Bernd Voigt

A common generalization of binomial coefficients, Stirling numbers and Gaussian Coefficientes

A COMMON GENERALIZATION OF BINOMIAL COEFFICIENTS, STIRLING NUMBERS AND GAUSSIAN COEFFICIENTS

B. Voigt

In this paper we present a common generalization of some basic enumeration problems. We show how well known recursion and inversion formulae fit into our model.

Let $A_0, A_1, A_2, \ldots$ be finite sets. For nonnegative integers n and k denote by $S_k^n(a_0, a_1, a_2, \ldots)$, where $a_i = |A_i|$, the number of words $w = (w_0, \ldots, w_{n-1})$ such that

(1) w contains k labels, say at positions $i_0, \ldots, i_{k-1}$,

(2) all entries in w before position i_0 belong to A_0, all entries in w between positions i_ℓ and $i_{\ell+1}$, where $\ell = 0, \ldots, k-2$, belong to $A_{\ell+1}$, all entries after position i_{k-1} belong to A_k.
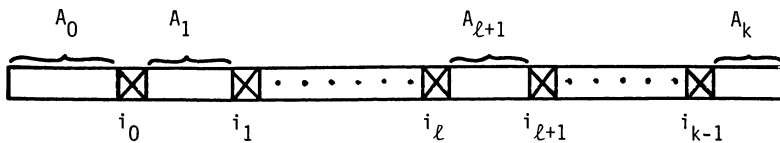
As $\displaystyle S_k^n(\vec{a}) = \sum_{0 \le i_0 < i_1 < \ldots < i_{k-1} < n} a_0^{i_0} \cdot a_1^{i_1 - i_0 - 1} \cdot \ldots \cdot a_k^{n - i_{k-1} - 1}$,

the numbers S_k^n can obviously be defined for sequences of complex numbers.

Examples:

(1) $S_k^n(1,1,\ldots) = \binom{n}{k}$ (Binomial coefficients)

(2) $S_k^n(0,1,2,\ldots) = S_k^n$ (Stirling numbers of the second kind)

(3) $S_k^n(1,q,q^2,\ldots) = \binom{n}{k}_q$ (Gaussian Binomial coefficients)

(4) $S_k^n(q,q^2,q^3,\ldots) =$ number of affine k-dimensional subspaces
                                          in the n-dimensional affine space over
                                          GF(q) .

(5) $S_k^n(2,3,4,\ldots) =$ number of Boolean sublattices $P(k)$ in
                                      $P(n)$
                                      ($P(n) \equiv$ lattice of subsets of an m-element
                                      set) .

Theorem 1

Let n be a nonnegative integer and let $a_0,\ldots,a_k$ be mutually distinct complex
numbers. Then

$$S_k^n(a_0,\ldots,a_k) = \frac{\det \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{k-1} & a_k \\ a_0^2 & a_1^2 & \cdots & a_{k-1}^2 & a_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{k-1} & a_1^{k-1} & \cdots & a_{k-1}^{k-1} & a_k^{k-1} \\ a_0^n & a_1^n & \cdots & a_{k-1}^n & a_k^n \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{k-1} & a_k \\ a_0^2 & a_1^2 & \cdots & a_{k-1}^2 & a_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{k-1} & a_1^{k-1} & \cdots & a_{k-1}^{k-1} & a_k^{k-1} \\ a_0^k & a_1^k & \cdots & a_{k-1}^k & a_k^k \end{bmatrix}}$$

$$= \sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k} (a_i - a_j)^{-1}$$

Proof:

As the determinant occuring in the denominator is van der Monde's determinant and
the determinant occuring in the numerator differs from van der Monde's determi-

nant only in the last row, the second equality follows immediately by expanding

the numerator with respect to the last row. Thus it suffices to show that

$$S_k^n(a_0,\ldots,a_k) = \sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k} (a_i-a_j)^{-1}$$

We proceed by induction on k, the case $k = 0$ being obviously valid. Let us

consider the case $k + 1$:

$$S_{k+1}^n(a_0,\ldots,a_{k+1}) = \sum_{-1=\mu_{-1}<\mu_0<\ldots<\mu_k<\mu_{k+1}=n} \frac{\overline{\hspace{3cm}}}{\overline{\hspace{3cm}}} \prod_{j=0}^{k+1} a_j^{\mu_j-\mu_{j-1}-1}$$

(distributivity) $\displaystyle = \sum_{i=0}^{n-k-1} a_0^i \cdot S_k^{n-i-1}$

(by induction) $\displaystyle = \sum_{i=0}^{n-k-1} a_0^i \cdot \sum_{\ell=1}^{k+1} a_\ell^{n-i-1} \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$

(changing summation) $\displaystyle = \sum_{\ell=1}^{k+1} \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1} \cdot \sum_{i=0}^{n-k-1} a_0^i \cdot a_\ell^{n-i-1}$

(distributivity) $\displaystyle = \sum_{\ell=1}^{k+1} a_\ell^k \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1} \cdot \sum_{i=0}^{n-k-1} a_0^i \cdot a_\ell^{n-k-i-1}$

(Cauchy-convolution) $\displaystyle = \sum_{\ell=1}^{k+1} a_\ell^k \cdot (a_0^{n-k}-a_\ell^{n-k}) \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$

$\displaystyle = \sum_{\ell=1}^{k+1} a_\ell^k \cdot a_0^{n-k} \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$

$\displaystyle - \sum_{\ell=1}^{k+1} a_\ell^n \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$

$\displaystyle = \sum_{i=1}^{k+1} a_i^n \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k+1} (a_i-a_j)^{-1}$

$\displaystyle - a_0^{n-k} \cdot \sum_{i=1}^{k+1} a_i^k \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k+1} (a_i-a_j)^{-1}$

