. 54-58).
(Let us remark that if we adopt the coordinates $(t, q^A, \ldots, q^A_k)$, then s is a path of ξ if and only if it verifies the following system of differential equations:

$$\frac{d^{k+1}}{dt^{k+1}} \, s^A = \bar{\xi}^A \ (s^A, (d/dt)s^A, \ldots, (d^k/dt^k)s^A) \quad ,$$

where ξ is locally given by

$$\xi = \sum_{i=0}^{k-1} q^A_{i+1} \frac{\partial}{\partial q^A_i} + \bar{\xi}^A \frac{\partial}{\partial q^A_k} \quad).$$

Let us remark that on T^kM there is defined an appropiate exterior calculus induced by J_1: an inner product on p-forms

$$i_{J_1} \omega(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega \ (X_1, \ldots, J_1X_i, \ldots, \ldots, X_p)$$

and an exterior differentiation d_{J_1} defined by $d_{J_1} = i_{J_1} d - d \, i_{J_1}$.

A proof of the following result may be found in |DLR2|, p.95-99.
Theorem. Let $L : T^kM \longrightarrow R$ be a regular Lagrangian (that is, the Hessian matrix $(\partial^2 L/\partial y^A_k \partial y^B_k)$ is of maximal rank everywhere). Consider the following closed 2-form on $T^{2k-1}M$

$$\omega_L = -dd_{J_1} L + \frac{1}{2!} d_T dd_{J_2} L - \frac{1}{3!} d_T{}^2 dd_{J_3} L + \ldots + (-1)^k \frac{1}{k!} d_T{}^{k-1} dd_{J_k} L$$

and the intrinsical equation

$$i_\xi \, \omega_L = d\, E_L \;, \qquad (2.5).$$

where

$$E_L = C_1 L - \frac{1}{2!} d_T (C_1 L) + \frac{1}{3!} d_T{}^2 (C_3 L) + \ldots + (-1)^{k-1} \frac{1}{k!} d_T{}^{k-1} (C_k L) - L.$$

Then

(1) ω_L is a symplectic form on $T^{2k-1}M$,

(2) The vector field ξ given by (2.5) is a semispray on $T^{2k-1}M$, that is, $J_1 \xi = C_1$,

(3) The paths of ξ are the solutions of the Lagrange equations

$$\sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} \left(\frac{\partial L}{\partial q_i^A}\right) = 0 \quad .$$

3.- The generalized evolution space.

We have remarked that $J^k(R,M)$ may be identified with $R\times T^kM$ ahd so we may transport the geometric structures needed to develop the autonomous Lagrangian formalism on T^kM to $J^k(R,M)$ via this identification. We call $J^k(R,M)$ the (generalized) evolution space. Thus we have the following induced endomorphisms on $J^k(R,M)$:

$$\bar{J}_r = J_r - C_r \otimes dt, \quad 1\leq r\leq k.$$

Locally :

$$\bar{J}_r = \sum_{i=0}^{k-r} \partial/\partial y^A_{r+i} \otimes dy^A_i - \left(\sum_{i=0}^{k-r} (i+1) y^A_{i+1}\, \partial/\partial y^A_{r+i}\right) \otimes dt \;,$$

and it is clear that we may define in a similar way as we do for the autonomous case the operators $i_{\bar{J}_r}$ and $d_{\bar{J}_r}$. The following equalities are easily obtained:

$$\bar{J}_r(\partial/\partial t) = - C_r,$$

$$\bar{J}_r(\partial/\partial y^A_i) = J_r(\partial/\partial y^A_i) = \begin{cases} \partial/\partial y^A_{r+i}, & r+i\leq k, \\ 0, & r+i\geq k. \end{cases}$$

Let $\bar{J}_r^*$ be the adjoint operator induced by $\bar{J}_r$ on the exterior algebra $\Lambda(J^k(R,M))$ of $J^k(R,M)$. Then we have

$$\bar{J}_r^*(dt) = 0,$$

$$\bar{J}_r^*(dy_i^A) = \begin{cases} 0, & i<r, \\ \bar{\theta}_{i-r}^A, & r\leq i\leq k+r-1, \end{cases}$$

where

$$\bar{\theta}_i^A = dy_i^A - (i+1)\, y_{i+1}^A\, dt, \quad 0\leq i\leq k-1. \qquad (3.1)$$

