# HORIZONTAL MAPS
# WITH HOMOGENEITY CONDITION

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1. Introduction. The first step towards a new development of differential geometry was accomplished by LEVI-CIVITA in 1917. Some years later - as S.KOBAYASHI pointed out in [11] - a sequence of CARTAN's papers on connections, published between 1923-25, "made the most remarkable progress in the history of differential geometry". Elie CARTAN's ideas was rigorously established by Ch. EHRESMANN, by the above-mentioned S. KOBAYASHI and others from a principal bundle viewpont in the fifties. - It seems to the author that since the seventies a "reinterpretation" of the theory has taken place from a purely vector bundle viewpoint. This remark is supported by such works as VILMS [18], [19], GRIFONE [6], DUC [3]. One of the sources of this approach is DOMBROWSKI's famous paper [2]. The utilisation of the vector bundle technique is simpler and more suitable in many questions and applications than the principal bundle technique. This is illustrated e.g. by the recent progress of the theory of FINSLER-connections (see [6], [10], [13], [14],) or of the differential geometry of higher order tangent bundles [12]. Taking for a basis a tensor product of vector bundles, we also have a particularly adequate setting for the formulation of BOMPIANI's tensorial connections; see e.g. [17].

In these notes we are going to sketch a foundation of the theory in this "new" spirit. Our main purpose is to discuss the so-called homogeneity condition imposed on a horizontal map. The results of these considerations are formulated in two theorems which characterize the linear connections and the homogeneous ones summarizing and supplementing some known results.

2. Notations. We will follow the terminology of the excellent monograph [5], as closely as feasible. All manifolds are finite dimensional, Hausdorff, second countable and smooth

i.e., infinitely differentiable . A differentiable map /or simply a map/ also means a smooth map, unless otherwise stated. B, i, and $\iota$ denote a fixed n-dimensional manifold, inclusion map, and identity map, respectively, throughout the paper.

/i/ If $M$ is a manifold, the $C^\infty(M)$ is the algebra of real-valued smooth functions on $M$, $T_xM$ is the tangent space at $x \in M$.

/ii/ $\xi = (E,\pi,B,F)$ will denote a vector bundle of rank $r$, where $E$, and $B$ are the total space, and the base space resp., $\pi : E \to B$ is the projection of $\xi$, $F$ and $F_x := \pi^{-1}(x)$ are the typical fibre, and the fiber at $x \in B$ resp.

/iii/ $\mathrm{Sec}\xi$ is the module over $C^\infty(B)$ of /smooth/ cross-sections of $\xi$. $\mathrm{Sec}\xi$ is a topological vector space with respect to the pointwise convergence, the differentiability of a map $\mathrm{Sec}\xi \to \mathrm{Sec}\xi$ means GATEAUX's differentiability of class $C^\infty$.

/iv/ $\tau_M = (TM,\pi_M,M,F)$ denotes the tangent bundle of the manifold $M$. Here $F_x := T_xM(x \in M)$, and $\dim F = \dim M$. $\mathfrak{X}(M) := \mathrm{Sec}\tau_M$ is the $C^\infty(M)$-module of vector fields on $M$.

/v/ $T\xi = (TE,d\pi,TB,\mathbb{R}^{2r})$ is the tangent fibration of the vector bundle $\xi$. For its definition see e.g. [7], Ch.IX.1 ; we shall give a local description of $T\xi$ in the proof of the Theorem 1.

/vi/ $A^p(B;\xi)$ is the module over the ring $C^\infty(B)$ of $\xi$-valued p-forms on $B(p \geq 0; A^0(B;\xi) := \mathrm{Sec}\xi)$,

$$i(X) : A^p(B;\xi) \to A^{p-1}(B;\xi)$$

is the substitution operator $(X \in \mathfrak{X}(B);\ p \geq 1)$.

/vii/ The indices $i,j,k,\ldots$ run over the range $\{1,2,\ldots,n\}$, while the indices $\alpha,\beta,\ldots$ run over the range $\{1,2,\ldots,r\}$. EINSTEIN's summation convention is applied accordingly.

Remark: We recall, that $\Omega \in A^p(B;\xi) :\Longleftrightarrow \Omega : x \in B \mapsto \Omega_x$ /smoothly/, where $\Omega_x : T_xB\times\ldots\times T_xB \to F_x$ is a skew-symmetric p-linear map. If specially $\Omega \in A^1(B;\xi)$, then $i(X)\Omega \in \mathrm{Sec}\xi$, namely $(i(X)\Omega)(x) := \Omega_x[X(x)]$.