Hence it remains to show that

$$0 = a_0^n \cdot \prod_{\substack{j=0 \\ j\neq 0}}^{k+1} (a_0-a_j)^{-1} + a_0^{n-k} \cdot \sum_{i=1}^{k+1} a_i^k \cdot \prod_{\substack{j=0 \\ j\neq i}}^{k+1} (a_i-a_j)^{-1}$$

Multiplying with $a_0^{k-n} \cdot \prod_{0\leq s<t\leq k+1} (a_s-a_t)$ yields the equivalent formulation

$$0 = \sum_{i=0}^{k+1} (-1)^i \cdot a_i^k \cdot \prod_{\substack{0\leq s<t\leq k+1 \\ s\neq i \\ t\neq i}} (a_s-a_t) \quad .$$

However, the expression on the right hand side is the determinant of the following matrix (expanded with respect to the first row):

$$\begin{bmatrix} a_0^k & a_1^k & \cdots & a_k^k & a_{k+1}^k \\ 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_k & a_{k+1} \\ a_0^2 & a_1^2 & \cdots & a_k^2 & a_{k+1}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^k & a_1^k & \cdots & a_k^k & a_{k+1}^k \end{bmatrix}$$

As the first and last row of this matrix are identical, the determinant is zero, thus completing the proof of the theorem.   ◻

It is wellknown that e.g.

$$\lim_{q\to 1} S_k^n(1,q,\ldots,q^k) = \lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k} \quad ,$$

thus it is desirable to have an explicit expression for the numbers $S_k^n(a_0,\ldots,a_k)$ also if some of the a_i's are equal.

Notation:

Let a be a complex number and let k be a positive integer. The k-tuple $(a,\ldots,a)$ consisting of precisely k a's is abbreviated by "$\langle a\rangle^k$" .

Theorem 2

Let n be a nonnegative integer and let $a_0,\ldots,a_\ell$ be mutually distinct complex

numbers. Let $k_0,\ldots,k_\ell$ be positive integers. Then

$$S^n_{(\Sigma k_i)-1}(<a_0>^{k_0},\ldots,<a_\ell>^{k_\ell}) = \sum_{i=0}^{\ell} \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[\prod_{\substack{j=0 \\ j\neq 1}}^{\ell} (a_i-a_j)^{-k_j} \right]$$

Remark:

By definition, the numbers $S^n_k(a_0,\ldots,a_k)$ are invariant under permutations of the

arguments, i.e. for every permutation $\tau : \{0,\ldots,k\} \to \{0,\ldots,k\}$ it follows that

$$S^n_k(a_0,\ldots,a_k) = S^n_k(a_{\tau(0)},\ldots,a_{\tau(k)}) \quad .$$

Thus the theorem gives an explicit characterization of the numbers $S^n_k(a_0,\ldots,a_k)$

in general.

Proof:

We proceed by induction on the sequence $(k_0,\ldots,k_\ell)$.

The beginning of the induction, viz. $k_0 = \ldots = k_\ell = 1$, has been established in

theorem 1.

For the inductive step it suffices to show that the validity of the assertion for

the sequence $(1,k_0,\ldots,k_\ell)$ implies the validity of the assertion for

$(k_0+1,k_1,\ldots,k_\ell)$.

Assume that for all x which are different from $a_0,\ldots,a_\ell$ it follows that

$$S^n_{\Sigma k_i}(x,<a_0>^{k_0},\ldots,<a_\ell>^{k_\ell}) = x^n \cdot \prod_{j=0}^{\ell} (x-a_j)^{-k_j}$$

$$+ \sum_{i=0}^{\ell} \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot$$

$$\cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[(a_i-x)^{-1} \cdot \prod_{\substack{j=0 \\ j\neq i}}^{\ell} (a_i-a_j)^{-k_j} \right]$$

The mapping $S_{\Sigma k_i}^n(\cdot, <a_0>^{k_0}, \ldots, <a_\ell>^{k_\ell}) : \mathbb{C} \to \mathbb{C}$ is continous, hence

$$\lim_{x \to a_0} S_{\Sigma k_i}^n(x, <a_0>^{k_0}, \ldots, <a_\ell>^{k_\ell}) = S_{\Sigma k_i}^n(<a_0>^{k_0+1}, <a_1>^{k_1}, \ldots, <a_\ell>^{k_\ell}) \quad .$$

We show that

$$\lim_{x \to a_0} S_{\Sigma k_i}^n(x, <a_0>^{k_0}, \ldots, <a_\ell>^{k_\ell}) = \sum_{\mu=0}^{k_0} \binom{n}{k_0-\mu} \cdot a_0^{n-k_0+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_0} \left[\prod_{j=1}^\ell (a_0-a_j)^{-k_j} \right]$$

$$+ \sum_{i=1}^\ell \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[(a_i-a_0)^{-k_0-1} \cdot \prod_{\substack{j=1 \\ j\neq i}}^\ell (a_i-a_j)^{-k_j} \right] \quad .$$

From elementary calculations it follows that

$$\lim_{x \to a_0} S_{\Sigma k_i}^n(x, <a_0>^{k_0}, \ldots, <a_\ell>^{k_\ell}) = \lim_{x \to a_0} (x-a_0)^{-k_0} \left(x^n \cdot \prod_{j=1}^\ell (x-a_j)^{-k_j} \right.$$

$$+ (x-a_0)^{k_0} \cdot \sum_{\mu=0}^{k_0-1} \binom{n}{k_0-1-\mu} \cdot a_0^{n-k_0+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_0} \left[(a_0-x)^{-1} \cdot \prod_{j=1}^\ell (a_0-a_j)^{-k_j} \right] \Bigg)$$

$$+ \sum_{i=1}^\ell \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[(a_i-a_0)^{-k_0-1} \cdot \prod_{\substack{j=1 \\ j\neq i}}^\ell (a_i-a_j)^{-k_j} \right] \quad .$$

