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by

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§1. Introduction.

C , a closed bounded convex nonempty subset of a Banach space X , has the <u>fixed point property</u> if every nonexpansive mapping $T: C \to C$ (i.e., $||Tx - Ty|| \le ||x - y||$ for all $x, y \in C$) has a fixed point. The study to determine those subsets of Banach spaces that have the fixed point property had its origins in four papers in 1965.

First, Browder [4], using concepts from the theory of monotone operators, showed that all closed bounded convex subsets of Hilbert space have the fixed point property. Then Browder [5] and, independently, Göhde [9] extended the result to all uniformly convex spaces. In the fourth seminal paper Kirk [11] further extended the result to all weakly compact convex subsets that have normal structure. (Recall that $x \in C$ is <u>diametral</u> if $\sup\{||x - y||: y \in C\} = \operatorname{diam} C$ and C has <u>normal structure</u> if every bounded convex subset of C with positive diameter has a non-diametral point.)

Compact convex subsets of arbitrary Banach spaces and closed bounded convex subsets of uniformly convex spaces have normal structure. Also, certain generalizations of uniform convexity imply normal structure, for example, uniformly convex in every direction [8, 20] and k-uniform rotundity [19]. Karlovitz [10], Odell and Sternfeld [15] and others extended the theory to spaces without normal structure. Lim [12] renormed ℓ_1 to obtain a nonexpansive self-mapping of a weak*-compact, convex set that is fixed point free.

The fundamental open question from the outset was whether or not weakly compact convex subsets of a Banach space have the fixed point property. Alspach [1] recently gave an example of such a subset of $L_1[0, 1]$ that is fixed point free. Namely, let $C = \{f \in L_1[0, 1]: \int_0^1 f = 1, 0 \le f \le 2 \text{ a.e.}\}$. Then C is a closed, convex subset of the order interval $[0, 2]$, hence C is weakly compact. Define $T: C \to C$ by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 2; & 0 \le t \le \frac{1}{2} \\ 2f(2t-1) - 2 \vee 0; & \frac{1}{2} < t \le 1 . \end{cases}$$

T is an isometry on C and has no fixed points. Subsequently, additional examples have been found, see, eg., Schechtman [17] and Sine [18].

The problem now is to determine which Banach spaces X have the <u>weak fixed point property</u> (wfpp), that is, every nonempty weakly compact convex subset of X has the fixed point property. In particular, is it true that all reflexive or superreflexive spaces have the wfpp? The purpose of this note is to indicate some recent results, which are lattice theoretic in spirit, concerning this problem.

§2. Banach Lattices.

For an introduction to Banach lattices the reader is referred to Lindenstrauss and Tzafriri [13] or Schaefer [16].

Let X be a Banach lattice. A sequence (x_n) in X is <u>weakly orthogonal</u> if $x_n \to x_0$ weakly and

$$\lim_n \lim_m \| |x_n - x_0| \wedge |x_m - x_0| \| = 0 .$$

$C \subseteq X$ is <u>weakly orthogonal</u> if every weakly convergent sequence in C is weakly orthogonal. The Banach lattice X is <u>weakly orthogonal</u> if every weakly compact convex subset of X is weakly orthogonal.

Clearly every weakly convergent monotone sequence is weakly orthogonal and every compact subset of X is weakly orthogonal. The following property is a sufficient condition for X to be weakly orthogonal.

A Banach lattice X has the <u>Riesz approximation property</u> (RAP) if there is a family $\mathcal{P}$ of linear projections with $P|x| = |Px|$, for all $P \in \mathcal{P}$, which satisfies:

i) $P(X)$ is a finite dimensional ideal;

ii) for each $x \in X$, $\inf\{\|Px - x\| : P \in \mathcal{P}\} = 0$.

<u>Proposition 2.1.</u> ([3]). If a Banach lattice X has the RAP, then X is weakly orthogonal.

Note that by taking $\mathcal{P}$ to be the standard bases projections it is easily seen that $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 \le p < \infty$, Γ any set, have the RAP. Also, RAP is preserved under lattice isomorphisms.

On the other hand, $\ell_\infty(\Gamma)$, $c(\Gamma)$ and $L_p[0, 1]$, $1 \le p \le \infty$, fail to have the RAP and, except for $c(\Gamma)$, fail to be weakly orthogonal.