3. Vertical subbundle. The differential of the projection $\pi : E \to B$ is a strong bundle map $d\pi : \tau_E \to \tau_B$ of constant rank, hence

$$\mathrm{Ker}\, d\pi := V_\xi = (VE,\pi_V,E,F)$$

/where $VE := \bigcup_{z\in E} \operatorname{Ker}(d\pi)_z$; $\pi_V : VE \to E$, $\operatorname{Ker}(d\pi)_z \ni a \mapsto z$/

is a vector subbundle of $\tau_E$, which is called vertical subbundle. The fibers $V_zE := \operatorname{Ker}(d\pi)_z$ are mentioned as vertical subspaces.

Let $i_x$ denote the inclusion $F_x \longrightarrow E$ and let $I_z$ be a /fixed/ identifying isomorphism $F_x \xrightarrow{\cong} T_zF_x$ /For example, the map $u \in F_x \mapsto I_z(u) := (dc_u)_o \left(\frac{d}{dt}\right)_o$, where $c_u : \mathbb{R} \to F_x$, $\tau \mapsto c_u(\tau) := z + \tau u$ is a straight line of $F_x$ through the point $z$ in the direction $u$./

The following well-known facts are collected for the readers convenience.

/a/ $V_zE = \operatorname{Im}(di_x)_z$ /[5], Vol.I, p. 280/.

/b/ The linear isomorphisms

$$\alpha_z := I_z^{-1} \circ (du_x)_z^{-1} : V_zE \to F_{\pi(z)}$$

generate a bundle map $\alpha : V_\xi \to \xi$ inducing $\pi$ as the map of base manifolds, that is which makes the diagram

$$\begin{array}{ccc} VE & \xrightarrow{\alpha} & E \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & B \end{array}$$

commutative /[5], Vol.I, p. 291/. This map $\alpha$ will be referred to as canonical map.

/c/ With the help of the canonical map, we can construct the vector field $C : E \to TE, z \mapsto C(z) := i \circ \alpha_z^{-1}(z)$. This vector field is also called canonical; cf. e.g. [9] or [14].

## 4. Horizontal map: the basic constructions

It is a simple, but fundamental fact, that the sequence

$$0 \to V_\xi \xrightarrow{i} \tau_E \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \longrightarrow 0$$

/where $\widetilde{d\pi}|_{T_zE} := (d\pi)_z$, and $\pi^*(\tau_B)$ is the pull-back of $\tau_B$ over $\pi$/ is a short exact sequence of vector bundles. A further basic result is that all short exact sequences of vector bundles are split: there always exists a strong bundle map $H : \pi^*(\tau_B) \to \tau_E$ such that $\widetilde{d\pi} \cdot H = 1 : \pi^*(\tau_B) \to \pi^*(\tau_B)$.

/The proof of this is much less trivial, the crucial point is the countability axiom presupposed for the manifolds./

The following notion, which is central in our treatment, is based on these observations.

<u>Definition 1</u>: The splittings of the short exact sequence

$$0 \longrightarrow V_\xi \xrightarrow{\ i\ } \tau_E \xrightarrow{\ \widetilde{d\pi}\ } \pi^*(\tau_B) \longrightarrow 0$$

are called <u>horizontal maps.</u>

Let $H$ be a horizontal map. It is easy to see that rank $H=$ $=$const., consequently one can consider the so-called <u>horizontal subbundle</u> $\mathrm{Im}H = (HE, \pi_H, E, \mathbb{R}^n)$ of $\tau_E$, where $HE$ /as a set/ is the union of the <u>horizontal subspaces</u>

$$H_zE := \mathrm{Im}H_z \ , \ H_z := H\big|_{T_{\pi(z)}B} : T_{\pi(z)}B \to T_zE \ .$$

It is very simple to prove the next

<u>Lemma 1</u>: $\tau_E = \mathrm{Im}H \oplus V_\xi$ /Whitney-sum/. □

<u>Lemma 2</u>: The map $\widetilde{d\pi}\big|_{\mathrm{Im}H} : \mathrm{Im}H \to \pi^*(\tau_B)$ is a strong bundle isomorphism.

<u>Proof</u>: By Lemma 1 the restricted maps

$$(d\pi)_z\big|_{H_zE} : H_zE \to T_{\pi(z)}B$$

are linear isomorphisms. On the common total space of the considered bundles the map $\widetilde{d\pi}\big|_{\mathrm{Im}H}$ induces the identity in the sense that the diagram

$$\begin{tikzcd}
HE \arrow[r, "\widetilde{d\pi}|_{\mathrm{Im}H}"] \arrow[d, "\pi_H"'] & \bigcup\limits_{z\in E} T_{\pi(z)}B \arrow[d, "\text{projection}"] \\
E \arrow[r, dashed, "1"] & E
\end{tikzcd}$$

commutes. These two facts together imply the assertion /cf. [5], Vol.I 2.1 Prop.I/. □

The next important constructions are partially based on Lemma 2.

