MICROFUNCTIONS WITH VALUES IN A CLIFFORD ALGEBRA 1

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Abstract. Inspired by the theory of microfunctions on the real line, we introduce a notion of microfunction defined in an open subset of $\mathbb{R}^m$ and taking values in a Clifford algebra $\mathcal{A}$, and this by making use of formal boundary values of left or right monogenic functions. Moreover we define a local Hilbert transform on the sheaf of $\mathcal{A}$-valued microfunctions, which is useful in order to study singularities of $\mathcal{A}$-valued hyperfunctions in a purely algebraic way.
In this way we obtain theorems relating the singular behaviour of an $\mathcal{A}$-valued hyperfunction to its values and we give an electromagnetic interpretation of these theorems.

Introduction. This paper is a continuation of our previous paper [8], in which Clifford algebra valued hyperfunctions in open subsets of $\mathbb{R}^m$ were represented as formal boundary values of left or right monogenic functions in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$. Hereby $\mathbb{R}^m$ is identified with a hyperplane in $\mathbb{R}^{m+1}$, which separates $\mathbb{R}^{m+1}$ into an " upper half space " $\mathbb{R}^{m+1}_+$ and a " lower half space " $\mathbb{R}^{m+1}_-$.
In the first section we define the space of $\mathcal{A}$-valued microfunctions $\mathcal{C}(\Omega;\mathcal{A})$ in an open subset Ω of $\mathbb{R}^m$. Moreover, when $\tilde{\Omega} \subsetneq \mathbb{R}^{m+1}$ is open such that Ω is relatively closed in $\tilde{\Omega}$, we show that $\mathcal{C}(\Omega;\mathcal{A})$ coincides with the quotient of the space of left or right monogenic functions in $\tilde{\Omega}\setminus\Omega$ with respect to the space of those functions f in $\tilde{\Omega}\setminus\Omega$ the restrictions of which to the upper and lower halfspaces admit monogenic extensions about each point of Ω. We thus obtain representations of $\mathcal{A}$-valued microfunctions as boundary values of monogenic functions, generalizing in this way the definition of microfunctions on the real line by means of boundary values of holomorphic functions (see [3], [5], [6], [7]). Furthermore we define left and right Hilbert transforms $\mathcal{H}_l$ and $\mathcal{H}_r$, acting on the space $\mathcal{C}(\Omega;\mathcal{A})$ of $\mathcal{A}$-valued microfunctions, and having a local nature, i.e. $\mathcal{H}_l$ and $\mathcal{H}_r$ commute with the restriction operators.

(*) Senior Research Assistant supported by N.F.W.O. Belgium

These transforms $\mathcal{H}_l$ and $\mathcal{H}_r$ are in fact extensions of the Hilbert - Riesz transform introduced in [9] for the $\mathcal{A}$-valued L_2-functions in $\mathbb{R}^m$, which itself is related to the classical Riesz transform (see [10]).
In this way, the four group of main involutions on the Clifford algebra together with $\mathcal{H}_l$ and $\mathcal{H}_r$ generate a finite group of thirty-two elements which enables us to express singularity properties of $\mathcal{A}$-valued hyperfunctions in an algebraic way. Moreover, as the Clifford algebra is a graded algebra, we may decompose $\mathcal{H}_l$ and $\mathcal{H}_r$ into a sum of boundary operators $\mathcal{H}_l^+$ and $\mathcal{H}_r^+$ and coboundary operators $\mathcal{H}_l^-$ and $\mathcal{H}_r^-$.
In the second section we introduce so called electric and magnetic projection operators ε_+, ε_- and μ_+, μ_- which are mutually orthogonal and which satisfy $\mathbb{1} = \varepsilon_+ + \varepsilon_- + \mu_+ + \mu_-$.
They are constructed by means of the Hilbert transforms and they describe microfunctions which are simultaneously upper or lower boundary values of left or right monogenic functions. Furthermore we obtain a non trivial generalization of the following theorem (see [5]) : " When a hyperfunction F on $\mathbb{R}$ is real valued, then for every representing holomorphic function f(z) in $\mathbb{C} \setminus \mathbb{R}$ and every $x \in \mathbb{R}$, $f \mid \mathbb{C}_+$ admits a holomorphic extension about x if and only if $f \mid \mathbb{C}_-$ admits a holomorphic extension about x ". We thus establish a relationship between the singular behaviour of $\mathcal{A}$-hyperfunctions and the nature of their values, a result which admits an electromagnetic interpretation as is explained at the end of the paper.
In the third section we introduce the so called logarithmic microfunctions λ_+ and λ_- which are generalizations of the classical logarithmic microfunctions.
In section four the convolution of microfunctions is calculated by means of intgrals of left and right monogenic functions over suitable orientable surfaces in a way which is inspired by [3] . This leads to a definition of micro-differential operators and we show that the micro-differential operator associated with the logarithmic microfunction $\lambda = \lambda_+ + \lambda_-$ is equal to the inverse of the Dirac operator D_o in $\mathbb{R}^m$.
In the final section we show that every microfunction φ may be decomposed into a unique sum $\varphi = \varphi_\varepsilon + \varphi_\mu$, where φ_ε satisfies the electric field equations