(we remark that, if we adopt the coordinates $(t, q^A, \ldots, q_k^A)$ introduced in section 2, then we put

$$\theta_i^A = dq_i^A - q_{i+1}^A\, dt, \quad 0\leq i\leq k-1,$$

and we have

$$\theta_i^A = i!\,(\bar{\theta}_i^A)).$$

Definition (3.1). A vector field ξ on $J^k(R,M)$ is said a semispray (or (k+1)-th differential equation), if and only if $<\xi, \theta_i^A> = 0$ and $<\xi, dt> = 1, 0\leq i\leq k-1$.

We can easily prove that a semispray ξ is locally given by

$$\xi = \partial/\partial t + \sum_{i=0}^{k-1} (i+1)\, y_{i+1}^A\, \partial/\partial y_i^A + \xi^A \partial/\partial y_k^A \qquad (3.2).$$

Therefore, we have

Proposition (3.1). A vector field ξ on $J^k(R,M)$ is a semispray if and only if $J_1\xi = C_1$ and $\bar{J}_1\xi = 0$.

Definition (3.2). Let ξ be a semispray on $J^k(R,M)$. A curve $s: R \longrightarrow M$ is said a path of ξ if s^k is an integral curve of ξ.

From (3.2) we deduce that s is path of ξ if and only if s satisfy the following system of differential equations:

$$\frac{1}{k!}\frac{d^{k+1}y^A}{dt^{k+1}} = \xi_k^A, \qquad (3.3).$$

(Let us remark that, if we adopt the coordinates $(t, q^A, \ldots, q_k^A)$, then s is a path of ξ if and only if it satisfies the following system of differential equations:

$$\frac{d^{k+1}}{dt^{k+1}} s^A = \bar{\xi}^A,$$

where

$$\xi = \partial/\partial t + \sum_{i=0}^{k-1} q^A_{i+1}\, \partial/\partial q^A_i + \bar{\xi}^A\, \partial/\partial q^A_k) .$$

4.-Dynamical connections on $J^k(R,M)$.

The tensor fields J_r and J_r on $J^k(R,M)$ permit us to give a characterization of a kind of connections for the fibration $J^k(R,M) \longrightarrow J^{k-1}(R,M)$.

Definition (4.1). By a dynamical connection on $J^k(R,M)$ we mean a tensor field Γ of type (1.1) on $J^k(R,M)$ satisfying

$$\Gamma \bar{J}_k = -\bar{J}_k, \Gamma J_k = -J_k, \bar{J}_1 \Gamma = J_1 \Gamma = J_1 . \qquad (4.1).$$

By a straigthforward computation from (4.1) we deduce that the local expressions of Γ are

$$\Gamma(\partial/\partial t) = -\sum_{i=1}^{k} i\, y^A_i\, \partial/\partial y^A_{i-1} + \Gamma^A \partial/\partial y^A_k ,$$

$$\Gamma(\partial/\partial y^A_i) = \partial/\partial y^A_i + \Gamma^{(i-1)B}_{A}\, \partial/\partial y^A_k , \quad 0 \leq i \leq k-1,$$

$$\Gamma(\partial/\partial y^A_k) = -\partial/\partial y^A_k .$$

The functions Γ^A, Γ^B_A will be called the components of Γ. From the local expressions above, it is easy to prove that $\Gamma^3 - \Gamma = 0$ and rank $\Gamma = 2km$. So Γ is an $f(3, -1)$-structure on $J^k(R,M)$ (see |YI|). Now, we associate to Γ two canonical operators l and m given by

$$l = \Gamma^2, \quad m = -\Gamma^2 + I.$$

Then we have

$$l^2 = l, m^2 = m, \; lm = ml = 0, l+m = I ,$$

and, so, l and m are complementary projectors locally given by

$$l(\partial/\partial t) = -\sum_{i=1}^{k} y^A_i\, \partial/\partial y^A_{i-1} - (\Gamma^B + \sum_{i=1}^{k} y^A_i \Gamma^{(i-1)B}_{A})\, \partial/\partial y^B_k ,$$