We apply the rule of de l'Hospital to the first summand, observing that

$$\frac{\delta^{k_0}}{\delta^{k_0} x} \left[(x-a_0)^{k_0} \cdot \sum_{\mu=0}^{k_0-1} \binom{n}{k_0-1-\mu} \cdot a_0^{n-k_0+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_0} \left[(a_0-x)^{-1} \cdot \prod_{j=1}^\ell (a_0-a_j)^{-k_j} \right] \right] = 0$$

Hence

$$\lim_{x \to a_0} S_{\Sigma k_i}^n(x, <a_0>^{k_0}, \ldots, <a_\ell>^{k_\ell}) = \frac{1}{k_0!} \cdot \lim_{x \to a_0} \cdot \frac{\delta^{k_0}}{\delta^{k_0} x} \left[x^n \cdot \prod_{j=1}^\ell (x-a_j)^{-k_j} \right]$$

$$+ \sum_{i=1}^\ell \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[(a_i-a_0)^{-k_0-1} \cdot \prod_{\substack{j=1 \\ j\neq i}}^\ell (a_i-a_j)^{-k_j} \right]$$

$$= \sum_{\mu=0}^{k_0} \binom{k_0}{k_0-\mu} \cdot \frac{n!}{k_0! \cdot (n-k_0+\mu)!} \cdot a_0^{n-k_0+\mu} \cdot \frac{\delta^\mu}{\delta^\mu a_0} \left[\prod_{j=1}^\ell (a_0-a_j)^{-k_j} \right]$$

$$+ \sum_{i=1}^{\ell} \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{\,n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[\prod_{\substack{j=1 \\ j \neq i}}^{\ell} (a_i - a_j)^{-k_j} \right]$$

$$= \sum_{\mu=0}^{k_0} \binom{n}{k_0-\mu} \cdot a_0^{\,n-k_0+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_0} \left[\prod_{j=1}^{\ell} (a_0 - a_j)^{-k_j} \right]$$

$$+ \sum_{i=0}^{\ell} \sum_{\mu=0}^{k_i-1} \binom{n}{k_i-1-\mu} \cdot a_i^{\,n-k_i+1+\mu} \cdot \frac{1}{\mu!} \cdot \frac{\delta^\mu}{\delta^\mu a_i} \left[\prod_{\substack{j=1 \\ j \neq i}}^{\ell} (a_i - a_j)^{-k_j} \right] \quad .$$

This completes the proof of the theorem.

□

Remark:

Theorems 1 and 2 show that the numbers $S_k^n(a_0,\ldots,a_k)$ are divided differences, see e.g. [6]. For a treatment based on the calculus of finite differences see the forthcoming paper [2]. In order to keep this paper self contained we continue to give elementary proofs.

For the remainder of this section let $\vec{a} = (a_0,a_1,a_2,\ldots)$ denote an infinite sequence of (complex) numbers.

For convenience put $S_{-1}^n(\vec{a}) = 0$ for every nonnegative integer n .

Theorem 3 (Pascal identity for the S - numbers of the second kind)

$$S_k^{n+1}(\vec{a}) = S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a}) \quad .$$

Proof: obvious.

□

Definition:

For nonnegative integers let the polynomial $p_k^{\vec{a}}(x) \in \mathbb{C}[x]$ be defined as follows:
$p_0^{\vec{a}}(x) = 1$, viz. the polynomial which is constantly 1 ,

$$p_{k+1}^{\vec{a}}(x) = (x-a_k) \cdot p_k^{\vec{a}}(x) \quad , \quad \text{i.e.} \quad p_{k+1}^{\vec{a}}(x) = (x-a_0) \cdot (x-a_1) \cdot \ldots \cdot (x-a_k) \quad .$$

Examples:

(1) For $\vec{a} = (1,1,1,\ldots)$ it is $p_k^{\vec{a}}(x) = (x-1)^k$,

(2) for $\vec{a} = (0,1,2,\ldots)$ it is $p_k^{\vec{a}}(x) = [x]_k$, the falling factorial,

(3) for $\vec{a} = (0,-1,-2,\ldots)$ it is $p_k^{\vec{a}}(x) = [x]^k$, the rising factorial,

(4) for $\vec{a} = (1,q,q^2,\ldots)$ the polynomial $p_k^{\vec{a}}(x) = (x-1)\cdot(x-q)\cdot\ldots\cdot(x-q^{k-1})$

   is the k.th Gaussian polynomial.

Theorem 4   (Inversion from $(x^n)_{n\in\mathbb{N}}$ to $(p_n^{\vec{a}})_{n\in\mathbb{N}}$)

$$x^n = \sum_{k=0}^{n} S_k^n(\vec{a}) \cdot p_k^{\vec{a}}(x) \qquad .$$

Lemma:

Let k be a positive integer and let $b_0,\ldots,b_k$ be mutually distinct (complex)
numbers. Then

$$\sum_{i=0}^{k} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 0 \qquad .$$

Proof:

We use induction on k , the case k = 1 is obviously valid. Let us consider the
case k + 1 :

$$\sum_{i=0}^{k+1} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 1 - \frac{b_0-b_k}{b_0-b_1} + \sum_{i=2}^{k+1} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_j}$$

$$= \frac{b_k-b_1}{b_0-b_1} + \frac{b_k-b_1}{b_0-b_1} \cdot \sum_{i=2}^{k+1} (-1)^{i-1} \cdot \frac{b_0-b_k}{b_0-b_2} \cdot \prod_{j=2}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 0$$

using the inductive hypothesis on $b_0,b_2,\ldots,b_k$.