$$[D_o, \varphi_\varepsilon] = D_o \varphi_\varepsilon - \varphi_\varepsilon D_o = 0$$

and where φ_μ satisfies the magnetic field equations

$$\{D_o, \varphi_\mu\} = D_o\varphi_\mu + \varphi_\mu D_o = 0 \ .$$

Furthermore we establish the relations

$$\varepsilon_+ + \varepsilon_- = 1/2 \ \{D_o, \cdot\} \circ D_o^{-1}$$
$$\mu_+ + \mu_- = 1/2 \ [D_o, \cdot] \circ D_o^{-1}$$

and

$$\mathcal{H}_r \mathcal{H}_l \psi = D_o^{-1} \psi D_o \ , \quad \psi \in \mathcal{E}(\Omega ; \mathcal{A}) \ ,$$

thus relating the Hilbert transforms to the Dirac operator. Finally we show that, when F is a free magnetic field in $\mathbb{R}^3 \setminus \mathbb{R}^2$ such that $F \mid \mathbb{R}^3_+$ extends to a free magnetic field about the origin, while $F \mid \mathbb{R}^3_-$ does not, then F is not extendable to a magnetic field about the origin, i.e. there is a magnetic charge at the origin. This explains the above mentioned relationship between the singularities and the values of an $\mathcal{A}$-hyperfunction.

Acknowledgement The author wishes to thank Prof.Dr. R. Delanghe and Dr. F. Brackx for their constant interest during the preparation of this paper.

Preliminaries. Let V be a complex m-dimensional vector space provided with a quadratic form and let $\{e_1, \ldots, e_m\}$ be an associated orthonormal basis. Then a basis for the universal Clifford algebra $\mathcal{A}$ constructed over V is given by $\{ e_A : A \subset \{1, \ldots, m\} \}$ where $e_A = e_{\alpha_1} \ldots e_{\alpha_h}$ with $A = \{\alpha_1, \ldots, \alpha_h\}$ and $1 \leq \alpha_1 < \ldots \leq \alpha_h \leq m$.
Obviously $e_{\{i\}} = e_i \quad (i = 1, \ldots, m)$.
The product in $\mathcal{A}$ is defined by the relations

$$e_i e_j + e_j e_i = -2 \delta_{ij} e_o \ , \ (i,j = 1, \ldots, m) \ ,$$

$e_o = e_\emptyset = 1$ being the identity of $\mathcal{A}$.
We define involutions in $\mathcal{A}$ as follows.
Let $a = \sum_A a_A e_A \in \mathcal{A}$, $a_A \in \mathbb{C}$; then we put $\bar{a} = \sum_A \bar{a}_A \bar{e}_A$, $\check{a} = \sum_A \bar{a}_A \check{e}_A$ and $\mathring{a} = \sum_A a_A \mathring{e}_A$, where $\bar{e}_A = \bar{e}_{\alpha_h} \ldots \bar{e}_{\alpha_1}$, $\bar{e}_j = -e_j = \check{e}_j$ ($j = 1, \ldots, m$) , $\bar{e}_o = e_o$, $\check{e}_A = e_{\alpha_h} \ldots e_{\alpha_1}$ and $\mathring{e}_A = \bar{\check{e}}_A = \check{\bar{e}}_A$.
The four group $\mathcal{G} = \{ \mathbb{I}, \alpha, \beta, \gamma \}$ of main involutions on $\mathcal{A}$ is then given by $\alpha(a) = \bar{a}$, $\beta(a) = \check{a}$ and $\gamma(a) = \mathring{a}$, $a \in \mathcal{A}$.
A norm on $\mathcal{A}$ is defined by $|a|_o^2 = 2^m [a.\bar{a}]_o = 2^m \sum_A |a_A|^2$, where for $b \in \mathcal{A}$ we put $[b]_o = b_o$.

