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Generalized inverses of elliptic systems of differential operators with constant coefficients and related REDUCE programs for explicit calculations

F.Brackx, R.Delanghe and J.Van hamme

In this paper it is shown how the theory of generalized inverses for closed densely defined linear operators $\mathcal{L} : H_1 \to H_2$, H_1 and H_2 being Hilbert spaces, may be applied to the case where $\mathcal{L} = \mathcal{L}(D)$ is an elliptic matrix differential operator with constant coefficients. For $\mathcal{L}(D)$ the gradient operator in $\mathbb{R}^3$ an example is worked out and the explicit solution is constructed by means of a REDUCE program.

1. Introduction.

Let H_1, H_2 be Hilbert spaces and let $\mathcal{L} : H_1 \to H_2$ be a closed densely defined linear operator with domain $\mathrm{dom}(\mathcal{L})$, kernel $\eta(\mathcal{L})$ and range $R(\mathcal{L})$. Furthermore call $\mathcal{C}(\mathcal{L}) = \mathrm{dom}(\mathcal{L}) \cap \eta(\mathcal{L})^{\perp}$; then $\mathrm{dom}(\mathcal{L}) = \mathcal{C}(\mathcal{L}) \oplus \eta(\mathcal{L})$ and $\mathcal{L}$ admits a generalized inverse $\mathcal{L}^{-1}$ with $\mathrm{dom}(\mathcal{L}^{-1}) = R(\mathcal{L}) \oplus R(\mathcal{L})^{\perp}$ and $R(\mathcal{L}^{-1}) = \mathcal{C}(\mathcal{L})$. As is well known $\mathcal{L}^{-1} : H_2 \to H_1$ is also a closed densely defined linear operator. Moreover $L = \mathcal{L}^*\mathcal{L}$ is a non-negative self-adjoint operator and $\mathcal{L}$ admits the polar decomposition $\mathcal{L} = R\sqrt{L}$ whereby $R : H_1 \to H_2$ is a partial isometry called the elementary operator associated with $\mathcal{L}$. Denoting by M the spectral measure associated with L, $\mathcal{L}$ and $\mathcal{L}^{-1}$ admit the following spectral decomposition (see [2])

$$\mathcal{L} = \int_0^{+\infty} \sqrt{t}\, dRM \quad , \quad \mathcal{L}^{-1} = \int_0^{+\infty} \frac{1}{\sqrt{t}}\, dMR^*$$

whereby RM and MR^* are so-called generalized spectral measures w.r.t. R (see also [3]).

Now assume that $L_r = L|\eta(\mathcal{L})^{\perp}$ is a positive definite operator having a pure point spectrum $(\lambda_j)_{j\in\mathbb{N}}$; then, if $\langle Lf,f\rangle \geq C\|f\|^2$ for all

$f \in \mathrm{dom}(L_r)$, there exists an orthonormal basis $(f_j)_{j\in\mathbb{N}}$ of eigenvectors of L_r with

$$Lf_j = \lambda_j f_j \ , \ j\in\mathbb{N}$$
$$0 < C \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_j \leqslant \dots$$

and

$$\lim_{j\to\infty} \lambda_j = +\infty \ .$$

The following theorem may then easily be proved :

Theorem. Let $\pounds$ be a closed densely defined operator from H_1 into H_2, such that, if $L = \pounds^*\pounds$, $L_r = L|\eta(\pounds)^\perp$ is a self-adjoint positive definite operator having a pure point spectrum $(\lambda_j)_{j\in\mathbb{N}}$. Furthermore, let $(f_j)_{j\in\mathbb{N}}$ be a corresponding orthonormal basis consisting of eigenvectors of L_r, let R be the elementary operator associated with $\pounds$ and let $g_j = Rf_j$, $j\in\mathbb{N}$. Then

(i) $(g_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $R(R) = R(\pounds)$

(ii) $\mathrm{dom}\ \pounds = \left\{f\in H_1 : \sum_{j\in\mathbb{N}} \lambda_j |\langle f,f_j\rangle|^2 < +\infty\right\}$

(iii) $\mathrm{dom}\ \pounds^* = \left\{g\in H_2 : \sum_{j\in\mathbb{N}} \lambda_j |\langle g,g_j\rangle|^2 < +\infty\right\}$

(iv) $\mathrm{dom}(\pounds^{-1}) = H_2$ and $\mathrm{dom}\ (\pounds^{*-1}) = H_1$