□

Proof of theorem 4

We prove the particular case, where the numbers $a_0,a_1,\ldots$ are mutually distinct.
The remaining cases follow from continuity reasons.

According to theorem 1 we show that

$$x^n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k} (a_i - a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (x - a_j)$$

As both sides of this equation are polynomials of degree n it suffices to show that

$$a_\ell^n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j \neq i}}^{k} (a_i - a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell - a_j)$$

for every $0 \leq \ell \leq n$. Fix any such ℓ . Consider, changing the summation:

$$(*) \qquad \sum_{i=0}^{n} a_i^n \cdot \left(\sum_{k=i}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{k} (a_i - a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell - a_j) \qquad .$$

For $i = 0, \ldots, n$ then consider each summand separately in order to find out what it contributes to the sum. We distinguish three cases:

Case 1:

$i > \ell$, then the contribution is zero, due to the factor $\prod\limits_{j=0}^{k-1} (a_\ell - a_j)$ which vanishes as $k \geq i > \ell$.

Case 2:

$i = \ell$, then we have the contribution a_ℓ^n , viz.

$$a_\ell^n \cdot \left(\sum_{k=\ell}^{n} \prod_{\substack{j=0 \\ j \neq \ell}}^{k} (a_\ell - a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell - a_j) = a_\ell^n \cdot \left(\prod_{j=0}^{\ell-1} (a_\ell - a_j)^{-1} \right) \cdot \prod_{j=0}^{\ell-1} (a_\ell - a_j)$$

$$= a_\ell^n \quad .$$

Case 3:

$i < \ell$, then the contribution is zero again, viz.

$$a_i^n \cdot \left(\sum_{k=i}^{n} \left(\prod_{j=0}^{i-1} (a_i - a_j)^{-1} \right) \cdot \left(\prod_{j=i+1}^{k} (a_i - a_j)^{-1} \right) \right) \cdot \prod_{j=0}^{k-1} (a_\ell - a_j) =$$

$$= a_i^n \cdot \prod_{j=0}^{i-1} (a_i - a_j)^{-1} \cdot \prod_{j=0}^{i-1} (a_\ell - a_j) \cdot \left[\sum_{k=i}^{\ell} (-1)^k \cdot \prod_{j=i}^{k-1} \frac{a_j - a_\ell}{a_i - a_{j+1}} \right] = 0$$

as according to the lemma the sum in the square brackets vanishes.

Hence the sum (*) takes the value a_ℓ^n , thus completing the proof of the theorem.

□

Definition:

The numbers $\delta_k^n(\vec{a})$ are defined by the following identities:

$$\sum_{j \geq 0} \delta_j^n(\vec{a}) \cdot S_k^j(\vec{a}) = \delta_k^n \qquad \text{(Kronecker-Symbol)} \quad ,$$

resp. $\sum_{j \geq 0} S_j^n(\vec{a}) \cdot \delta_k^j(\vec{a}) = \delta_k^n$, resp. $p_n^{\vec{a}}(x) = \sum \delta_k^n \cdot x^k$.

Remark: From the classical theory we know:

$$\delta_k^n(1,\ldots,1) = (-1)^{n+k} \binom{n}{k} \qquad \text{(Binomial inversion)} \quad ,$$

$$\delta_k^n(0,1,2,\ldots,k) = s_k^n \qquad \text{(Stirling numbers of the first kind)},$$

$$\delta_k^n(1,q,q^2,\ldots,q^k) = (-1)^{n+k} \cdot q^{\binom{n-k}{2}} \cdot \binom{n}{k}_q \quad \text{(Gaussian inversion)}$$

Theorem 5 (Pascal identity for the δ - numbers of the first kind)

$$\delta_k^{n+1}(\vec{a}) = \delta_{k-1}^n(\vec{a}) - a_n \cdot \delta_k^n(\vec{a}) \quad ,$$

where again for convenience $\delta_{-1}^n(\vec{a}) = 0$ for every nonnegative integer n .

Proof:

Consider the polynomial $p_{n+1}^{\vec{a}}(x) = x \cdot p_n^{\vec{a}}(x) - a_n \cdot p_n^{\vec{a}}(x)$ and compare the coefficients of x^k in the expansion $\sum_{j \geq 0} \delta_j^{n+1} \cdot x^j =$

$$= \sum_{j \geq 0} \delta_{j-1}^n \cdot x^j - \sum_{j \geq 0} a_n \cdot \delta_j^n \cdot x^j \quad .$$

□

Theorem 6

$$\delta_k^n = (-1)^{n+k} \cdot \sum_{0 \leq \mu_0 < \mu_1 < \ldots < \mu_{n-k-1} < n} \prod_{i=0}^{n-k-1} a_{\mu_i} \qquad \text{for every } k < n \ .$$

Proof:

We use induction on n . For $n = 0$ there is nothing to show. Thus consider the case $n + 1$. The particular case $k = n$ is treated separately, viz.

$$\delta_n^{n+1} = \delta_{n-1}^n - a_n \cdot \delta_n^n = - \sum_{i=0}^{n-1} a_i - a_n \cdot \delta_n^n = - \sum_{i=0}^{n} a_i$$

in accordance with the assertion.

Now let be $k < n$. Then

$$\delta_k^{n+1} = \delta_{k-1}^n - a_n \cdot \delta_k^n$$

$$= (-1)^{n+k-1} \cdot \sum_{0 \le \mu_0 < \ldots < \mu_{n-k} < n} \prod_{i=0}^{n-k} a_{\mu_i}$$

$$- (-1)^{n+k} \cdot a_n \cdot \sum_{0 \le \mu_0 < \ldots < \mu_{n-k-1} < n} \prod_{i=0}^{n-k-1} a_{\mu_i}$$

$$= (-1)^{n+k+1} \cdot \sum_{0 \le \mu_0 < \ldots < \mu_{n-k} < n+1} \prod_{i=0}^{n-k} a_{\mu_i}$$

in accordance with the assertion.