<u>Definition 2</u>: Let a fixed horizontal map $H : \pi^*(\tau_B) \longrightarrow \tau_E$ be given.

/1/ The map

$$\mathfrak{l}^h := \left(\widetilde{d\pi}\big|_{\mathrm{Im}H}\right)^{-1} : \pi^*(\tau_B) \to \mathrm{Im}H$$

is the <u>horizontal lift</u> belonging to $H$.

/2/ The horizontal lift of a vector field $X \in \mathfrak{X}(B)$ is the vector field

$$X^h \in \mathfrak{X}_H(E) := \mathrm{Sec}\,\mathrm{Im}H,$$

given by

$$X^h(z) := \ell_z^h[X(\pi(z))],\ \ell_z^h := \ell^h|_{T_{\pi(z)}B}\quad (z \in E).$$

/$\mathcal{X}_H(E)$ is the module of all horizontal vector fields on $E$./

/3/ $h := H \circ \widetilde{d\pi}$ , $v := \iota - h$ , $K := \alpha \circ v$ are the horizontal projection, the vertical projection, and the DOMBROWSKI - /or connection -/ map respectively, induced by $H$.

/4/ We say that $H$ satisfies the homogeneity condition, if $\forall t \in \mathbb{R} : d\mu_t \circ h = h \circ d\mu_t$ where $\mu_t : E \to E$ , $z \mapsto tz$.

For the introduced maps a simple calculation shows that the following relations hold.

Proposition 1: $h^2 = h,\ \ v^2 = v,\ \ h \circ v = v \circ h = 0;$
$K \circ \ell^h = \sigma,\ \ K \circ C = \iota;$
$K \circ h = \sigma,\ \ K \circ v = K.$ □

Remarks:

1/ The definition of $\ell^h$ adopted here agrees with the definition given in [19]. It is a natural generalization of the construction known from the differential geometry of tangent bundles /cf. e.g. [2]/.

2/ Fixing some suitable properties of the vertical projection $v$ as axioms, one can obtain a formally different, alternative approach to the theory. So for example the definition in [3], Ch.II.8 formulates that peculiarity of $v$ that it assigns a vertical vector to an arbitrary vector of $TE$, while the vertical vectors remain fixed /see the Proposition/. The so-called almost product structure $P := 2h - \iota = h - v$ /for which $P^2 = \iota$/ can play a similar role /cf. [6] , Prop.I.15, where, however, $\xi = \tau_B$/.

3/ Starting from a linear connection of the tangent bundle, the connection map was constructed by P.DOMBROWSKI in his above-mentioned paper [ 2 ]. The importance - and what is more: the "career" - of the DOMBROWSKI - map in the foundation of the theory /especially in the infinite - dimensional generalizations/ is well-known; see e.g. [4]/.

5. Local description. Suppose that the open set $U \subset B$ is a trivializing neighbourhood for $\xi$ with the trivializing map $\psi_U : U \times F \to \pi^{-1}(U)$. Let $u^1,\dots,u^n$ be a local coordinate system defined on $U$, and let a basis of the conjugate space $L(F)$ be denoted by $\ell^1,\dots,\ell^r$. Then the system of functions

$$x^i := u^i \circ \pi \ , \ y^\alpha := \ell^\alpha \circ pr_2 \circ \psi_U^{-1} \quad (pr_2 : U \times F \longrightarrow F)$$

is a local coordinate system on $\pi^{-1}(U)$, which will be fixed in the sequel. Obviously, the restrictions of the functions $y^\alpha$ to an arbitrary fiber $F_x$ $(x \in U)$ constitute a basis of $L(F_x)$. If its dual is $\{e_\alpha(x)\}$, then the maps

$$e_\alpha : U \longrightarrow E \ , \ x \longmapsto e_\alpha(x)$$

are such cross-sections over $U$, which provide a <u>framing</u> for $\xi$ over $U$; this framing is called <u>induced</u> by the local coordinate system $\{x^i, y^\alpha\}$.

Fixing $\{x^i, y^\alpha\}$, we have the basis $\left\{\left(\frac{\partial}{\partial x^i}\right)_z, \left(\frac{\partial}{\partial y^\alpha}\right)_z\right\}$ of the tangent space $T_zE$ at an arbitrary point $z \in \pi^{-1}(U)$. Here the vectors $\left(\frac{\partial}{\partial y^\alpha}\right)_z$ are basis vectors of the vertical subspace $V_zE$, so it is easy to see that the local form of the canonical vector field is $C = y^\alpha \frac{\partial}{\partial y^\alpha}$ .

