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A FIELD EQUATION DEFINED BY A HURWITZ PAIR *

Jerzy Kalina, Julian Ławrynowicz (Łódź), and Osamu Suzuki (Tokyo)**

Contents

* This paper is in final form and no version of it will be submitted for publication elsewhere.

Summary

A field equation of the Dirac type is introduced by using Hurwitz pairs and its basic properties are investigated. A physical meaning of Hurwitz pairs is given. Isospectral deformations of the field equation define a soliton model of a Hurwitz pair. Main results are stated in Theorems I and II.

Introduction

In [3] and [4], A. Hurwitz considered the following problem and introduced a special kind of a pair of vector spaces: Find a pair (V, S) of Euclidean vector spaces which admits a bilinear mapping $f: V \times S \to V$ with the condition $\|f(x, y)\| = \|x\| \, \|y\|$ for $x \in V$ and $y \in S$, where $\| \ \|$ denotes the euclidean norm on V or S, respectively. We set $n = \dim V$ and $p = \dim S \leq n$. Hurwitz [4] discovered a remarkable fact that such a pair (V, S) satisfies very strong conditions on (n, p). We call such a pair a Hurwitz pair provided that it satisfies some additional conditions listed in Section 1. Moreover, he essentially discovered the following fact: In terms of a Hurwitz pair, we can obtain $p-1$ hermitian $n \times n$-matrices $\gamma_1, \ldots, \gamma_{p-1}$ with the following anticommutation rela-

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tions:

(0.1) $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2 I_n \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, p-1,$

where I_n is the identity $n \times n$-matrix. Then by recalling the relationship between Dirac matrices and the Dirac equation, we may hope to obtain field equations by the relation (0.1). This is a motivation of this paper. By extending the discussion about the Dirac equation, we can obtain a deep relationship between Hurwitz pairs and the field theory. Then the restriction to (p, n) may give a "selection rule" or "exclusion principle" for choosing elementary particles.

In [8] and [9], one of the authors contributed to discussing a relationship between the complex analysis and Hurwitz pairs. In [5], we have discussed a relationship between Hurwitz pairs and a special solution of the original K.-P. system (= Kadomtsev-Petviašvili system).

In this paper we propose two field equations, linear and non-linear, which are defined by a Hurwitz pair. The first equation is given in the following manner: For any positive integer $n \geq 2$, using matrices $\gamma_1, \gamma_2, \ldots, \gamma_{p-1}$, we can define a field equation with the following Hamiltonian $\mathcal{H}$ on $\mathbb{R}^{p-1}$:

(0.2) $$\mathcal{H} = i\hbar \sum_{\alpha=1}^{p-1} \gamma_\alpha (\partial/\partial x^\alpha).$$

This may be regarded as an analogue of the Dirac equation with zero mass. Since $\mathcal{H}^2 = \Delta_{p-1}$, we obtain a wave equation, which may be regarded as an analogue of the Klein - Gordon equation. In special cases $n = p = 2$ and $n = p = 4$ our equation is reduced to the Cauchy -Riemann equations (2.4) and to the Dirac equation (2.5), respectively. Moreover, we can obtain an interpretation of the Hurwitz pair in terms of solutions (2.6) of our field equation. Finally we can give the second quantization of our field and obtain a Fermion field. Next we define a soliton model by using isospectral deformations of our Hamiltonian. This is an analogue of the K.-P. system. Our isospectral deformations are defined in a special group whose Lie algebra is generated by $\gamma_1, \gamma_2, \ldots, \gamma_{p-1}$. Then by using the framework of generalised K.-P. systems which are given in [5], we can obtain soliton models. In the special case $n = p = 2$, our equation is reduced to the original K.-P. system of a special type.

We make some remarks on this paper. Here we are concerned only with an introductory part of field equations being obtained by

Hurwitz pairs. Detailed considerations will be given in a forthcoming paper. In particular, relationships between linear and nonlinear soliton equations will be discussed in connection with the relationship between the generalized Dirac equation and the nonlinear Heisenberg equation [6].