$$l(\partial/\partial y^A_i) = \partial/\partial y^A_i , \; m(\partial/\partial y^A_i) = 0 , \qquad (4.2)$$

$$m(\partial/\partial t) = \partial/\partial t + \sum_{i=1}^{k} i\, y^A_i\, \partial/\partial y^A_{i-1} + (\Gamma^B + \sum_{i=1}^{k} i y^A_i\, \Gamma^{(i-1)B}_{A})\, \partial/\partial y^B_k,$$

$0 \leq i \leq k, \ 1 \leq A, B \leq m$.

If we put $L = \mathrm{Im}\, l$, $M = \mathrm{Im}\, m$, then we have that L and M are complementary distributions on $J^k(R,M)$, that is,

$$T J^k(R,M) = M \oplus L .$$

Furthermore, from (4.2), we deduce that L is $(k+1)m$-dimensional and M 1-dimensional. In fact, L is locally spanned by $\{\partial/\partial y_i^A, \ 0 \leq i \leq k\}$ and M is globally spanned by the vector field

$$\xi = m(\partial/\partial t) = \partial/\partial t + \sum_{i=1}^{k} i y_i^A \ \partial/\partial y_{i-1}^A + (\Gamma^B + \sum_{i=1}^{k} i y_i^A \ \Gamma^{(i-1)B}_{A}) \ \partial/\partial y_k^A . \qquad (4.3)$$

From (4.3) we deduce that ξ is a semispray of type 1 on $J^k(R,M)$ which will be called the canonical semispray associated to Γ.

Since we have $\Gamma^2 l = l$ and $\Gamma m = 0$, then Γ acts on L as an almost product structure operator and trivially on M . Because $M = \mathrm{Ker}\, \Gamma$, Γ is said to be an $f(3,-1)$-structure of rank $(k+1)m$ and paralleliza ble kernel.

Now, we put

$$h = \frac{1}{2} (I+\Gamma)\, l, \ v = \frac{1}{2} (I-\Gamma)\, l.$$

Then we have

$$h\xi = 0, h(\partial/\partial y_i^A) = \partial/\partial y_i^A + (1/2)\Gamma^{(i)B}_{A} \ \partial/\partial y_k^B, h(\partial/\partial y_k^A) = 0,$$

$$v\xi = 0, v(\partial/\partial y_i^A) = (-1/2)\Gamma^{(i)B}_{A} \ \partial/\partial y_k^B, v(\partial/\partial y_k^A) = \partial/\partial y_k^A, \qquad (4.4)$$

$0 \leq i \leq k-1, \ 1 \leq A , B \leq m$.

If we put $H = \mathrm{Im}\, h$ and $V = \mathrm{Im}\, v$, then we have $L = H \oplus V$, where V is the vertical distribution defined by the fibration $J^k(R,M) \longrightarrow J^{k-1}(R,M)$. Hence, we deduce that

$$T J^k(R,M) = M \oplus L = M \oplus H \oplus V .$$

(So, Γ defines, in fact, a connection on the fibration $J^k(R,M) \longrightarrow J^{k-1}(R,M)$).

Let $H_i^A = h(\partial/\partial y_i^A), V^A = \partial/\partial y_k^A$, $0 \leq i \leq k-1$. Then, from (4.4), we have

$$\Gamma\xi = 0, \ \Gamma H_i^A = H_i^A , \ \Gamma V_i^A = - V_i^A ,$$

$$h\xi = 0, \ hH_i^A = H_i^A , \ hV_i^A = 0 , \qquad (4.5)$$

$$v\xi = 0, \ vH_i^A = 0, vV_i^A = V_i^A .$$

From (4.5) we deduce that a dynamical connection Γ on $J^k(R,M)$ induces an almost product structure on $J^k(R,M)$ given by three complementary distributions for the eigenvalues 0,+1 and -1. Furthermore, $\{\xi, H_i^A, V_i^A\}$ is a local basis of vector fields on $J^k(R,M)$. In fact, $M = <\xi>$, $H = <H_i^A>$, $V = <V_i^A>$; $\{\xi, H_i^A, V_i^A\}$ is called an adapted basis to the f(3,-1)-structure defined by Γ. An easy computation in local coordinates whows that the dual basis of 1-forms is given by