Notice that $|a.b|_0 \leq |a|_0 \, | b|_0$, $a,b \in \mathcal{A}$.
Let $a,b \in \mathcal{A}$; then we use the notations $[a, b]$ for $a\,b - b\,a$, $\{a, b\}$ for $a\,b + b\,a$.
Similar notations will be used for the commutator and the anti-commutator of operators.
An element $a \in \mathcal{A}$ which is of the form $a = \sum_A a_A e_A$ with $a_A = 0$ whenever $\# A \neq k$, $k \leq m$, will be called a k-vector. The space of all k-vectors is denoted by $\mathcal{A}_k$. Moreover, it is well known that each element $a \in \mathcal{A}$ admits a unique decomposition of the form $a = \sum_{j=0}^{m} a_j$, with $a_j \in \mathcal{A}_j$. Hence there exist projection operators θ_j from $\mathcal{A}$ onto $\mathcal{A}_j$ which are given by $\theta_j\,(a) = a_j$.
Let $\Omega \subseteq \mathbb{R}^{m+1}$ be open. Then $M_{(r)}(\Omega\,;\mathcal{A})$ and $M_{(l)}(\Omega\,;\mathcal{A})$ are the right and left $\mathcal{A}$-modules of functions $f \in C_1(\Omega\,;\mathcal{A})$ satisfying respectively $Df = \sum_{j=0}^{m} e_j \partial_{x_j} f = 0$ and $fD = \sum_{j=0}^{m} \partial_{x_j} f\, e_j = 0$, where $\sum_{j=0}^{m} e_j \partial_{x_j}$ is a generalized Cauchy-Riemann operator. The elements of $M_{(r)}(\Omega\,;\mathcal{A})$ and $M_{(l)}(\Omega\,;\mathcal{A})$ are respectively called left and right monogenic functions in Ω . Moreover the $\mathcal{A}$-modules $M_{(r)}(\Omega\,;\mathcal{A})$ and $M_{(l)}(\Omega\,;\mathcal{A})$ are provided with the topology of uniform convergence on compact subsets of Ω ; in this way they become right (resp. left) Fréchet $\mathcal{A}$-modules.
By D_0 we denote the operator $\sum_{j=1}^{m} e_j \partial_{x_j}$, acting on $\mathcal{A}$-valued functions in open subsets of $\mathbb{R}^m$.
In the sequel arbitrary elements of $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ will be denoted respectively by $\vec{x} = (\,x_1,\ldots,\,x_m)$ and $x = x_0 + \vec{x} = (\,x_0,\,x_1,\ldots,\,x_m)$ while $|.|$ denotes the Euclidean norm.
Moreover $x \in \mathbb{R}^{m+1}$ and $\vec{x} \in \mathbb{R}^m$ will be identified with the elements $x = \sum_{j=0}^{m} x_j\, e_j$ and $\vec{x} = \sum_{j=1}^{m} x_j\, e_j$ in $\mathcal{A}$. Hence we have that $\bar{x} = x_0 - \vec{x} = (\,x_0,-x_1,\ldots,\,-x_m)$.
For any open subset Ω of $\mathbb{R}^{m+1}$ we put $\Omega_{\pm} = \{x \in \Omega : x_0 \gtrless 0\}$ and $S_0(\Omega\,) = \{x \in \mathbb{R}^{m+1} : -\bar{x} \in \Omega\}$.
In the sequel we also use the notations $B(x,\,r) = \{y \in \mathbb{R}^{m+1} : |y - x| < r\}$ and $B_m(\vec{x},\,r) = \{\vec{y} \in \mathbb{R}^m : |\vec{y} - \vec{x}| < r\}$ while for any subset U of $\mathbb{R}^{m+1}$, the characteristic function on U is denoted by χ_U .
The following $\mathcal{A}$-modules will be used currently.
Let $S \subseteq \mathbb{R}^{m+1}$ be closed. Then $M_{(r)}(\,S;\mathcal{A})$ denotes the right