1. Hurwitz pairs

In this section we state the definition of a Hurwitz pair and its basic properties. Let V and S be n-dimensional and p-dimensional vector spaces, respectively, $n \geq p$, with the Euclidean norms $\|\cdot\|$. We consider a mapping $f: V \times S \to V$ with the following conditions:

(1) f is a bilinear mapping,

(2) $\|f(x, y)\| = \|x\| \cdot \|y\|$ for $x \in V$ and $y \in S$ (the so-called *Hurwitz condition*),

(3) there exists the unit $\varepsilon \in S$ such that $f(x, \varepsilon) = x$ for $x \in V$.

In order to exclude a trivial f, we introduce a concept of irreducibility of f. V is f-*irreducible*, if f does not leave invariant proper subspaces of V. In the following we assume:

(4) V is f-irreducible.

Definition (1.1). A pair (V, S) is called a *Hurwitz pair* if there exists a mapping f satisfying the conditions (1)-(4).

In 1898 Hurwitz [3] considered the above mentioned pairs in the case $p = n$ and proved the n of the Hurwitz pair to be 1, 2, 4 or 8 only. In 1923 he treated [4] general cases $n \geq p$ and listed up all the possibilities admissible. The most remarkable fact is that a Hurwitz pair gives a very strict condition on a pair of dimensions (n, p). We choose systems of orthonormal bases e_j, $j = 1, \ldots, n$, and ε_α, $\alpha = 1, \ldots, p$, of V and S, respectively. Then we have the following $n \times n$-matrices $C_\alpha = (C^k_{j\,\alpha})$, $\alpha = 1, \ldots, p$, by

$$f(e_j, \varepsilon_\alpha) = \sum_{k=1}^{n} c^k_{j\,\alpha} e_k .$$

The Hurwitz condition (2) in (1.1) can be stated as follows:

$$(1.2) \qquad {}^tC_\alpha C_\beta + C_\beta {}^tC_\alpha = 2\,\delta_{\alpha\beta} I_n, \quad \alpha, \beta = 1, \ldots, p,$$

$$(1.3) \qquad {}^tC_\alpha C_\beta = I_n, \quad \alpha = 1, \ldots, p,$$

where I_n is the $n \times n$-identity matrix [4], [7]. We define $\gamma_1, \dots, \gamma_{p-1}$ by $\gamma_\alpha = i C_p C_\alpha$, $\alpha = 1, \dots, p-1$. Then (1.2) and (1.3) can be written as (0.1) and

$$(1.4) \qquad {}^t\gamma_\alpha = -\gamma_\alpha, \qquad \operatorname{re} \gamma_\alpha = 0, \qquad \alpha = 1, \dots, p-1.$$

We remark that γ_α is an hermitian matrix. The following lemma is obtained [8]:

LEMMA (1.4). *The problem of classifying Hurwitz pairs is equivalent to the classification problem for real Clifford algebras with real antisymmetric generators* $i\gamma_\alpha$.

2. A field equation of the Dirac type

In this section we introduce and discuss a field equation related to a given Hurwitz pair (V, S). We define a field equation with n components on the p-dimensional space-time $x_1, \dots, x_{p-1}$, t by using $\gamma_1, \dots, \gamma_{p-1}$.

Let (V, S) be a Hurwitz pair. Then we obtain $\gamma_1, \dots, \gamma_{p-1}$ in (0.1) and (1.4). We define the Hamiltonian $\mathcal{H}$ by (0.2), so it is a self-adjoint operator on $\mathbb{R}^{p-1} = (x_1, \dots, x_{p-1})$.

Definition (2.1). By the field equation defined by a Hurwitz pair (V, S) or, simply, by the *Hurwitz equation* we mean the differential equation

$$(2.2) \qquad (i\hbar\, \partial/\partial t - \mathcal{H})\psi = 0, \qquad {}^t\psi = (\psi_1, \psi_2, \dots, \psi_n)$$

where t is a time parameter.