The <u>Riesz angle</u> of a Banach lattice X is defined to be $\alpha(X) = \sup\{\| |x| \vee |y| \| : \|x\| \le 1, \|y\| \le 1\}$. Recall that the Banach-Mazur distance between two Banach spaces X and Y is $d(X, Y) = \inf \|U\| \cdot \|U^{-1}\|$ where the inf is over all linear isomorphisms U of X onto Y. If the inf is restricted to lattice isomorphisms, it is denoted $d_R(X,Y)$ and called the Riesz distance.

<u>Proposition 2.2.</u> ([3]).

a) For any Banach lattice X, $1 \le \alpha(X) \le 2$.

b) $\alpha(X) = 1$ if and only if X is an M-space.

c) If X is an abstract L_p space, $1 \le p \le \infty$, then $\alpha(X) = 2^{\frac{1}{p}}$.

d) For every pair of Banach lattices X and Y, $\alpha(Y) \le d_R(X, Y)\alpha(X)$.

§3. Weak Fixed Point Property.

Maurey [14], using ultrafilter techniques, obtained the following results.

<u>Theorem 3.1</u>. (Maurey) Every reflexive subspace of $L_1[0, 1]$ has the wfpp.

<u>Theorem 3.2.</u> (Maurey) c_0 has the wfpp.

Borwein and Sims [3], using classical methods, obtain Theorem 3.2 and most of the other known results and more concerning the wfpp.

<u>Theorem 3.3</u>. (Borwein and Sims) A Banach space X has the wfpp if there is a weakly orthogonal Banach lattice Y such that $d(X, Y)\,\alpha(Y) < 2$.

Some of the consequences of this surprisingly wide reaching theorem are given in the following corollaries.

<u>Corollary 3.4</u>. Let X be a weakly orthogonal lattice such that $\alpha(X) < 2$, then X has the wfpp.

<u>Corollary 3.5</u>. A Banach space X has the wfpp if, for some Γ and $1 < p < \infty$, $d(X, \ell_p(\Gamma)) < 2^{\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Corollary 3.6</u>. A Banach space X has the wfpp if either

i) $d(X, c(\Gamma)) < 2$, or

ii) $d(X, c\ (\Gamma)) < 2$.

It is known [7] that $d(c_0(\Gamma), c(\Gamma)) = 3$ so part (i) does not imply (ii) and conversely.

<u>Corollary 3.7</u>. $X_\lambda^{p,r} = (\ell_p(\Gamma), ||\cdot||_\lambda)$, where $||\cdot||_\lambda = (\lambda||\cdot||_p) \vee ||\cdot||_r$ and $1 < p < r \le \infty$, $\lambda > 0$, has the wfpp. Indeed, $X^{p,r}$ is a weakly orthogonal lattice and $\alpha(X_\lambda^{p,r}) \le \max\{\alpha(\lambda||\cdot||_p), \alpha(||\cdot||_r)\} = 2^{\frac{1}{p}} < 2$.

<u>Corollary 3.8</u>. All Banach spaces X such that $d(X, X_\lambda^{2,\infty}) < 2^{-\frac{1}{2}}$ have the wfpp.

The above two corollaries encompass several known results due to Karlovitz [10], Baillon and Schoneberg [2] and Bynum [6].

The next theorem, also due to Borwein and Sims, characterizes those order complete M spaces with the wfpp.

<u>Theorem 3.9</u>. Let X be a countably order complete M space. The following are equivalent:

(i) X has the wfpp.

(ii) X is isometric and lattice isomorphic to $c_0(\Gamma)$ for some index set Γ .

(iii) X is order continuous.

(iv) X has weakly compact order intervals.

(v) X contains no (lattice or norm) copy of L_∞ .

(vi) X contains no isometric copy of $L_1[0, 1]$.

Corollary 3.10. An abstract L_p space, $1 \leq p \leq \infty$, has the wfpp if and only if X contains no isometric copy of $L_1[0, 1]$.

These last two results suggest the conjecture that Corollary 3.10 obtains for arbitrary Banach spaces. This is further reinforced by Theorem 3.1.

Every Banach space X embeds isometrically in some $\ell_\infty(\Gamma)$. So the wfpp for X reflexive, respectively superreflexive, can be established by showing that all such subspaces of $\ell_\infty(\Gamma)$ are weakly orthogonal. Is this true? In particular, is it true for separable subspaces of ℓ_∞ ? If so, this would include Maurey's result Theorem 3.1.

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