$\mathcal{A}$-module of all left monogenic functions in an open neighbourhood of S .
Let $\mathcal{Q}(\Omega)$ and $\mathcal{B}(\Omega)$ be respectively the space of analytic functions and the space of Sato-hyperfunctions in $\Omega \subseteq \mathbb{R}^m$. Then the spaces of $\mathcal{A}$-valued analytic functions and $\mathcal{A}$-valued Sato-hyperfunctions are denoted by $\mathcal{Q}(\Omega;\mathcal{A})$ and $\mathcal{B}(\Omega;\mathcal{A})$ respectively. The corresponding left (resp. right) $\mathcal{A}$-modules are denoted by $\mathcal{Q}_{(1)}(\Omega;\mathcal{A})$, $\mathcal{B}_{(1)}(\Omega;\mathcal{A})$ (resp. $\mathcal{Q}_{(r)}(\Omega;\mathcal{A})$, $\mathcal{B}_{(r)}(\Omega;\mathcal{A})$) .
By $\mathcal{C}(\Omega)$ we denote the space $\mathcal{B}(\Omega) / \mathcal{Q}(\Omega)$. The elements of $\mathcal{C}(\Omega)$ are called $\mathbb{C}$-valued microfunctions.
If Ω is relatively compact in $\mathbb{R}^m$, then $\mathcal{Q}(\overline{\Omega})$ and $\mathcal{Q}'(\overline{\Omega})$ stand respectively for the space of $\mathbb{C}$-valued analytic functions in a neighbourhood of $\overline{\Omega}$ and the space of $\mathbb{C}$-valued analytic functionals on $\overline{\Omega}$, while $\mathcal{Q}'_{(1)}(\overline{\Omega};\mathcal{A})$ is the right $\mathcal{A}$-module of all left $\mathcal{A}$-linear analytic functionals on $\overline{\Omega}$.
Let $\Omega \subseteq \mathbb{R}^m$, $\widetilde{\Omega} \subseteq \mathbb{R}^{m+1}$ be open such that Ω is relatively closed in $\widetilde{\Omega}$. Then in [8] we used the notations $\mathcal{B}_1(\Omega;\mathcal{A})$ (resp. $\mathcal{B}_r(\Omega;\mathcal{A})$) for the $\mathcal{A}$-modules of formal boundary values in Ω of left (resp. right) monogenic functions in $\widetilde{\Omega} \setminus \Omega$. The elements of these modules were called left and right $\mathcal{A}$-hyperfunctions for short. For the definition of semi-monogenic $\mathcal{A}$-hyperfunctions and the corresponding $\mathcal{A}$-modules $\mathcal{B}_{\pm,1}(\Omega;\mathcal{A})$ and $\mathcal{B}_{\pm,r}(\Omega;\mathcal{A})$ we refer to [8] .
By $\mathcal{Q}_{o,m}$ (resp. $\mathcal{B}_{o,m}$) we denote the space of $\mathcal{A}$-valued analytic functions (resp. hyperfunctions) in a neighbourhood of the origin.
The definitions of the signature $s(F)$ of $F \in \mathcal{B}_{o,m}$ as well as the definitions of the singular spectra $SS_{\pm,1}F$, $SS_{\pm,r}F$, $r_{\pm}(F)$, $k_{\pm}(F)$, $t_{\pm,1}(F)$, $t_{\pm,r}(F)$ and $w(F)$, $F \in \mathcal{B}(\Omega;\mathcal{A})$ may be found in [8] . Moreover it was shown in [8] that the image of s acting on $\mathcal{B}_{o,m}$ consists of ten matrices $\{s_0,\dots, s_9\}$, leading up to the definition of special subspaces $\mathfrak{S}_q$ of $\mathcal{B}_{o,m}$ ($q = 0,\dots, 9$) .
Recall that the Cauchy-kernel of the generalized Cauchy-Riemann operator D is given by $\frac{1}{\omega_{m+1}} \cdot \frac{\overline{x}}{|x|^{m+1}}$, ω_{m+1} being the area of the unit sphere in $\mathbb{R}^{m+1}$.
For the definition of the Cauchy-transform $\hat{T}$ of an analytic functional $T \in \mathcal{Q}'_{(1)}(\overline{\Omega};\mathcal{A})$ we refer to [1] , [2] and [8] .

1. Microfunctions with values in a Clifford algebra

In this section we give the basic definitions and study the elementary properties concerning microfunctions with values in a Clifford algebra.