$\{dt, \theta_i^A, \psi^A\}$, where

$$\psi^A = -(\Gamma^A + \frac{1}{2}\sum_{i=1}^{k} i y_i^B \Gamma^{(i-1)A}_{B})dt - \frac{1}{2}\sum_{i=1}^{k}\Gamma^{(i-1)A}_{B} dy^B_{i-1} + dy^A_k \qquad (4.4).$$

Definition (4.2). H (resp. $M \oplus H$) will be called the strong (resp. weak) horizontal distribution.

Remark. Since $J^k(R,M)$ is a fibred manifold over $J^r(R,M)$, $1 \leq r \leq k-1$, we may consider connections on the fibration $J^k(R,M) \longrightarrow J^r(R,M)$, $1 \leq r \leq k-1$. The study of this type of connections will be elaborated in a forthcoming paper

5.- Paths of a dynamical connection.

Definition (5.1). A curve s in M is called a path of a dynamical connection Γ on $J^k(R,M)$ if and only if s^k is a weak horizontal curve in $J^k(R,M)$, that is, the tangent vector $s^k(t)$ belongs to $(M \oplus H)_{s^k(t)}$, for every $t \in R$.

Since a tangent vector X to $J^k(R,M)$ is in $M \oplus H$ if and only if $\psi^A(X) = 0$, we deduce, from (4.4), that s is a path of Γ if and only if satisfy the following system of differential equations:

$$\frac{1}{k!}\frac{d^{k+1}y^A}{dt^{k+1}} = \Gamma^A + \sum_{i=1}^{k}\frac{1}{(i-1)!}\Gamma^{(i-1)A}_{B}\frac{d^i y^B}{dt^i} \qquad (5.1)$$

From (3.3), (4.3) and (5.1), we easily deduce the following Proposition (5.1). A dynamical connection Γ on $J^k(R,M)$ and its associated semispray ξ have the same paths.

6.- Semisprays and dynamical connections on $J^k(R,M)$.

In this section, we prove that to each semispray ξ of type 1 on $J^k(R,M)$ there exists canonically associated a dynamical connection. Let ξ be a semispray of type 1 on $J^k(R,M)$ and suppose that ξ is locally given by

$$\xi = \partial/\partial t + y_1^A \partial/\partial y^A + 2y_2^A \partial/\partial y_1^A + \ldots + k y_k^A \partial/\partial y_{k-1}^A + \xi^A \partial/\partial y_k^A \qquad (6.1).$$

Then a direct computation from (6.1) shows that

$$|\xi, \partial/\partial t| = -\partial\xi^A/\partial t \ \partial/\partial y_k^A ,$$

$$|\xi, \partial/\partial y_i^A| = -\partial\xi^B/\partial y^A \ \partial/\partial y_k^B , \qquad (6.2).$$

$$|\xi, \partial/\partial y_i^A| = -i\partial/\partial y_{i-1}^A - \partial\xi^B/\partial y_i^A \ \partial/\partial y_k^B , \ 1 \leq i \leq k.$$

Now, put

$$\Gamma = -\frac{2}{k+1} L_\xi \mathfrak{J}_1 + \frac{k-1}{k+1} (I - \xi \otimes dt).$$