Acting on (2.2) with the operator

$$i\hbar\, \partial/\partial t + \mathcal{H},$$

we obtain the wave equation

$$(\partial^2/\partial t^2 - \Delta_{p-1})\Psi = 0,$$

where Δ_{p-1} is the usual Laplacian. This may be regarded as an analogue of the Klein-Goldon equation with zero mass. We are going to treat the special cases $n = p = 2$ or 4 and to compare our equations with the well known field equations.

PROPOSITION (2.3). *In the case* $n = p = 2$, *the equation* (2.2) *is reduced to the Cauchy-Riemann equations.*

Proof. In the case $n = p = 2$ we can choose

$$i\hbar \frac{\partial \Psi}{\partial t} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \hbar \frac{\partial \Psi}{\partial x}, \qquad {}^t\Psi = (\Psi_1, \Psi_2),$$

as the Hurwitz equation. Hence, if we set ${}^t\Psi = (i\,\Psi_1, \Psi_2)$, we obtain

$$(\partial/\partial t)\Psi_1 = -(\partial/\partial x)\Psi_2 \quad \text{and} \quad (\partial/\partial t)\Psi_2 = (\partial/\partial x)\Psi_1,$$

which proves the assertion.

Next, we consider the case $n = p = 4$. In this case, the following γ_1, γ_2, and γ_3 are admissible:

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & \end{bmatrix}.$$

Then we obtain the following

PROPOSITION (2.4). *In the case* $n = p = 4$, *the Hurwitz equation is the massless Dirac equation,* i.e., *the equation of neutrino.*

Proof. It is well known that the Dirac equation is defined by using the so called Dirac matrices γ_j, $j = 1,2,3,4$, as follows [12]:

$$i\hbar \frac{\partial\Psi}{\partial t} = i\hbar \Sigma_{\alpha=1}^{3} \gamma_\alpha \frac{\partial\Psi}{\partial x^\alpha} + m\,\gamma_4\,\Psi.$$

Hence we may regard our Hurwitz equation as the massless Dirac equation.

Now we proceed to restate the Hurwitz condition in terms of solutions of the Hurwitz equation. We consider the eigenfunction Ψ of the form

(2.5) $\quad \Psi = u_o\, e^{i\,k\cdot x}, \qquad {}^t u_o = (u_o^1 \ldots u_o^n),$

where $k \cdot x = k_1 x_1 + \ldots + k_{p-1} x_{p-1}$. Then we obtain

LEMMA (2.6). *We choose a system of* $n \times n$-*matrices* $\gamma_1, \ldots, \gamma_{p-1}$ *and consider the Hamiltonian* $\mathcal{H}$ *given by* (0.2). *Then the following statements are equivalent:*

(i) (V, S) *is a Hurwitz pair,*

(ii) $\gamma_1, \ldots, \gamma_{p-1}$ *satisfy the conditions* (0.1) *and* (1.4),

(iii) $\mathcal{H}$ *is a self-adjoint operator,* $\|\mathcal{H}\bar\Psi\|^2 = 2\hbar^2 \|k\|^2 \|\Psi\|^2$, *and* $\mathcal{H}$ *does not leave invariant proper subspaces of* V.

Proof. (i)$\Leftrightarrow$(ii) follows by Proposition (1.4). (ii) $\Rightarrow$(iii) is a consequence of $(\mathcal{H}\bar\Psi, \mathcal{H}\bar\Psi) = (\mathcal{H}^2\bar\Psi, \bar\Psi) = 2\hbar^2 \|k\|^2 \|\bar\Psi\|^2$. It remains to prove (iii) $\Rightarrow$(ii). Since $\bar\Psi$ is of the form (2.5), we have

$$\sum_i \Big| \sum_{j,\alpha} \gamma_{ij}^\alpha\, k_\alpha\, u_o^j \Big|^2 = 2(\Sigma\, k_\alpha^2)(\Sigma\, |u_o^j|^2).$$

so we obtain (ii), as desired.