□

The connection between the matrices $[S_k^n(\vec{a})]$, $[\delta_k^n(\vec{a})]$ and the sequence $[p_n^{\vec{a}}]^T$ written as a column vector is given by the following two inversion formulae:

$$(*) \quad \left\{ \begin{array}{l} [p_n^{\vec{0}}]^T = [S_k^n(\vec{a})] \cdot [p_n^{\vec{a}}]^T \quad , \\[2mm] [p_n^{\vec{a}}]^T = [\delta_k^n(\vec{a})] \cdot [p_n^{\vec{0}}]^T \end{array} \right.$$

Of course, we also could have started from a given ascending sequence of normalized polynomials $(p_n)_{n \in \mathbb{N}}$, where ascending means that p_n divides (w.r.t. the ring $\mathbb{C}[x]$) p_{n+1}. As $\mathbb{C}$ is algebraically closed then $p_{n+1}(x) = p_n(x) \cdot (x-a_n)$ for some complex number a_n. As $p_n^{\vec{a}} = p_n$ for the sequence $\vec{a}$ of roots we again obtain the inversion formulae $(*)$.

Particularly the sequence $\vec{0} = (0,0,\ldots)$ yields that $p_n^{\vec{0}}(x) = x^n$ for every non-negative integer n.

Now let $\vec{a}$ and $\vec{b}$ be two infinite sequences. From the inversion formulae $(*)$ we immediately obtain inversion formulae for transforming the polynomials $(p_n^{\vec{a}})_{n \in \mathbb{N}}$

into $(p_n^{\vec{b}})_{n \in \mathbb{N}}$ and vice versa:

$$(**) \quad \begin{cases} [p_n^{\vec{a}}]^T = [\delta_k^n(\vec{a})] \cdot [S_k^n(\vec{b})] \cdot [p_n^{\vec{b}}]^T \\[2ex] [p_n^{\vec{b}}]^T = [\delta_k^n(\vec{b})] \cdot [S_k^n(\vec{a})] \cdot [p_n^{\vec{a}}]^T \end{cases}$$

Let us denote the numbers occuring in the matrices $[\delta_k^n(\vec{a})] \cdot [S_k^n(\vec{b})]$ by $S_k^n(\vec{a},\vec{b})$. Analogous let us denote the entries of $[\delta_k^n(\vec{b})] \cdot [S_k^n(\vec{a})]$ by $\delta_k^n(\vec{a},\vec{b})$, more precisely:

Definition:

$$S_k^n(\vec{a},\vec{b}) = \sum_{j \geq 0} \delta_j^n(\vec{a}) \cdot S_k^j(\vec{b}) \quad \text{and} \quad \delta_k^n(\vec{a},\vec{b}) = \sum_{j \geq 0} \delta_j^n(\vec{b}) \cdot S_k^j(\vec{a})$$

An well known example are the Lah numbers [4], let us consider here the unsigned Lah numbers:

One immediately verifies that $\delta_k^n(0,-1,-2,-3,\ldots) = |s_k^n|$, the absolute Stirling numbers of the first kind. Hence

$$L'^n_k = \sum_{j \geq 0} |s_j^n| \cdot S_k^j = S_k^n((0,-1,-2,\ldots),(0,1,2,\ldots)) \quad,$$

the signless Lah numbers, satisfy the identity

$$[x]^n = \sum_{k \geq 0} L'^n_k \cdot [x]_k \quad.$$

The following recursion for the unsigned Lah numbers is well known:

$$L'^{n+1}_k = L'^n_{k-1} + (k+n) \cdot L'^n_k \quad.$$

This recursion can be generalized as follows:

Theorem 7 (Pascal identity for $S_k^n(\vec{a},\vec{b})$ and $\delta_k^n(\vec{a},\vec{b})$)

$$S_k^{n+1}(\vec{a},\vec{b}) = S_{k-1}^n(\vec{a},\vec{b}) + (b_k - a_n) \cdot S_k^n(\vec{a},\vec{b})$$

$$\delta_k^{n+1}(\vec{a},\vec{b}) = \delta_k^n(\vec{a},\vec{b}) + (a_k - b_n) \cdot \delta_k^n(\vec{a},\vec{b}) \quad.$$

Proof:

We proceed by induction on n :

$$
\begin{aligned}
S_k^{n+1}(\vec{a},\vec{b}) &= \sum_{j \geq 0} \delta_j^{n+1}(\vec{a}) \cdot S_k^j(\vec{b}) \\
&= \sum_{j \geq 0} \left(\delta_{j-1}^n(\vec{a}) \cdot S_k^j(\vec{b}) \right) - a_n \cdot \sum_{j \geq 0} \delta_j^n(\vec{a}) \cdot S_k^j(\vec{b}) \\
&= \sum_{j \geq 0} \left(\delta_j^n(\vec{a}) \cdot S_k^{j+1}(\vec{b}) \right) - a_n \cdot S_k^n(\vec{a},\vec{b}) \\
&= \sum_{j \geq 0} \left(\delta_j^n(\vec{a}) \cdot S_{k-1}^j(\vec{b}) \right) + b_k \cdot \sum_{j \geq 0} \left(\delta_j^n(\vec{a}) \cdot S_k^j(\vec{b}) \right) - a_n \cdot S_k^n(\vec{a},\vec{b}) \\
&= S_{k-1}^n(\vec{a},\vec{b}) + (b_k - a_n) \cdot S_k^n(\vec{a},\vec{b}) \qquad .
\end{aligned}
$$

The second recursion follows from the first one, as $S_k^n(\vec{a},\vec{b}) = \delta_k^n(\vec{b},\vec{a})$.