For each horizontal map $H : \pi^*(\tau_B) \longrightarrow \tau_E$ it is possible to find in a unique way the differentiable functions $\Gamma_i^\alpha : \pi^{-1}(U) \longrightarrow \mathbb{R}$ such that the linear map $H_z : T_{\pi(z)}B \longrightarrow T_zE$ / $z \in \pi^{-1}(U)$ is an arbitrary point/ is represented by the matrix

$$\begin{array}{c} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ & \boxed{-\Gamma_i^\alpha(z)} & \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\ \vdots\\ 0\end{matrix}}\right\} n \\ \left.\vphantom{\boxed{x}}\right\} r \end{array} \\ \underbrace{\hphantom{xxxxxxxxxx}}_{n} \end{array}$$

of type $(n+r) \times n$ with respect to the basis-pair $\left\{\left(\frac{\partial}{\partial u^i}\right)_{\pi(z)}\right\}$ , $\left\{\left(\frac{\partial}{\partial x^i}\right)_z , \left(\frac{\partial}{\partial y^\alpha}\right)_z\right\}$. These functions $\Gamma_i^\alpha$ are called the <u>connection parameters</u> for $H$ /the minus sign in the defining matrix is traditional/. Having them, a simple calculation yields the local form of the maps defined in the previous section.

/a/ The vector $a \in T_zE(z \in \pi^{-1}(U))$ is a horizontal one iff $a = a^i\left[\left(\frac{\partial}{\partial x^i}\right)_z - \Gamma_i^\alpha(z)\left(\frac{\partial}{\partial y^\alpha}\right)_z\right]$ $(a^i \in \mathbb{R})$. In particular, the vectors $b_i = \left(\frac{\partial}{\partial x^i}\right)_z - \Gamma_i^\alpha(z)\left(\frac{\partial}{\partial y^\alpha}\right)_z$ constitute a basis of $H_zE$.

If $X = X^i \frac{\partial}{\partial u^i}$ is a vector field over $U$, then $X^h = (X^i \circ \pi)\left(\frac{\partial}{\partial x^i} - \Gamma_i^\alpha \frac{\partial}{\partial y^\alpha}\right)$, thus $b_i = \left(\frac{\partial}{\partial u^i}\right)^h$. It follows, that $\left\{\left(\frac{\partial}{\partial u^i}\right)_z^h, \left(\frac{\partial}{\partial y^\alpha}\right)_z\right\}$ is a basis of $T_zE$, which is called <u>adapted basis</u>. /This is analogous to the adapted frame, used in the case of $\xi = \tau_B$ ; cf. [20]./

/b/ The projectors $h_z := h|_{T_zE}$ , and $v_z := v|_{T_zE}$ are represented by diagonal matrices of rank $n$ and $r$, resp., with respect to the adapted basis.

/c/ Let $a = a^i\left(\frac{\partial}{\partial x^i}\right)_z + a^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)_z \in T_zE$ be arbitrary and let the restriction of the DOMBROWSKI-map $K$ to the tangent space $T_zE$ be denoted by $K_z$. The action of $K_z$ is given by the formula

$$a \longmapsto K_z(a) = (a^\alpha + a^i\Gamma_i^\alpha(z))e_\alpha(x), \quad x = \pi(z).$$

## 6. Connections.

<u>Definition 3</u>: Let $K$ be the DOMBROWSKI-map, belonging to the horizontal map $H$. The map

$$\nabla : \mathrm{Sec}\,\xi \longrightarrow A^1(B;\xi) \ , \ \sigma \longmapsto \nabla\sigma := K \circ d\sigma$$

is called the <u>general connection</u> induced by $H$, and the maps

$$\nabla_X := i(X) \circ \nabla : \mathrm{Sec}\,\xi \longrightarrow \mathrm{Sec}\,\xi \quad \left(X \in \mathcal{H}(B)\right)$$

are called the <u>covariant derivatives</u> by $X$ with respect to $\nabla$. We talk about <u>linear connection</u>, if $\nabla$ is induced by a horizontal map, satisfying the homogeneity condition.

Retaining the notations of the previous section, we have

<u>Proposition 2</u>: The covariant derivatives with respect to the general connection $\nabla$ act by the formula

$$\sigma = \sigma^\alpha e_\alpha \longmapsto \nabla_X\sigma = X^i\left(\frac{\partial\sigma^\alpha}{\partial u^i} + \Gamma_i^\alpha \circ \sigma\right)e_\alpha \ . \quad \square$$

As we approached the linearity of the connection by the homogeneity condition it is important that we should have the best possible complex view of the power of this condition. In connection with this a summarization of the known results and their partial supplementation are given in

Theorem 1: For a horizontal map H the following properties are equivalent:

/1/ H satisfies the homogeneity condition.