By referring to $\mathcal{H}^2 = \Delta_{p-1}$, we can see that the eigenvalues of $\mathcal{H}$ are obtained by extracting square roots of the eigenvalues of Δ_{p-1}. Hence we may say that wave functions of the Hurwitz equation have the spin of half-integers [12].

By applying this fact we can define the second quantization of the Hurwitz equation and obtain a Fermion field. For simplicity sake, we treat our Hamiltonian on a $(q-1)$-dimensional torus T_{q-1} and consider the eigenvalue problem of $\mathcal{H}$. Then we can obtain a system of orthonormal bases of $L^2(T_{q-1}, V)$, the L^2-space of vector-valued functions on T_{q-1}. By labelling the bases in a suitable way, we denote them by $\{\Psi_{\mathbb{k}}\}$ (cf. e.g. [5], [12]). For $\Psi_{\mathbb{k}}$, we define creation and annihilation operators $a_{\mathbb{k}}$ and $a_{\mathbb{k}}^*$, respectively. We denote the Fock vacuum by $|0\rangle$ and we can obtain the anti-symmetric Fock space and a Fermion field as in the usual manner [12]. In a similar way we can define the second quantization of operators [5], [12].

Summarizing the above discussion we obtain the following

THEOREM 1. *Any Hurwitz pair defines a Hurwitz equation* (2.2) *on the* q-*dimensional space-time, which includes the Cauchy-Riemann equations* (2.3) *and the massless Dirac equation* (2.4) *as special cases. The Hurwitz condition can be expressed in terms of wave functions* (2.5). *The second quantization is defined and a Fermion field is obtained in a standard* (*above described*) *way.*

3. A soliton model

In this section we obtain a soliton model defined by a Hurwitz pair. M. and Y. Sato have obtained soliton equations, called K. - P. -systems by considering isospectral deformations of ∂x. We follow their idea by considering an isospectral deformation of $\mathcal{H}$ (0.2) in the case $n = p = 2$. In this case a Hurwitz equation is defined on $\mathbb{R}$, so we can generalize Satos' theory in reference to our situation. In the case $n = p = 2$, the Hamiltonian of the Hurwitz equation can be written as

$$(3.1)\qquad \mathcal{H} = \begin{bmatrix} 0 & i\,\partial x \\ -i\,\partial x & 0 \end{bmatrix} .$$

We define an isospectral deformation of $\mathcal{H}$ as follows: We set

$$P = I_2 + U_{-1}\partial x^{-1} + U_{-2}\partial x^{-2} + \dots + U_{-n}\partial x^{-n} + \dots ,$$

where

$$U_{-n} \in gl(2, \mathbb{C}).$$

We introduce infinitely many parameters t_n $(n = 1,2,\dots)$ and consider $P(t)$ whose element U_{-n} depends on t_n. We consider the following *isospectral deformation* L of $\mathcal{H}$:

(3.2) $L(t) = P(t)\mathcal{H}P(t)^{-1}$.

Following M. and Y. Sato [2], we define *a generalized K.-P.-system* for $\mathcal{H}$ by

(3.3) $(\partial/\partial t_n)L = [(L^n)_+, L]$,

where

$$(L^n)_+ = (P\mathcal{H}^n P^{-1})_+$$

and $(\ )_+$ means the part of non-negative order of the operator ∂x.

D e f i n i t i o n (3.4). The system (3.3) is called a *K.-P. system of Hurwitz type* and any isospectral deformation L, which satisfies the system (3.3), is called its *solution*.