(v) $\pounds f = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j}\ \langle f,f_j\rangle\ Rf_j \ , \quad f \in \mathrm{dom}(\pounds)$

(vi) $\pounds^* g = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j}\ \langle g,g_j\rangle\ R^* g_j \ , \quad g \in \mathrm{dom}(\pounds^*)$

(vii) $\pounds^{-1} g = \sum_{j\in\mathbb{N}} \frac{1}{\sqrt{\lambda_j}}\ \langle g,g_j\rangle\ R^* g_j \ , \quad g \in H_2$

(viii) $\pounds^{*-1} f = \sum_{j\in\mathbb{N}} \frac{1}{\sqrt{\lambda_j}}\ \langle f,f_j\rangle\ Rf_j \ , \quad f \in H_1$

(ix) $\pounds^{-1}$ and $\pounds^{*-1}$ are compact operators.

Corollary. For each $j\in\mathbb{N}$, $g_j = \frac{1}{\sqrt{\lambda_j}} \pounds f_j$ and $f_j = \frac{1}{\sqrt{\lambda_j}} \pounds^* g_j$. Moreover

$$Rf = \sum_{j\in\mathbb{N}} \langle f,f_j\rangle\ g_j \ , \quad f \in H_1$$

and

$$R^* g = \sum_{j\in\mathbb{N}} \langle g,g_j\rangle\ f_j \ , \quad g \in H_2$$

Remark. In this context the results and examples of M.R. Hestenes in [6] should also be mentioned.

2. Elliptic systems of differential operators

In what follows $\mathcal{L} = \mathcal{L}(D)$ stands for an elliptic system of differential operators with constant coefficients in $\mathbb{R}^n$, i.e.

$$\mathcal{L}(D) = [\mathcal{L}_{jk}(D)] \quad , \quad j = 1,\dots,M \; ; \; k = 1,\dots,N$$

whereby

(i) $$\mathcal{L}_{jk}(D) = \sum_{|\alpha| \leq r_{jk}} c_{jk\alpha} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}$$

with $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $c_{jk\alpha} \in \mathbb{C}$, $r_{jk} \in \mathbb{N}$

(ii) If $r = \max r_{jk}$,

$$\mathring{\mathcal{L}}_{ij}(D) = \sum_{|\alpha| = r} c_{jk\alpha} \frac{\partial^{r}}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}$$

and

$$\mathring{\mathcal{L}}(D) = [\mathring{\mathcal{L}}_{ij}(D)] \; ,$$

then the equation

$$\mathring{\mathcal{L}}(iy)\, \vec{\xi} = 0 \; , \quad \vec{\xi} \in \mathbb{C}^{N \times 1} \; ,$$

admits for each $y \in \mathbb{R}^n \setminus \{0\}$ the unique solution $\vec{\xi} = 0$.

Hereby $\mathring{\mathcal{L}}(iy)$ is the matrix obtained from $\mathring{\mathcal{L}}(D)$ by replacing $\frac{\partial^{\alpha_l}}{\partial x_l^{\alpha_l}}$ by $(iy_l)^{\alpha_l}$, $l = 1,\dots,n$.

Putting $\mathcal{L}^+(-D) = [\mathcal{L}^+_{kj}(-D)]$ whereby

$$\mathcal{L}^+_{kj}(-D) = \sum_{|\alpha| \leq r_{jk}} (-1)^{|\alpha|} \, \overline{c}_{jk\alpha} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}$$

we then have that

$$L(D) = \mathcal{L}^+(-D)\; \mathcal{L}(D)$$

is a strongly elliptic operator of order $2r$ (see [4]). Moreover, if for $\Omega \subset \mathbb{R}^n$ open and $J \in \mathbb{N}$, $L_{2,J}(\Omega)$ stands for the space of $\mathbb{C}^{J \times 1}$-valued L_2-functions in Ω, then we put $H_1 = L_{2,N}(\Omega)$, $H_2 = L_{2,M}(\Omega)$ and $V = \mathring{W}^r_{2,N}(\Omega)$.

In the sequel we assume that $r = 1$, i.e. $L(D)$ is a second order strongly elliptic operator, that Ω is bounded and of the class C^1 and that the Dirichlet problem for the operator $L(D)$ is well-posed in $\mathcal{N} \subset L_{2,N}(\Omega)$.