/2/ $\forall t \in \mathbb{R} : d\mu_t \circ v = v \circ d\mu_t$ .

/3/ $\forall t \in \mathbb{R} : K \circ d\mu_t = \mu_t \circ K$ .

/4/ The map $K : T\xi \longrightarrow \xi$ is a bundle map.

/5/ $\nabla : \mathrm{Sec}\xi \longrightarrow A^1(B;\xi)$ is $\mathbb{R}$-linear:

$$\nabla(\sigma_1 + \sigma_2) = \nabla\sigma_1 + \nabla\sigma_2 , \quad \nabla(\mu_t \circ \sigma) = t\nabla\sigma$$

$$(\sigma_1, \sigma_2 \in \mathrm{Sec}\xi, \quad t \in \mathbb{R}).$$

/6/ $\forall \sigma \in \mathrm{Sec}\xi, f \in C^\infty(B) : \nabla(f\sigma) = \delta f \wedge \sigma + f\nabla\sigma$ / $\delta$ is the operator of the exterior derivative/.

/7/ The Lie-product $[C,X^h]$ vanishes for each vector field $X \in \mathfrak{X}(B)$.

/8/ The restrictions of the connection parameters to the fibers are linear functions, namely

$$\Gamma_i^\alpha = y^\beta(\Gamma_{i\beta}^\alpha \circ \pi)$$

where

$$\Gamma_{i\beta}^\alpha : U \longrightarrow \mathbb{R} , \ x \longmapsto \Gamma_{i\beta}^\alpha(x) := \frac{\partial \Gamma_i^\alpha}{\partial y^\beta}(z) , \ z \in \pi^{-1}(x).$$

Proof:

/a/ First we briefly recall the notion of homogeneous maps. - Let F and H be /real/ vector spaces. A map $F \longrightarrow H$ , $z \longmapsto z'$ is called homogeneous if $\forall t \in \mathbb{R} , z \in F : (tz)' = tz'$ . The generalization of this concept to vector bundles is obvious. /Of course, the horizontal map H is homogeneous in this sense and the meaning of the homogeneity condition is totally different./ In particular a function f on the total space E of $\xi$ is homogeneous if $\forall t \in \mathbb{R} : f \circ \mu_t = tf$ . If follows easily from the classical EULER relation for a homogeneous function that $f : \pi^{-1}(U) \longrightarrow \mathbb{R}$ is homogeneous iff $f = y^\alpha \frac{\partial f}{\partial y^\alpha}$ . - As regards the maps $\mu_t$, it is useful to note that they are strong bundle endomorphisms of $\xi$ and their derivatives act as follows:

$$(d\mu_t)_z\left(\frac{\partial}{\partial x^i}\right)_z = \left(\frac{\partial}{\partial x^i}\right)_{tz}, \quad (d\mu_t)_z\left(\frac{\partial}{\partial y^\alpha}\right)_z = t\left(\frac{\partial}{\partial y^\alpha}\right)_{tz}. \qquad (*)$$

Now we have a look at the tangent fibration $T\xi = (TE, d\pi, TB, \mathbb{R}^{2r})$ of $\xi$. Its fiber at the point $a = a^i\left(\frac{\partial}{\partial u^i}\right)_x \in T_xB$ is the set

$$(d\pi)^{-1}(a) = \{a^i\left(\frac{\partial}{\partial x^i}\right)_z + b^\alpha\left(\frac{}{\partial y^\alpha}\right)_z \,\Big|\, \pi(z) = x,\ b^\alpha \in \mathbb{R}\},$$

equipped with the vector space operations

$$\left[a^i\left(\frac{\partial}{\partial x^i}\right)_z + b^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)_z\right] + \left[a^i\left(\frac{\partial}{\partial x^i}\right)_{z'} + \tilde{b}^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)_{z'}\right] =$$

$$= a^i\left(\frac{\partial}{\partial x^i}\right)_{z+z'} + (b^\alpha + \tilde{b}^\alpha)\left(\frac{\partial}{\partial y^\alpha}\right)_{z+z'},$$

$$\lambda\left[a^i\left(\frac{\partial}{\partial x^i}\right)_z + b^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)_z\right] = a^i\left(\frac{\partial}{\partial x^i}\right)_{\lambda z} + \lambda b^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)_{\lambda z}.$$

If $\sigma = \sigma^\alpha e_\alpha$ is a section over $U$ and $a = a^i\left(\frac{\partial}{\partial x^i}\right)_x \in T_xB$ $(x \in U)$, then

$$(d\sigma)_x(a) = a^i\left(\frac{\partial}{\partial x^i}\right)_{\sigma(x)} + a^i\left(\frac{\partial}{\partial u^i}\right)_x\left(\frac{\partial}{\partial y^\alpha}\right)_{\sigma(x)},$$

which means that $(d\sigma)_x(a)$ is an element of the $T\xi$-fiber $(d\pi)^{-1}(a)$. Conversely, every element of the $T\xi$-fibers has such a form.