Definition 1. Let $\Omega \subseteq \mathbb{R}^m$ and $\tilde{\Omega} \subseteq \mathbb{R}^{m+1}$ be open such that Ω is relatively closed in $\tilde{\Omega}$.
Then by $M_{(r)\pm}(\tilde{\Omega},\Omega)$ (resp. $M_{(l)\pm}(\tilde{\Omega},\Omega)$) the right (resp. left) $\mathcal{A}$-module is denoted, consisting of the left (resp. right) monogenic functions in $\tilde{\Omega}_\pm$, which admit a left (resp. right) monogenic extension about each point of Ω.
By $M_{(r)}(\tilde{\Omega},\Omega)$ (resp. $M_{(l)}(\tilde{\Omega},\Omega)$) we denote the space of left (resp. right) monogenic functions f in $\tilde{\Omega}\setminus\Omega$ such that $f|\tilde{\Omega}_\pm \in M_{(r)\pm}(\tilde{\Omega},\Omega)$ (resp. $f|\tilde{\Omega}_\pm \in M_{(l)\pm}(\tilde{\Omega},\Omega)$).

Definition 2. The right (resp. left) $\mathcal{A}$-module of upper and lower left (resp. right) $\mathcal{A}$-microfunctions is given by

$$\mathcal{C}_{\pm,l}(\Omega;\mathcal{A}) = M_{(r)}(\mathbb{R}^{m+1}_\pm;\mathcal{A})/M_{(r)\pm}(\mathbb{R}^{m+1}\setminus\partial\Omega,\Omega)$$
$$\mathcal{C}_{\pm,r}(\Omega;\mathcal{A}) = M_{(l)}(\mathbb{R}^{m+1}_\pm;\mathcal{A})/M_{(l)\pm}(\mathbb{R}^{m+1}\setminus\partial\Omega,\Omega) .$$

The space of $\mathcal{A}$-valued microfunctions is defined by

$$\mathcal{C}(\Omega;\mathcal{A}) = \mathcal{B}(\Omega;\mathcal{A}) / \mathcal{Q}(\Omega;\mathcal{A}) ,$$

while the space of $\mathcal{A}$-microfunctions at the origin, called $\mathcal{A}$-micros for short, is given by

$$\mathcal{C}_{o,m} = \mathcal{B}_{o,m} / \mathcal{Q}_{o,m} .$$

Throughout this paper, monogenic functions will be denoted by letters f, g, h ...; hyperfunctions by capital letters F, G, H ..., and microfunctions by Greek letters φ, ψ, ...
As in the one dimensional case one may easily show

Theorem 1. $\mathcal{C}_{\pm,l}(\Omega;\mathcal{A}) \cong M_{(r)}(\tilde{\Omega}_\pm;\mathcal{A}) / M_{(r)\pm}(\tilde{\Omega},\Omega)$
and

$$\mathcal{C}_{\pm,r}(\Omega;\mathcal{A}) \cong M_{(l)}(\tilde{\Omega}_\pm;\mathcal{A}) / M_{(l)\pm}(\tilde{\Omega},\Omega) .$$

Furthermore we have

Theorem 2. The following isomorphisms of $\mathcal{A}$-modules hold :

(i) $\mathcal{C}_{\pm,\{\frac{1}{r}\}}(\Omega;\mathcal{A}) \cong \mathcal{B}_{\{\frac{1}{r}\}}(\Omega;\mathcal{A}) \,/\, \mathcal{B}_{\mp,\{\frac{1}{r}\}}(\Omega;\mathcal{A})$,

while the following isomorphisms of vector spaces hold :

(ii) $\mathcal{C}(\Omega;\mathcal{A}) \cong \mathcal{C}_{+,\{\frac{1}{r}\}}(\Omega;\mathcal{A}) \oplus \mathcal{C}_{-,\{\frac{1}{r}\}}(\Omega;\mathcal{A})$

(iii) $\mathcal{C}(\Omega;\mathcal{A}) \cong M_{(r)}(\widetilde{\Omega}\setminus\Omega\;;\mathcal{A}) \,/\, M_{(r)}(\widetilde{\Omega},\Omega)$
$\cong M_{(1)}(\widetilde{\Omega}\setminus\Omega\;;\mathcal{A}) \,/\, M_{(1)}(\widetilde{\Omega},\Omega)$

(iv) $\mathcal{C}(\Omega;\mathcal{A}) \cong \sum_A\oplus\, \mathcal{C}(\Omega)\, e_A$, $\mathcal{C}_{o,m} \cong \sum_A\oplus\, \mathcal{C}_{o,m}(\mathbb{C})e_A$,

i.e. for every $\varphi \in \mathcal{C}(\Omega;\mathcal{A})$ there exist unique $\varphi_A \in \mathcal{C}(\Omega)$ such that $\varphi = \sum_A \varphi_A\, e_A$.