From (6.2), we have

$$\Gamma(\partial/\partial t) = -\sum_{i=1}^{k} i y_i^A \ \partial/\partial y_{i-1}^A + \frac{3-k}{k+1} \xi^A - \frac{2}{k+1} \sum_{i=1}^{k} i y_i^B \ \partial\xi^A/\partial y_i^B \ \ \partial/\partial y_k^A ,$$

$$\Gamma(\partial/\partial y_i^A) = \partial/\partial y_i^A + \frac{2}{k+1} \partial\xi^B/\partial y_{i+1}^A \ \partial/\partial y_k^B , \ 0 \leq i \leq k-1 , \qquad (6.3).$$

$$\Gamma \ \partial(\partial/\partial y_k^A) = -\partial / \partial y_k^A .$$

From (6.3), we deduce that Γ is a dynamical connection on $J^k(R,M)$ whose associated semispray $\tilde{\xi}$ is locally given by

$$\tilde{\xi} = \partial/\partial t + \sum_{i=1}^{k} i y_i^A \ \partial/\partial y_{i-1}^A + \tilde{\xi}^A \ \partial/\partial y_k^A ,$$

where

$$\tilde{\xi}^A = \frac{3-k}{k+1} \xi^A .$$

Let us remark that, if $k=1$, then $\Gamma = -L_\xi \mathfrak{J}$ and $\tilde{\xi} = \xi$. This case has been discussed in |DLR1| ; in the sequel we only consider the case $k \geq 2$.

Since $\tilde{\xi}$ is different from ξ, it is neccessary to modify Γ in order to obtain a dynamical connection $\tilde{\Gamma}$ whose associated semispray is, precisely, ξ . To do this, we put

$$\tilde{\Gamma} = \Gamma - (\tilde{\xi} - \xi) \otimes dt .$$

A simple computation shows that

$$((\tilde{\xi}-\xi)\otimes dt)(\partial/\partial t) = \frac{2(1-k)}{k+1} \xi^A \ \partial/\partial y_k^A , \qquad (6.4).$$

$$((\tilde{\xi}-\xi) \otimes dt)(\partial/\partial y_i^A) = 0 , \quad 0 \leq i \leq k .$$

From (6.4), we easily deduce the following.

Proposition (6.1). $\tilde{\Gamma}$ is a dynamical connection on $J^k(R,M)$ whose associated semispray is, precisely, ξ. (Obviously, for $k=1$, we have $\tilde{\Gamma} = \Gamma = - L_\xi \tilde{J}$) .

7.- The generalized time-depending Poincaré-Cartan form.

Let $L: J^k(R,M) \longrightarrow R$ be a non-autonomous regular Lagrangian of order k on M, that is, the Hessian matrix $(\partial^2 L/\partial y_k^A \partial y_k^B)$ is non-singular.

As it is well-known, the Poincaré-Cartan 1-form determined by L is the 1-form Θ_L on $J^{2k-1}(R,M)$ given by

$$\Theta_L = \sum_{i=1}^{k} p_A^i \, dq_{i-1}^A - E_L \, dt \quad , \tag{7.1}$$

where

$$p_A^i = \sum_{j=0}^{k-i} (-1)^j (d_T)^j (\partial L/\partial q_{i+j}^A) \quad , \quad 1 \leq i \leq k \; , \tag{7.2}$$

and

$$E_L = \sum_{i=1}^{k} q_i^A \, p_A^i - L \quad .$$

Here, $\{p_A^i \, , \, 1 \leq i \leq k\}$ are the generalized Jacobi-Ostrogradsky momentum coordinates and E_L is the Hamiltonian energy corresponding to L. Taking into account (2.1) and (3.1), we easily deduce that Θ_L can be re-written as

$$\Theta_L = d_{\bar{J}_1} L - \frac{1}{2!} d_T(d_{\bar{J}_2} L) + \frac{1}{3!} d_T{}^2 (d_{\bar{J}_3} L) - \ldots + (-1)^k \frac{1}{(k-1)!} d_T^{k-1}(d_{\bar{J}_k} L) + L \, dt,$$

and the Hamiltonian energy becomes

$$E_L = C_1 L - \frac{1}{2!} d_T(C_2 L) + \frac{1}{3!} d_T^2 \, (C_3 L) - \ldots + (-1)^k \frac{1}{(k-1)!} d_T^{k-1} \, (C_k L) - L \quad .$$