□

The next recursions generalizes an identity for Gaussian binomial coefficients which has been discovered by Carlitz [1] .

Notation:

Let ℓ be a complex number. By $\vec{a} - \ell$ we denote the sequence $(a_0 - \ell, a_1 - \ell, \ldots)$, i.e. ℓ is subtracted from each component of $\vec{a}$.

Theorem 8

(i) $S_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a} - \ell)$

(ii) $S_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot S_j^n(\vec{a} - \ell)$

(iii) $\delta_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot \delta_k^j(\vec{a} + \ell)$

(iv) $\delta_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot \delta_j^n(\vec{a} + \ell)$.

Proof:

We prove (i) , the remaining cases can be handled analogously. Proceed by induction on n . The case n = 0 is obviously valid, thus let us consider the case n + 1 :

$$S_k^{n+1}(\vec{a}) = S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a})$$

$$= \sum_{j \geq 0} \left(\binom{n}{j} \cdot \ell^{n-j} \cdot S_{k-1}^j(\vec{a}-\ell) \right) + a_k \cdot \sum_{j \geq 0} \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a}-\ell)$$

$$= \sum_{j \geq 0} \left(\binom{n}{j} \cdot \ell^{n-j} \cdot (S_{k-1}^j(\vec{a}-\ell) + (a_k-\ell) \cdot S_k^j(\vec{a}-\ell)) \right)$$

$$+ \ell \cdot \sum_{j \geq 0} \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a}-\ell)$$

$$= \ell \cdot \binom{n}{0} \cdot \ell^n \cdot S_k^0(\vec{a}-\ell) + \sum_{j \geq 1} \left(\binom{n}{j} \cdot \ell^{n-j+1} + \binom{n}{j-1} \cdot \ell^{n-j+1} \right) \cdot S_k^j(\vec{a}-\ell)$$

$$= \binom{n+1}{0} \cdot \ell^{n+1} \cdot S_k^0(\vec{a}-\ell) + \sum_{j \geq 1} \binom{n+1}{j} \cdot \ell^{n+1-j} \cdot S_k^j(\vec{a}-\ell)$$

$$= \sum_{j \geq 0} \binom{n+1}{j} \cdot \ell^{n+1-j} \cdot S_k^j(\vec{a}-\ell) \qquad ,$$

completing the proof.

□

Remark:

The identity of Carlitz [1] appears considering (i) with $\vec{a} = (1, q^2, q^3, \ldots)$ and $\ell = 1$, viz. $\binom{n}{k}_q = \sum_{j \geq 0} \binom{n}{j} \cdot A_j^{(k)}$, where $A_j^{(k)} = S_k^j(0, q-1, q^2-1, \ldots)$ counts the number of k-dimensional linear subspaces W of the n-dimensional vector space $(GF(q))^n = \{(x_1, \ldots, x_n) | x_i \in GF(q)\}$, such that every projection $\pi_i : V \to GF(q)$ is surjective, where the projection π_i is defined by $\pi_i(x_1, \ldots, x_n) = x_i$ [5].

We should mention that theorem 8 (i) applied to Stirling numbers of the second kind does not yield the familiar recursion $S_k^{n+1} = \sum_{j \geq 0} \binom{n+1}{j} S_{k-1}^j$, simply because generally the numbers S_{k-1}^j and $S_k^j(-1, 0, \ldots, k-1)$ are not the same. However, the recursion $S_k^{n+1} = \sum_{j \geq 0} \binom{n+1}{j} \cdot S_{k-1}^j$ is a unique feature of the Stirling numbers of the second kind, more precisely:

Observation:

Let $a = (a_0, a_1, \ldots)$ be an infinite sequence of complex numbers such that

$$S_k^{n+1}(\vec{a}) = \sum_{j \geq 0} \binom{n+1}{j} \cdot S_{k-1}^j(\vec{a})$$

holds for every pair of nonnegative integers k and n. Then it follows that $a_i = i$, i.e. $S_k^n(\vec{a}) = S_k^n$.

Proof:

As $S_0^{n+1}(\vec{a}) = a_0^{n+1} = \binom{n+1}{0} \cdot S_{-1}^{n+1}(\vec{a}) = 0$ it follows that $a_0 = 0$. Assume by induction that $a_{k-1} = k-1$ and consider $S_k^{n+1}(\vec{a})$, viz.

$$S_k^{n+1}(\vec{a}) = S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a})$$

$$= \sum_{j \geq 0} \binom{n}{j} \cdot S_{k-2}^j(\vec{a}) + \sum_{j \geq 0} \binom{n}{j} \cdot a_k \cdot S_{k-1}^j(\vec{a})$$

$$= \sum_{j \geq 0} \left(\binom{n}{j} \cdot (S_{k-2}^j(\vec{a}) + (k-1) \cdot S_{k-1}^j(\vec{a})) \right) + (a_k - k + 1) \cdot \sum_{j \geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a})$$

$$= \left(\sum_{j \geq 1} \binom{n}{j-1} \cdot S_{k-1}^j(\vec{a}) \right) + \binom{n}{0} \cdot S_{k-1}^0(\vec{a}) + \left(\sum_{j \geq 1} \binom{n}{j} \cdot S_{k-1}^j(\vec{a}) \right)$$

$$+ (a_k - k) \cdot \sum_{j \geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a})$$

$$= \sum_{j \geq 0} \binom{n+1}{j} \cdot S_{k-1}^j(\vec{a}) + (a_k - k) \cdot \sum_{j \geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a}) ,$$

hence $(a_k - k) \cdot S_k^n(\vec{a}) = 0$, which shows that $a_k = k$. This completes the proof of the observation.