/b/ After these preparations, we come to the point.

- First (1) <=> (2) is evident by the substitution $v = \iota - h$. By $(*)$

$$\alpha \circ d\mu_t\left(\frac{\partial}{\partial y^\rho}\right)_z = \alpha\left[te_\rho(x)\right] = te_\rho(x) = (\mu_t \circ \alpha)\left(\frac{\partial}{\partial y^\rho}\right)_z,$$

so $\mu_t \circ K = \mu_t \circ \alpha \circ v = \alpha \circ d\mu_t \circ v$, from which we immediately get the equivalence (2) <=> (3).

Now, let us consider the statements

/4a/ $K : T\xi \longrightarrow \xi$ is a homogeneous map;

/5a/ $\nabla : \mathrm{Sec}\,\xi \longrightarrow A^1(B;\xi)$ is $\mathbb{R}$-homogeneous;

/8a/ the connection parameters $\Gamma_i^\alpha$ are homogeneous on fibers $F_x$ $(x \in U)$.

By the observation given in /a/ it is clear that

$$\forall t \in \mathbb{R} : t[(d\sigma)_x(a)] = d(\mu_t \circ \sigma)_x(a) ,$$

hence

$$K[t(d\sigma)_x(a)] = K[d(\mu_t \circ \sigma)_x(a)] =$$

$$= K[(d\mu_t)_{\sigma(x)} \circ (d\sigma)_x(a)] = K \circ d\mu_t[(d\sigma)_x(a)]$$

and consequently /4a/ <=> /3/. On account of $\nabla(\mu_t \circ \sigma) := K \circ d(\mu_t \circ \sigma) = K \circ d\mu_t \circ d\sigma$ we get at once /5a/<=> /3/, while from relation $(*)$ we infer by a simple calculation the equivalence /8a/ <=> /1/.

By section 5,

$$\left[C, X^h\right] = \left[y^\alpha \frac{\partial}{\partial y^\alpha} , \left(X^i \circ \pi\right)\left(\frac{\partial}{\partial x^i} - \Gamma_i^\beta \frac{\partial}{\partial y^\beta}\right)\right] =$$

$$= \left(X^i \circ \pi\right)\left(\Gamma_i^\alpha - y^\beta \frac{\partial \Gamma_i^\alpha}{\partial y^\beta}\right)\frac{\partial}{\partial y^\alpha} ,$$

hence $\left[C, X^h\right] = 0 \iff \Gamma_i^\alpha = y^\beta \dfrac{\partial \Gamma_i^\alpha}{\partial y^\beta}$, therefore /applying the EULER relation/ /7/ <=> /8a/.

Again, with the help of a direct computation we get

$$\left[\nabla(f\sigma) - f\nabla\sigma - \delta f \wedge \sigma\right]_x \left(\frac{\partial}{\partial u^i}\right)_x =$$

$$= \left(\Gamma_i^\alpha\left[f(x)\sigma(x)\right] - f(x)\Gamma_i^\alpha[\sigma(x)]\right)e_\alpha(x) ,$$

so the equivalence /6/ <=> /8a/ also holds.

/c/ We have already proved:

$$\begin{array}{ccccccc}
 & & /1/ & \Longleftrightarrow & /2/ & \Longleftrightarrow & /3/ & \Longleftrightarrow & /4a/ \\
 & & \Updownarrow & & & & \Updownarrow & & \\
/6/ & \Longleftrightarrow & /8a/ & \Longleftrightarrow & /7/ & & /5a/ & &
\end{array}$$

To complete the reasoning, finally we apply DOMBROWSKI's "clever observation" /M. SPIVAK's words, see [16], p. 8.58/. This guarantees: /8a/ $+\exists D\Gamma_i^\alpha(0)$ /FRECHET-derivative/ <=> $\Gamma_i^\alpha|_{F_x}$ is linear, from which we obtain the desired form of the functions $\Gamma_i^\alpha$ as well. So /8/ <=> /8a/. This equivalence also implies /e.g. by the local form of $K$ and $\nabla$ resp./ that /4a/ <=> /4/, /5a/ <=> /5/. Since the implications /4/ ==> /4a/, /5/ ==> /5a/ are true automaticly, the theorem is proved. □

7. Homogeneous connections. In this section we are going to investigate an "intermediate class" between the general connections

and the linear ones.