<u>Proof.</u> Define operators $\Pi_{\pm,1}$ from $\mathcal{B}_1(\Omega;\mathcal{A})$ into $\mathcal{C}_{\pm,1}(\Omega;\mathcal{A})$ as follows.

Let $F \in \mathcal{B}_1(\Omega;\mathcal{A})$ be represented by $f \in M_{(r)}(\widetilde{\Omega}\setminus\Omega\;;\mathcal{A})$. Then we put

$$\Pi_{\pm,1}(F) = f\,|\,\widetilde{\Omega}_{\pm} + M_{(r)\pm}(\widetilde{\Omega},\Omega) .$$

Notice that the operators $\Pi_{\pm,1}$ are well defined and surjective. Moreover, $\Pi_{\pm,1}(F) = 0$ if and only if $F \in \mathcal{B}_{\mp,1}(\Omega;\mathcal{A})$, which implies the isomorphism (i) .

(ii) follows from (i) . Moreover for any $F \in \mathcal{B}(\Omega;\mathcal{A})$ we denote $\Pi(F) = F + \mathcal{A}(\Omega;\mathcal{A})$, while for $F \in \mathcal{B}_{o,m}$ we put $\Pi(F) = F + \mathcal{A}_{o,m}$.

As to (iii) , notice that by Mittag-Leffler's theorem (see [2]), $\mathcal{A}(\Omega;\mathcal{A}) \cong M_{(r)}(\widetilde{\Omega},\Omega)$.

Furthermore in view of [8] ,

$$\mathcal{B}(\Omega;\mathcal{A}) \cong M_{(r)}(\widetilde{\Omega}\setminus\Omega\;;\mathcal{A}) \,/\, M_{(r)}(\widetilde{\Omega};\mathcal{A})$$

whence

$$\mathcal{C}(\Omega;\mathcal{A}) \cong \mathcal{B}(\Omega;\mathcal{A}) \,/\, \mathcal{A}(\Omega;\mathcal{A})$$
$$\cong M_{(r)}(\widetilde{\Omega}\setminus\Omega\;;\mathcal{A}) \,/\, M_{(r)}(\widetilde{\Omega},\Omega) .$$

As to (iv) , let $\varphi \in \mathcal{C}(\Omega;\mathcal{A})$. Then for some $F \in \mathcal{B}(\Omega;\mathcal{A})$, $\Pi(F) = \varphi$ or $\varphi = F + \mathcal{A}(\Omega;\mathcal{A})$.

Now, let $F_A \in \mathcal{B}(\Omega)$, $A \subset \{1,\dots, m\}$, be the unique hyperfunctions such that $F = \sum_A F_A\, e_A$ (see [8]) and put $\varphi_A = F_A + \mathcal{A}(\Omega)$. Then obviously $\varphi_A \in \mathcal{C}(\Omega)$ and

$$\varphi = F + \mathcal{A}(\Omega;\mathcal{A}) = \sum_A (\, F_A + \mathcal{A}(\Omega))\, e_A = \sum_A \varphi_A\, e_A .$$

Moreover, as the maps $\varphi \longrightarrow \varphi_A$ from $\mathcal{C}(\Omega;\mathcal{A})$ into $\mathcal{C}(\Omega)$ are well defined by the above construction, it follows easily that φ admits a unique decomposition $\varphi = \sum_A \varphi_A\, e_A$ with $\varphi_A \in \mathcal{C}(\Omega)$. ∎

The spaces $\mathcal{C}_{\pm,\{\frac{1}{r}\}}(\Omega;\mathcal{A})$ and $\mathcal{C}(\Omega;\mathcal{A})$ have the following sheaf properties.

Theorem 3. The $\mathcal{A}$-modules $\mathcal{C}_{\pm,\{\frac{1}{r}\}}(\Omega;\mathcal{A})$ and the spaces $\mathcal{C}(\Omega;\mathcal{A})$, $\Omega \subseteq \mathbb{R}^m$ open, are flabby sheafs in $\mathbb{R}^m$.