□

Let us apply theorem 8 (i) to Stirling numbers of the second kind and see what happens:

Consider the sequence $(-1, 0, 1, 2, 3, \ldots)$. Call the numbers

$$S_k^n(-1, 0, 1, 2, \ldots) = \hat{S}_k^n$$

the reduced Stirling numbers. By 8 (i) then

$$S_k^n = \sum_{j \geq 0} \binom{n}{j} \cdot \hat{S}_k^n .$$

The following table contains some values of the reduced Stirling numbers:

n \ k	0	1	2	3	4	5	6	7	8
0	1	0							
1	-1	1	0						
2	+1	-1	1	0					
3	-1	+1	0	1	0				
4	+1	-1	1	2	1	0			
5	-1	+1	0	5	5	1	0		
6	+1	-1	1	10	20	9	1	0	
7	-1	+1	0	21	70	56	14	1	0
8	+1	-1	1	42	231	294	126	21	1

For $k \geq 3$ the reduced Stirling numbers admit the following combinatorial interpretation:

Theorem 9

For $k \geq 3$ it follows that

$\hat{S}_k^n \equiv$ number of surjections $f : \{0,\ldots,n-1\} \to \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$ such that there exists an even nonnegative integer $\ell < n$ satisfying

(*) $\begin{cases} f^{-1}(\lambda_0) = \{0,\ldots,\ell\} \ , \\ f(\ell+1) = \lambda_1 \text{ and } f(\ell+2) = \lambda_2 \text{ and} \\ \min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_{i+1}) \quad \text{for every } i = 0,\ldots,k-2 \ . \end{cases}$

Proof: One immediately verifies (using Pascal identity) that

$\hat{S}_2^n = 1 \qquad$ iff $n \geq 2$ and $n \equiv 0 \pmod 2$

$\quad = 0 \qquad$ iff $n < 2$ or $n \not\equiv 1 \pmod 2$.

Now we use induction on n. The case $n = 3$ is obviously valid, hence consider $n+1$, viz. $\hat{S}_k^{n+1} = \hat{S}_{k-1}^n + (k-1) \cdot \hat{S}_k^n$.

Let $f : \{0,\ldots,n\} \to \{\lambda_0,\ldots,\lambda_{k-1}\}$ be any surjection satisfying (*) .

If still $f]\{0,\ldots,n-1\}$ acts surjectively onto $\{\lambda_0,\ldots,\lambda_{k-1}\}$ there exist pre-
cisely $k-1$ possibilities for $f(n)$, viz. $\lambda_1,\ldots,\lambda_k$. This explaines the
right summand. If $f]\{0,\ldots,n-1\}$ does not act surjectively, then $f(n) = \lambda_{k-1}$.
But in this case $f]\{0,\ldots,n-1\}$ acts surjectively onto $\{\lambda_0,\ldots,\lambda_{k-2}\}$. If still
$k-1 \geq 3$, $f]\{0,\ldots,n-1\}$ also satisfies (*) . This explaines the first summand.
If $k-1 = 2$ and f satisfies (*) , then it follows that $f^{-1}(\lambda_1) = \{n-1\}$, i.e.
$n-2$ is even and hence $\hat{S}_2^n = 1$. In both cases $\hat{S}_k^{n+1}$ turns out to be the right
number.

$\square$

We give two more examples applying theorem 8.

Example 1 (homogenous Boolean sublattices)

Let $P(n)$ denote the Boolean lattices of subsets of an n-element set. A $P(k)$-
sublattice L of $P(n)$ is a homogenous sublattice provided that $\min L = \min P(n)$.
By hB_k^n we denote the number of homogenous $P(k)$-sublattices of $P(n)$.
A homogenous $P(k)$-sublattice L of $P(n)$ is determined by its atoms, viz.
by k mutually disjoint and nonempty subsets $A_0,\ldots,A_{k-1}$. Without restriction
say that $\min A_1 < \min A_2 < \ldots < \min A_k$. L can be represented by a mapping
$f : \{0,\ldots,n-1\} \to \{\lambda_0,\ldots,\lambda_{k-1}\} \cup \{0\}$, where $f(i) = \lambda_j$ iff $i \in A_j$ and
$f(i) = 0$ in all other cases. Then

(*) $\min f^{-1}(\lambda_0) < \min f^{-1}(\lambda_1) < \ldots < \min f^{-1}(\lambda_{k-1})$.

On the other hand, every function $f : \{0,\ldots,n-1\} \to \{\lambda_0,\ldots,\lambda_{k-1}\} \cup \{0\}$
satisfying (*) determines uniquely a homogenous $P(k)$-sublattice of $P(n)$.
This establishes a bijection between homogenous sublattices and such functions f .
Hence $hB_k^n = S_k^n(1,2,3,\ldots)$.

Corollary 10

$$hB_k^n = \sum_j \binom{n}{j} \cdot S_k^j$$

$$\sum_k hB_k^n = \sum_k S_k^{n+1} = B_{n+1} \qquad \text{(Bell-number)} \qquad .$$

Proof:

The first equality is 8 (i) . Concerning the second inequality one observes that

$$\sum_k hB_k^n = \sum_k \sum_j \binom{n}{j} \cdot S_k^j$$

$$= \sum_j \binom{n}{j} \sum_k S_k^j$$

$$= \sum_j \binom{n}{j} \cdot B_k = B_{n+1} \qquad ,$$

where the last equality is well known.