Definition 4: Let $H$ be a continuous splitting of the sequence $0 \longrightarrow V_\xi \xrightarrow{i} \tau_E \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \longrightarrow 0$ . Suppose that the horizontal projection $h$ belonging to $H$ is differentiable over the total space

$$T\dot{E} \; , \; \dot{E} := \bigcup_{x \in B} (F_x \setminus \{\sigma\})$$

of the "slit bundle" made from $\tau_E$ and it is not differentiable on the nullsection. - We say that the general connection induced by $H$ is homogeneous, if

$$\forall t \in \mathbb{R} : \underline{h \circ d\mu_t = d\mu_t \circ h} \; .$$

THEOREM 2:

/1/ Let $\nabla : \mathrm{Sec}\xi \longrightarrow A^1(B;\xi)$ be a homogeneous connection. Then

$\forall \; f \in C^\infty(B), \sigma \in \mathrm{Sec}\xi : \nabla(f\sigma) = \delta f \wedge \sigma + f\nabla\sigma,$

that is

$1^\circ \; \forall \; X \in \mathcal{X}(B) : \nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma$ holds, furthermore the relations

$2^\circ \; \nabla_{X+Y}\sigma = \nabla_X\sigma + \nabla_Y\sigma$ ,

$3^\circ \; \nabla_{fX}\sigma = f\nabla_X\sigma$ ,

$4^\circ \; [\nabla_X(\sigma_1+\sigma_2)](x) = (\nabla_X\sigma_1)(x) + (\nabla_X\sigma_2)(x)$, if $\sigma_1(x) = 0$ are satisfied. Finally

$5^\circ \; \forall \; X \in \mathcal{X}(B)$, $\sigma_o \in \mathrm{Sec}\xi \setminus \{0\}$ : the map $\nabla_X : \sigma \longmapsto \nabla_X\sigma$ is differentiable at $\sigma_o$ and $\nabla_X\sigma_o \in \mathrm{Sec}\xi$ /i.e. $\nabla_X\sigma_o$ is a differentiable section, too/.

/2/ Conversely, let a map

$$\mathcal{X}(B) \times \mathrm{Sec}\xi \longrightarrow \mathrm{Sec}\xi \; , \quad (X,\sigma) \longmapsto \nabla_X\sigma$$

be given, satisfying the conditions $1^\circ$-$5^\circ$. Then there exists a unique horizontal map with properties listed in Definition 4 and the covariant derivatives belonging to this horizontal map are exactly the operators $\nabla_X$.

Proof:

/a/ Since the equivalence /1/ <=> /5/ in the preceding Theorem is independent of the "clever observation", the statement $1^\circ$ is an immediate consequence of our earlier considerations. By Proposition 2 the validity of $2^\circ$ and $3^\circ$ is evident. On account of the equivalence /1/ <=> /8a/ mentioned in the above proof, $\sigma_1(x) = 0$ implies that $\Gamma_i^\alpha(\sigma_1(x)) = 0$ . Hence

$$\left[\Gamma_i^\alpha\circ(\sigma_1+\sigma_2)\right](x) = \Gamma_i^\alpha(\sigma_2(x)) = \left(\Gamma_i^\alpha\circ\sigma_1 + \Gamma_i^\alpha\circ\sigma_2\right)(x) ,$$

thus

$$\sigma_1(x) = 0 \Rightarrow \left[\nabla_X(\sigma_1+\sigma_2)\right](x) = \left(\nabla_X\sigma_1\right)(x) + \left(\nabla_X\sigma_2\right)(x) ,$$

so $4^\circ$ is also satisfied. The verification of $5^\circ$ is an easy computation, too.

/b/ In this step /cf. [15] / we briefly show that the postulates $1^\circ$-$5^\circ$ uniquely determine continuous functions $\Gamma_i^\alpha : \pi^{-1}(U) \longrightarrow \mathbb{R}$ such that the assumptions

/i/ $\forall\sigma = \sigma^\alpha e_\alpha \in \mathrm{Sec}\xi\big|_U , \quad X = X^i\frac{\partial}{\partial u^i} \in \mathfrak{X}(B)\big|_U :$

$$\nabla_X\sigma = X^i\left(\frac{\partial\sigma^\alpha}{\partial u^i} + \Gamma_i^\alpha\circ\sigma\right)e_\alpha \quad ;$$

/ii/ $\Gamma_i^\alpha\circ\mu_t = t\Gamma_i^\alpha \quad (t \in \mathbb{R})$;

/iii/ the $\Gamma_i^\alpha$-s are differentiable on $\pi^{-1}(U\cap\dot{E})$ are satisfied.