Taking $\mathrm{dom}(\mathcal{L}) = W_{2,N}(\Omega)$, we thus obtain that $L = \mathcal{L}^*\mathcal{L}$ is a positive definite self-adjoint operator with $\mathrm{dom}(L) = \mathcal{N}$ (see also [2]). Moreover, as the embedding of $\mathring{W}^1_{2,N}(\Omega)$ into $L_{2,N}(\Omega)$ is compact, L is an operator having a pure point spectrum, whence the results from section 1 may be applied. Hence we have a.o. that $\mathcal{L}^{-1} : L_{2,M}(\Omega) \to L_{2,N}(\Omega)$ is a bounded operator such that for each $g \in L_{2,M}(\Omega)$

$$\mathcal{L}^{-1}(g) = \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j} \langle g, \mathcal{L} f_j\rangle f_j \quad , \qquad (*)$$

$(f_j)_{j\in\mathbb{N}}$ being an orthonormal basis of $L_{2,N}(\Omega)$ consisting of eigenfunctions of L with corresponding eigenvalues λ_j.

3. The gradient operator in $\mathbb{R}^3$

3.1 Take Ω to be the unit ball B in $\mathbb{R}^3$, $H_1 = L_2(B)$, $H_2 = L_{2,3}(B)$

and $\mathcal{L}(D) = \text{grad} = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix}$, with $\text{dom}[\mathcal{L}(D)] = V = \mathring{W}_2^1(B)$. Then

$\mathcal{L}^+(-D) = -\begin{bmatrix} \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \end{bmatrix} = -\text{div}$ and $L(D) = \mathcal{L}^+(-D).\mathcal{L}(D) = (-\Delta)$.

An orthogonal basis of $L_2(B)$ consisting of eigenfunctions of $(-\Delta)$ is given (using spherical co-ordinates) by :

$$u_{l,m,k} = e^{i \sin\varphi}\, P_l^m(\cos\theta)\, r^{-1/2}\, J_{l+\frac12}(\mu_k^{(l+\frac12)} r) ,$$

$$l = 0,1,2,\ldots \quad ; \quad m = 0,1,\ldots,l \quad ; \quad k = 1,2,\ldots ,$$

the corresponding eigenvalues being given by

$$\lambda_{l,m,k} = \left[\mu_k^{(l+\frac12)}\right]^2 ,$$

where $\mu_k^{(l+\frac12)}$, $k = 1,2,\ldots$, represent the positive zeros of the Bessel function $J_{l+\frac12}$.

Putting

$$a_{l,m,k}^2 = \langle u_{l,m,k}, u_{l,m,k}\rangle$$

$$= \frac{2\pi}{2l+1}\,\frac{(l+m)!}{(l-m)!}\left[J_{l+\frac32}(\mu_k^{(l+\frac12)})\right]^2$$

and

$$b_{l,m,k} = \langle g, \text{grad } u_{l,m,k}\rangle \quad , \quad g \in L_{2,3}(B) ,$$

the unique solution f in V of the system

$$\text{grad } f = g \quad , \quad g \in L_{2,3}(B)$$

is given, accordingly to formula (*) of the previous section, by

$$f = \pounds^{-1}(g) = \sum_{l=0}^{\infty} \sum_{n=0}^{l} \sum_{k=1}^{\infty} \frac{1}{a^2_{l,m,k}} \frac{1}{\lambda_{l,m,k}} b_{l,m,k} u_{l,m,k} .$$

Given an arbitrary $g \in L_{2,3}(B)$ the computation of the solution f by the above formula is practically unfeasible. Therefore we developed REDUCE programs, which take over the by hand calculations; we used the version 3.2 of REDUCE [5] implemented on a VAX 750 computer. For a brief introduction on the nature of REDUCE see also [1].

3.2 The correctitude of the REDUCE programs had first to be checked on a case where the computation of the solution was possible by hand.

Therefore we focussed on the special case where $g \in L_{2,3}(B)$ is spherically symmetric, i.e. has the specific form $g = g(r)e_r$. In this case the constants $b_{l,m,k}$ are easily seen to be zero unless $l = m = 0$, which reduces the form the solution takes to

$$f = \pounds^{-1}(g) = \frac{\sqrt{2}}{4\pi} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} b_{0,0,k} \frac{\sin(k\pi r)}{k\pi r}$$

In this way the (known) potential of the unit ball B with homogeneous electrical charge may easily be computed. The electrostatic field is radial and of magnitude proportional to the distance to the origin, say $g = re_r$. This yields for the potential vanishing on the sphere ∂B :

$$v = -f = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \frac{\sin(k\pi r)}{k\pi r}$$

where the series is uniformly convergent in [0,1].