Consequently , the Poincaré-Cartan 2-form is given by

$$\Omega_L = d \, \Theta_L \; .$$

Then, from (7.1), we have

$$(\Omega_L)^{km} \wedge dt \neq 0 \tag{7.3}$$

Hence Ω_L and dt define a contact structure on $J^{2k-1}(R,M)$ (see |BL|). Thus, there exists a unique vector field ξ_L on $J^{2k-1}(R,M)$ satisfying

$$i_{\xi_L} \Omega_L = 0 \quad , \quad dt(\xi_L) = 1 \tag{7.4}$$

Since $dt(\xi_L) = 1$, then ξ_L is locally given by

$$\xi_L = \partial/\partial t + \sum_{i=1}^{2k} X_i^A \partial/\partial q_{i-1}^A + \xi^A \partial/\partial q_{2k}^A \tag{7.5}$$

Because $i_{\xi_L} \Omega_L = 0$, we have

$$0 = \Omega_L(\xi_L, \partial/\partial q_{2k}^B) = d\,\Theta_L(\xi_L, \partial/\partial q_{2k}^B)$$

$$= -(\partial^2 L/\partial q_k^A \partial q_k^B)(q_{2k}^A - X_{2k}^A) .$$

Then, the regularity of L implies that

$$X_{2k}^A = q_{2k}^A ,$$

Now, let us suppose that $X_i^A = q_i^A$, $1 \leq i \leq s \leq 2k-2$. Then we have

$$0 = \Omega_L(\xi_L, \partial/\partial q_{s-1}^A) = d\,\Theta_L(\xi_L, \partial/\partial q_{s-1}^A)$$

$$= -(\partial^2 L/\partial q_k^A \partial q_k^B)(q_{s-1}^A - X_{s-1}^A) .$$

Therefore, we also have $X_{s-1}^A = q_{s-1}^A$. Hence (7.5) becomes

$$\xi_L = \partial/\partial t + \sum_{i=1}^{2k} q_i^A \partial/\partial q_{i-1}^A + \xi^A \partial/\partial q_k^A ,$$

or, equivalently,

$$\xi_L = \partial/\partial t + \sum_{i=1}^{2k} i y_i^A \partial/\partial y_{i-1}^A + \bar{\xi}^A \partial/\partial y_k^A \tag{7.6}$$

Then, from (7.6), we deduce that ξ_L is a semispray on $J^{2k-1}(R,M)$. Moreover, we have

$$\xi_L(p_A^1) - \partial L/\partial q^A = 0 \tag{7.7}$$

Now, taking into account (7.2), (7.7) becomes

$$\xi_L(\partial/\partial q_1^A) - \xi_L(d_T(\partial L/\partial q_2^A)) + \ldots + (-1)^k \xi_L(dt_T^{k-1}(\partial L/\partial q_k^A)) - \partial L/\partial q^A = 0 \tag{7.8}$$

Hence, if s is a path of ξ_L, then, from (7.8), we have

$$\sum_{i=0}^{k} (-1)^i \frac{1}{i!} \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial q_i^A}\right) = 0$$

along the canonical prolongation s^{2k-1} of s to $J^{2k-1}(R,M)$. Therefore, we have proved the following.

Proposition (7.1). Let $L: J^k(R,M) \longrightarrow R$ be a non-autonomous regular Lagrangian of order k on M. Then the vector field ξ_L satisfying (7.4) is a semispray on $J^{2k-1}(R,M)$ whose paths are the solutions of the generalized Lagrange equations (7.9).

We call ξ_L the Lagrange vector field for L.

Now, taking into account Proposition 5.1 and 6.1, we have

Theorem (7.1). Let $L : J^k(R,M) \longrightarrow R$ be a non-autonomous regular Lagrangian of order k on M and let ξ_L be the Lagrange vector field for L. Then there exists a dynamical connection Γ_L on $J^{2k-1}(R,M)$ whose paths are the solutions of the generalized Lagrange equations corresponding to L. This connection is given by

$$\Gamma_L = \Gamma - (\tilde{\xi}-\xi_L) \otimes dt ,$$

where $\Gamma = -\frac{1}{k} L_{\xi_L} \mathfrak{J}_1 + \frac{k-1}{k} (I-\xi_L \otimes dt)$, and $\tilde{\xi}$ is the associated semispray to Γ. (Here, $\mathfrak{J}_1$ is the canonical tensor field of type (1,1) on $J^{2k-1}(R,M)$).