In order to get the solutions, we have to introduce a special class of P: We choose a system of functions $u_{ij}(t)$ $(1 \leqq i, j \leqq 2)$, where $u_{ij}(t)$ is a formal power series of t_n. We denote the order of $u_{ij}(t)$ by $\mathrm{ord}(u_{ij}(t))$ and $\mathrm{ord}\, t_1^{j_1} \cdots t_n^{j_n} = \Sigma_{k=1}^{n} k\, j_k$. We denote the order of $U(t) = (u_{ij}(t))$ by

$$\mathrm{ord}\, U(t) = \min_{1 \leqq i, j \leqq 2} \mathrm{ord}\, u_{ij}(t).$$

We consider the following sets:

$$G = \{\Sigma_{n=-\infty}^{\infty} U_n(t)\, \partial x^n : \ U_n(t) \in gl(2, C) \text{ and there exists } N_0 \text{ such that } \mathrm{ord}\, U_n(t) > n - N_0 \text{ for any } n \text{ and } U_0(t) \text{ is invertible}\},$$

$$G_+ = \{\Sigma_{n=0}^{\infty} U_n(t)\, \partial x^n \in G\},$$

and

$$G_- = \{\Sigma_{n=-\infty}^{0} U_n(t)\, \partial x^n \in G : \ U_0(t) = I_2\}.$$

Then we can prove the following

LEMMA (3.5). G, G_+ *and* G_- *are groups and the following decomposition holds:*

$$G = G_- \cdot G_+ .$$

P r o o f: see [13].

To get the solutions of (3.3), we introduce the *linearization* of (3.3): we set

(3.6) $(\partial/\partial t_n)U = [\mathcal{H}^n, U]$.

LEMMA (3.7). *There exists a one-to-one correspondence between the solutions of* (3.3) *and those of* (3.6). *By a solution* U *of* (3.6), *we obtain a solution* L *of* (3.3) *as follows. We decompose* U:

$$U = W^{-1} V$$

by Lemma (3.5). *Then* $L = W^{-1} \mathcal{H} W$ *gives a solution of* (3.3).

Proof: see [5] (also [13]).

We investigate the solutions of (3.3) and show that they include solutions of well known equations. Firstly, we notice that

$$\mathcal{H}^{2n} = \begin{bmatrix} \partial x^{2n} & 0 \\ 0 & \partial x^{2n} \end{bmatrix} \quad \text{and} \quad \mathcal{H}^{2n+1} = \begin{bmatrix} 0 & i\,\partial x^{2n+1} \\ -i\,\partial x^{2n+1} & 0 \end{bmatrix}.$$

Hence we consider an isospectral deformation of the following form:

$$P = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \quad \text{or} \quad = \begin{bmatrix} 0 & i\,v \\ -i\,v & 0 \end{bmatrix},$$

where

$$u = \Sigma_{j=0}^{\infty} u^{(j)}\,\partial x^{-j} \qquad v = \Sigma_{j=0}^{\infty} v^{(j)}\,\partial x^{-j}.$$

Then we can see that (3.6) implies

$$\frac{\partial u}{\partial t_{2n}} = [\partial x^{2n}, u] \quad \text{resp.} \quad \frac{\partial v}{\partial t_{2n+1}} = [\partial x^{2n+1}, v].$$

Hence we can see that our equations include special type of the original K.-P. system.

Then we proceed to obtain solutions of some well known non-linear equations. We assume that L of (3.3) satisfies

$$L = \mathcal{H} + U, \quad \text{where} \quad U = \begin{bmatrix} u & i\,v \\ i\,v & u \end{bmatrix}.$$

Then the isospectral deformation of L implies that

$$L\,\Phi = \lambda\,i\,\Phi, \quad \Phi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

and λ is dependent of t_N. This can be written as

$$(3.8) \quad \begin{aligned} &i(\partial/\partial x)\psi_1 - i\,v\,\psi_1 - u\,\psi_2 = -\lambda\,i\,\psi_2, \\ &i(\partial/\partial x)\psi_2 + i\,v\,\psi_2 + u\,\psi_1 = \lambda\,i\,\psi_1. \end{aligned}$$