□

Example 2   (Boolean sublattices)

By aB_k^n we denote the number of arbitrary (viz. affine) $P(k)$ - sublattices of $P(n)$. As a $P(k)$ - sublattice of P is determined by k nonempty subsets $A_0, \ldots, A_{k-1}$ which have pairwise the same intersection, it follows that

$$aB_k^n = S_k^n(2,3,4,\ldots) \quad .$$

Corollary 11

$$aB_k^n = \sum_j \binom{n}{j} \cdot 2^{n-j} \cdot S_k^j$$

$$\sum_k aB_k^n = B_{n+2} \qquad .$$

Proof: proceed as before.

□

We conclude with an application of the inversion formula for the δ - numbers of the first kind, deriving a recursion formula for Mac Mahon numbers.

Example   (Mac Mahon numbers)

The Mac Mahon numbers B_k^n , where $n \geq 1$ and $0 \leq k \leq \binom{n}{2}$, are defined by the following identities:

$$(*) \qquad \sum_{k=0}^{\binom{n}{2}} B_k^n \cdot q^k = \prod_{i=1}^{n} \frac{1-q^i}{1-q} \qquad .$$

Foata [3] gives a combinatorial interpretation for the numbers B_k^n .

Let $[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}$ be the   q-analogue of the nonnegative integer n . Then (*) can be rewritten as

$$(**) \qquad \sum_{k=0}^{\binom{n}{2}} B_k^n \cdot q^k = \prod_{i=1}^{n} [i]_q \qquad .$$

Consider any sequence $\vec{a} = (a_0, a_1, \ldots)$ of complex numbers such that for every nonnegative integer n the numbers $a_{\binom{n}{2}}, a_{\binom{n}{2}+1}, \ldots, a_{\binom{n}{2}+n-1}$ are the $(n+1)$-st roots of unity different from 1 . Say

$$\vec{a} = (e^{\frac{1}{2} \cdot \sqrt{-1} \cdot 2 \cdot \pi}, e^{\frac{1}{3} \cdot \sqrt{-1} \cdot 2 \cdot \pi}, e^{\frac{2}{3} \cdot \sqrt{-1} \cdot 2 \cdot \pi}, e^{\frac{1}{4} \cdot \sqrt{-1} \cdot 2 \cdot \pi}, \ldots)$$

i.e.

$$a_{\binom{n}{2}+j} = e^{\frac{1+j}{n+1} \cdot \sqrt{-1} \cdot 2 \cdot \pi} \qquad \text{for every } n \geq 1 \text{ and } 0 \leq j < n .$$

Then

$$(***) \qquad [n+1]_x = 1 + x + \ldots + x^n = \prod_{i=\binom{n}{2}}^{n-1} (x - a_i)$$

and thus

$$p_{\binom{n}{2}}^{\vec{a}}(q) = \prod_{i=1}^{n} [i]_q \qquad .$$

According to the definition of the δ - numbers of the first kind it follows that

$$\sum_{k=0}^{\binom{n}{2}} \delta_k^{\binom{n}{2}} \cdot q^k = \prod_{i=1}^{n} [i]_q \qquad ,$$

hence $B_k^n = \delta_k^{\binom{n}{2}}$ for every $n \geq 1$.

From theorem 6 we have the following explicit characterization for the numbers B_k^n:

$$B_k^n = (-1)^{\binom{n}{2}+k} \cdot \sum_{0 \leq \mu_0 < \ldots < \mu_{\binom{n}{2}-k-1} < \binom{n}{2}} \prod_{i=0}^{\binom{n}{2}-k-1} a_i \quad ,$$

where the complex numbers a_i have been defined above.

However, the Pascal-identity for the δ - numbers of the first kind (theorem 5)

yields a recursion for the Mac Mahon numbers:

Theorem (Recursion for Mac Mahon numbers)

$$B_k^{n+1} = B_{k-1}^{n+1} + B_k^n - B_{k-1-n}^n \quad ,$$

where we put $B_0^0 = 1$ and $B_k^n = 0$ if k or n (or both) are negative.

Proof: using induction it follows from theorem 5 that

$$\delta_k^{n+m} = \sum_{i=0}^{m} (-1)^i \left(\sum_{0 \leq \mu_0 < \ldots < \mu_{i-1} \leq m} \prod_{\nu=0}^{i-1} a_{n+\mu_\nu} \right) \cdot \delta_{k-m+i}^n$$

From (***) it follows that

$$(-1)^i = \sum_{0 \leq \mu_0 < \ldots < \mu_{i-1} < n} \prod_{\nu=0}^{i-1} a_{\binom{n}{2}+\mu_\nu} \quad ,$$

hence

$$B_k^{n+1} = \delta_k^{\binom{n+1}{2}} = \delta_k^{\binom{n}{2}+n} = \sum_{i=0}^{n} \delta_{k-n+1}^n$$

$$= \sum_{i=0}^{n} B_{k-i}^n \quad ,$$

and the desired recursion follows immediately.   □

BINOMIAL COEFFICIENTS

References:

[1]   L. Carlitz: On abelian fields, Trans. AMS, 35, 1933, 122 - 136 .

[2]   W. Dahmen und B. Voigt: The use of divided differences for basic
          enumeration problems, in preparation .

[3]   D. Foata: Distributions eulériennes et mahoniennes sur le groupe des
          permutations, in Higher Combinatorics, ed. M. Aigner, 1977, 27 - 49 .

[4]   I. Lah: Eine neue Art von Zahlen, ihre Eigenschaften und Anwendungen
          in der math. Statistik, Mitteilungsblatt f. math. Statistik, 7, 1955,
          203 - 212 .

[5]   S.C. Milne: Mappings of subspaces into subsets, JCT(A), 33, 1982,
          36 - 47 .

[6]   N.E. Nörlund: Vorlesungen über Differenzenrechnung, Chelsea, New York
          1959 (reprint) .

Address:

B. Voigt
Fakultät für Mathematik
Universität Bielefeld
4800 Bielefeld 1
West-Germany