Remark.- Obviously, if $k=1$, we have

$$\Gamma_L = L_{\xi_L} \mathfrak{J} \quad \text{(see |DLR1|)} .$$

References

|AA1| V. Aldaya & J. de Azcárraga, Geometric formulation of Classical Mechanics and field theory, Rev. Nuovo Cimento, 3, (3), (10), (1980), 1-66.

|AA2| V. Aldaya & J. de Azcárraga, Higher order Hamiltonian formalism in field theory, J. Phys. A ' Math. Gen. 13, (1982), 2545-2551.

|BL| D. Blair, Contact manifolds in Riemannian geometry. L. N. M.#509, Springer-Verlag, Berlin, 1976.

|CB| R. Clark & M. Bruckheimer, Sur les structures presque tangentes, C.R. Acad. Sc. Paris, 251, (1960), 627-629.

|CPT| M. Crampin, G.E. Prince & G. Thompson, A geometrical version of the Helmholtz conditions in time-depending Lagrangian dynamics, J. Phys. A : Math. Gen., 17, (1984), 1437-1447.

|CSC| M. Crampin, M. Sarlet & F. Cantrijn, Higher order differential equations and higher order Lagrangian mechanics, Math. Proc. Cambridge Phil. Soc., 99,3 (1986), 565-587.

|DLR1| M. de León & P.R. Rodrigues, Geometric structures in non-autonomous Lagrangian dynamics, Preprint, (1986).

|DLR2| M. de León & P.R. Rodrigues, Generalized Classical Mechanics and Field Theory, Notas de Matematica # 112, North-Holland, 1985.

|FK| M. Francaviglia & D. Krupka, On the globalizations of Lagrangian and Hamiltonian formalisms in higher order mechanics, IUTAM-ISIUM, Symposium on Modern Developments in Analytical Mechanics, Turin, June, 7-11, (1982), Acad. Sc. of Turin, 117, (1983), 109-12.

|GM| P.L. García & J. Muñoz, Higher order Analytical Dynamics, Procee dings of the VII ELAM, Univ. Simón Bolívar, Caracas, (1984),19-48.

|G1| J. Grifone, Structure presque tangente et connexions I, Ann, Inst. Fourier, Grenoble, 22 (3), (1972), 287-334.

|G2| J. Grifone, Structure presque tangente et connexions II, Ann. Inst. Fourier, Grenoble, 22(4), (1972), 291-338.

|HK| M. Hórak & I. Kolár, On the higher order Poincaré-Cartan forms, Czech. Math. J.,33,103, (1983), 467-475.

|K| I. Kolár, A geometrical version of the higher order Hamiltonian formalism in fibred manifolds, Preprint (1984).

|KR| D. Krupka, On the higher order Hamiltonian theory in fibred spaces, Geom. Methods in Physics, Proc. of the Conf. on Diff. Geom. and its

Applications, Czech., September, 5-9, (1985), 167-183.

|L| P. Libermann, On sprays and higher order connections, Proc. Nat. Acad. Sci. U.S.A., 49, (1963), 459-462.

|S| W.F. Shadwick, The Hamiltonian formulation of regular rth-order Lagrangian field theories, L.N. Ph.# 6, (1982), 409-416.

|T1| W. Tulczjew, Sur la différentielle de Lagrange, C.R.Acad. Sc. Paris, 280 A, (1975), 1295-1298.

|T2| W. Tulczyjew, The Lagrange complex, Bull. Soc. Math. France, 105, (1977), 419-431.

|Y1| K. Yano & S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.

MANUEL DE LEON
CECIME
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTIFICAS
SERRANO ,123
28006 MADRID
SPAIN

PAULO R. RODRIGUES
DEPARTAMENTO DE GEOMETRIA
INSTITUTO DE MATEMATICA
UNIVERSIDADE FEDERAL FLUMINENSE
24.000 - NITEROI, RJ
BRAZIL