At first we notice that (3.8) is of a similar form as the so-called Abrowitz type ([10]). In fact we set

$$L_\sigma = \sigma \cdot L, \quad \text{where} \quad \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

then

$$L_\sigma \Phi = \lambda \Phi$$

is nothing but an equation of *Abrowitz type*. Here we choose $u = 0$. Then the equations (3.8) are reduced to

$$\psi_{1xx} + (\lambda^2 - v^2 - v_x)\psi_1 = 0,$$
$$\psi_{2}xx + (\lambda^2 - v^2 + v_x)\psi_2 = 0.$$

We set $v = iv'$. Then isospectral deformations of the equations of Schrödinger type give

$$v'_t + 6 v'^2 v'_x + v'_{xxx} = 0$$

which is nothing but a *modified K.-d V. equation* ([10]).

Next we consider an isospectral deformation of the following type:

$$L = \mathcal{K}_{\mathbb{C}} + U + U/\lambda, \tag{3.9}$$

where

$$\mathcal{K}_{\mathbb{C}} = \begin{bmatrix} 0 & 2i\,\partial_z \\ -2i\,\partial_{\bar z} & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & iu \\ -iu & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v^* & 0 \\ 0 & -v \end{bmatrix}.$$

Hence we consider an isospectral deformation of $\mathcal{K}_{\mathbb{C}}$ on the complex plane. In the following we denote the coordinate of $\mathbb{C}$ by $z = \eta + i\xi$.

We consider

$$L\psi = i\lambda\psi. \tag{3.10}$$

This implies

$$\begin{aligned} &2i(\partial\psi_1/\partial\xi) - 2(\partial\psi_1/\partial\eta) - iu\psi_2 + (v/\lambda + \lambda i)\psi_1 = 0, \\ &2i(\partial\psi_2/\partial\xi) + 2(\partial\psi_2/\partial\eta) - iu\psi_1 - (v^*/\lambda + \lambda i)\psi_2 = 0. \end{aligned} \tag{3.11}$$

Hence, if we choose a solution of the following isospectral deformation

$$2i(\partial/\partial\eta)\psi + \begin{bmatrix} u & \lambda \\ \lambda & -u \end{bmatrix}\psi = 0,$$

$$2i(\partial/\partial\xi)\psi + \begin{bmatrix} 0 & v/\lambda \\ v^*/\lambda & 0 \end{bmatrix}\psi = 0,$$

we obtain a special solution of (3.11).

If we choose $v = e^{iv'}$, then (3.12) implies that

$$u_\eta = v' \qquad v'_\xi + \sin u = 0.$$

Thus we conclude that the *Sine Gordon equations* are included in our deformations of complex form.

Finally we prove the following

PROPOSITION (3.13). *Any solution of* (3.3) *can be obtained by using a solution of the Hurwitz equation.*

Proof. By Lemma (3.7), we may *restrict* a solution U of (3.6). Then

$$(\partial/\partial t_1)U = [\mathcal{H}, U].$$

Hence U_n of $U = \Sigma_{n=-\infty}^{\infty} U_n \partial x^n$ satisfies

$$(\partial/\partial t_n)U_j = \mathcal{H} U_j \qquad (j = 0, \pm 1, \pm 2),$$

what proves the assertion.

From the above discussion we can obtain the following

THEOREM II. *Soliton equations are obtained by isospectral deformations of the Hurwitz equation in the case* $n = p = 2$. *Solutions of the soliton equations include special solutions of the original K.-P. system and the modified K.-d V. equations. There exist linearization equations of the soliton equations and the solution of the linear equations can be obtained by solutions of the Hurwitz equation.*

Remark 1. In the case of a general Hurwitz pair, we can define soliton models in a similar manner. It is probably preferable to treat isospectral deformations by using deformations of supercomplex structures. This will de discussed in the forthcoming paper.

Remark 2. The isospectral deformations in the case $n = p = 4$ will give us probably new understanding of the relationship between generalized Dirac equations and non-linear Heisenberg equations [6]. Another possibility is connected with a study of relativistic radial equations for spin-1/2 particles with a static interaction. Further investigations will be carried out in the forthcoming paper.

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