- For this purpose we first define for all $\sigma \in \mathrm{Sec}\xi\big|_U$ the section $G_i(\sigma) : U \longrightarrow E$ by

$$G_i(\sigma) := \nabla_{\frac{\partial}{\partial u^i}}\sigma - \frac{\partial\sigma^\alpha}{\partial u^i}e_\alpha .$$

With the help of the framing $\{e_\alpha\}$ it can be written as

$$G_i(\sigma) = \overset{*}{\Gamma}{}_i^\alpha(\sigma)e_\alpha , \quad \overset{*}{\Gamma}{}_i^\alpha(\sigma) : U \longrightarrow \mathbb{R} .$$

We claim: $\sigma_1(x) = \sigma_2(x) \Rightarrow \overset{*}{\Gamma}{}_i^\alpha(\sigma_1)(x) = \overset{*}{\Gamma}{}_i^\alpha(\sigma_2)(x)$.

By postulate $4^\circ$

$$\left[\nabla_{\frac{\partial}{\partial u^i}}(\sigma_1 + (\sigma_2-\sigma_1))\right](x) = \left(\nabla_{\frac{\partial}{\partial u^i}}\sigma_1\right)(x) + \left(\nabla_{\frac{\partial}{\partial u^i}}(\sigma_2-\sigma_1)\right)(x) .$$

Again by the same postulate

$$\left[\nabla_{\frac{\partial}{\partial u^i}}\sum_{\rho=1}^{r}(\sigma_2^\rho-\sigma_1^\rho)e_\rho\right](x) = \left(\sum_{\rho=1}^{r}\nabla_{\frac{\partial}{\partial u^i}}(\sigma_2^\rho-\sigma_1^\rho)e_\rho\right)(x)$$

/where we had to omit the summation convention/. Since by $1^\circ$

$$\left[\nabla_{\frac{\partial}{\partial u^i}}(\sigma_2^\rho-\sigma_1^\rho)e_\rho\right](x) = \frac{\partial(\sigma_2^\rho-\sigma_1^\rho)}{\partial u^i}(x)e_\rho(x) + (\sigma_2^\rho-\sigma_1^\rho)(x)\left(\nabla_{\frac{\partial}{\partial u^i}}e_\rho\right)(x) =$$

$$= \frac{\partial(\sigma_2^\rho-\sigma_1^\rho)}{\partial u^i}(x)e_\rho(x) \qquad /\ \rho \text{ is fixed}/$$

we get

$$\left[\nabla_{\frac{\partial}{\partial u^i}}(\sigma_1+(\sigma_2-\sigma_1))\right](x) = \left(\nabla_{\frac{\partial}{\partial u^i}}\sigma_1\right)(x) + \left(\frac{\partial\sigma_2^\alpha}{\partial u^i} - \frac{\partial\sigma_1^\alpha}{\partial u^i}\right)(x)e_\alpha(x) =$$

$$= \overset{*}{\Gamma}{}_i^\alpha(\sigma_1)(x) + \frac{\partial\sigma_2^\alpha}{\partial u^i}(x)e_\alpha(x) .$$

On the other hand

$$\left[\nabla_{\frac{\partial}{\partial u^i}}(\sigma_1+(\sigma_2-\sigma_1))\right](x) = \left(\nabla_{\frac{\partial}{\partial u^i}}\sigma_2\right)(x) =$$

$$= \left(\overset{*}{\Gamma}{}_i^\alpha(\sigma_2)\right)(x) + \frac{\partial\sigma_2^\alpha}{\partial u^i}(x)e_\alpha(x) ,$$

proving the equality $\overset{*}{\Gamma}{}_i^\alpha(\sigma_1)(x) = \overset{*}{\Gamma}{}_i^\alpha(\sigma_2)(x)$ .

The remaining detailes are left to the reader.

/c/ Having the functions $\Gamma_i^\alpha$, we <u>define</u> the splitting $H : \pi^*(\tau_B) \longrightarrow \tau_E$ locally with the help of the matrices described in section <u>5</u>. Then $H$ will be the desired horizontal map, which is guaranteed by the above mentioned equivalence /1/ <=> /8a/. □

<u>Remark</u>: A non-linear connection /i.e. in our terminology, a homogeneous connection/ was characterized by means of some postulates imposed on covariant derivatives by KANDATU [8] in the case of $\xi = \tau_B$. Later P.T. NAGY pointed out in [15] that KANDATU's postulates can be weakened. As far as we know, it was only M. AKO, who treated the problem somewhat more generally ([1]), but also on the basis of KANDATU's original axioms.

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