From the Fourier series of the function (r^3-r) it follows at once that the obtained series converges to the function $(1-r^2)/2$, of course the expected potential.

Our REDUCE program calculated exactly the terms of the above series; we show the first seven terms :

```
term(0,0,1) := ( - 6*SIN(R*PI))/(R*PI**3)$
term(0,0,2) := (3*SIN(2*R*PI))/(4*R*PI**3)$
term(0,0,3) := ( - 2*SIN(3*R*PI))/(9*R*PI**3)$
term(0,0,4) := (3*SIN(4*R*PI))/(32*R*PI**3)$
term(0,0,5) := ( - 6*SIN(5*R*PI))/(125*R*PI**3)$
term(0,0,6) := SIN(6*R*PI)/(36*R*PI**3)$
term(0,0,7) := ( - 6*SIN(7*R*PI))/(343*R*PI**3)$
```

3.3 Next by the same REDUCE program we solved the system

$$\text{grad } f = g \ , \ f \in V$$

for $g = 2r\cos\theta\, e_r + (1-r^2)/r \sin\theta\, e_\theta$, the solution of which is seen to be $f = (r^2-1)\cos\theta$. We computed the first seven terms of the series solution and found, after having introduced the zeros of $J_{3/2}$ with their numerical values :

```
term(1,0,1) := (COS(TH)*( - 0.0875874*SIN(4.493409*R) +
               0.3935659*COS( 4.493409*R)*R))/R**2$
term(1,0,2) := (COS(TH)*( - 0.0059394*SIN(7.725252*R) +
               0.0458836*COS( 7.725252*R)*R))/R**2$
term(1,0,3) := (COS(TH)*( - 0.0040778*SIN(10.90412*R) +
               0.0444648*COS( 10.90412*R)*R))/R**2$
term(1,0,4) := (COS(TH)*( - 0.0011643*SIN(14.06619*R) +
               0.0163777*COS( 14.06619*R)*R))/R**2$
term(1,0,5) := (COS(TH)*( - 0.0009338*SIN(17.22075*R) +
               0.0160806*COS( 17.22075*R)*R))/R**2$
term(1,0,6) := (COS(TH)*( - 0.0004089*SIN(20.3713*R) +
               0.0083294*COS( 20.3713*R)*R))/R**2$
term(1,0,7) := (COS(TH)*( - 0.0003496*SIN(23.51945*R) +
               0.008222*COS( 23.51945*R)*R))/R**2$
```

The value on the sphere ∂B of this partial sum turned out to be 0. with an error less than 10^{-6}.

3.4 We conclude this section by showing an excerpt of the REDUCE program to give a flavour of what it looks like. The complete programs can be obtained on simple request.

```
comment : This program computes the bessel-functions of order
          n+(1/2);
operator J;
J(1/2):=(2/(PI*z))**(1/2)*SIN(z);
J(3/2):=(2/(PI*z))**(1/2)*(SIN(z)/z - COS(z));
for i:=N1 step 2 until N2 do
<< begin scalar u;
   u:=i/2;
   J(u):=2*(u-1)*J(u-1)/z - J(u-2);
   end >>;
;end;
```

```
comment : this program computes the inner product of the
function G = GR e(r) + GT e(th) + GF e(fi)
with grad u(l,m,k);

procedure b(l,m);
    begin scalar e1,e2,e3,e4,e5,e6,e7,e8,e9;
    e1:= GR*dru(l,m)+(1/r)*GT*dthu(l,m)+(GF*dfiu(l,m))/(r*SIN(TH));
    e2:=INT(e1,FI);
    e3:=sub(FI=2*PI,e2) - sub(FI=0,e2);
    e4:=SIN(TH)*e3;
    e5:=INT(e4,TH);
    e6:=sub(TH=PI,e5) - sub(TH=0,e5);
    e7:=r**2*e6;
    e8:=INT(e7,r);
    e9:=sub(r=1,e8) - hosp(e8);
    return e9
    end;
;end;
```

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