In view of Theorem 3 it is clear how the supports of the micro-functions under consideration are defined.
One has that for every $F \in \mathcal{B}(\Omega;\mathcal{A})$

$$SS_{\pm,\frac{1}{r}} F = \operatorname{supp} \Pi_{\pm,\frac{1}{r}}(F) \quad \text{and} \quad \operatorname{sing\,supp} F = \operatorname{supp} \Pi(F),$$

where the definition of $\Pi_{\pm,r}$ is similar to the one of $\Pi_{\pm,1}$. Furthermore when $F \in \mathcal{B}_{o,m}$, then for every $a \in \mathcal{A}_{o,m}$, $s(F + a) = s(F)$ (see [8]), whence it makes sense to define the signature of an $\mathcal{A}$-micro as follows.

Definition 3. Let $\varphi \in \mathcal{C}_{o,m}$ and let $F \in \mathcal{B}_{o,m}$ be such that $\Pi(F) = \varphi$. Then we put $s(\varphi) = s(F)$.

Hence it also makes sense to introduce the decomposition of the support of an $\mathcal{A}$-valued microfunction as has been done for hyperfunctions in [8].
We have

Definition 4. Let $\varphi \in \mathcal{C}(\Omega;\mathcal{A})$ and let $F \in \mathcal{B}(\Omega;\mathcal{A})$ be such that $\Pi(F) = \varphi$. Then we put $r_{\pm}(\varphi) = r_{\pm}(F)$, $k_{\pm}(\varphi) = k_{\pm}(F)$, $t_{(\pm,\frac{1}{r})}(\varphi) = t_{(\pm,\frac{1}{r})}(F)$ and $w(\varphi) = w(F)$.

Furthermore the group $\mathcal{G} = \{\mathbb{I}, \alpha, \beta, \gamma\}$ of main involutions on $\mathcal{A}$ acts on $\mathcal{C}(\Omega;\mathcal{A})$ as follows. We put $\alpha(\varphi) = \Pi(\alpha(F))$ $\beta(\varphi) = \Pi(\beta(F))$ and $\gamma(\varphi) = \Pi(\gamma(F))$.

Notice that when $\varphi = \sum_A \varphi_A e_A$ with $\varphi_A \in \mathcal{C}(\Omega)$, $\alpha(\varphi) = \sum_A \overline{\varphi}_A \bar{e}_A$, $\beta(\varphi) = \sum_A \overline{\varphi}_A \check{e}_A$ and $\gamma(\varphi) = \sum_A \varphi_A \mathring{e}_A$.
Moreover the operators

$$A_{\pm} = \frac{1}{2}(\mathbb{I} \pm \alpha), \quad B_{\pm} = \frac{1}{2}(\mathbb{I} \pm \beta), \quad C_{\pm} = \frac{1}{2}(\mathbb{I} \pm \gamma)$$

are projection operators such that

$$A_+ + A_- = B_+ + B_- = C_+ + C_- = \mathbb{I}$$

and

$$A_+ A_- = B_+ B_- = C_+ C_- = A_- A_+ = B_- B_+ = C_- C_+ = 0 .$$

These operators will play an essential role in describing the relation between the values and the singularities of an $\mathcal{A}$-valued

hyperfunction or microfunction.
Another essential tool is the notion of left (resp. right) Hilbert transform $\mathcal{H}_l(\varphi)$ (resp. $\mathcal{H}_r(\varphi)$) of an $\mathcal{A}$-valued microfunction φ.

Definition 5. Let $\varphi \in \mathcal{C}(\Omega;\mathcal{A})$ be represented by $f \in M_{(r)}(\tilde{\Omega}\setminus\Omega;\mathcal{A})$ (resp. by $f \in M_{(l)}(\tilde{\Omega}\setminus\Omega;\mathcal{A})$). Then the left (resp. right) Hilbert transform $\mathcal{H}_l(\varphi)$ (resp. $\mathcal{H}_r(\varphi)$) of φ is the microfunction associated to the left (resp. right) monogenic function h, given by

$$h(x) = \begin{cases} f(x), & \text{if } x \in \tilde{\Omega}_+ \\ -f(x), & \text{if } x \in \tilde{\Omega}_- \end{cases} .$$

Clearly $\mathcal{H}_l^2 = \mathcal{H}_r^2 = \mathbb{I}$, whence the operators

$$Q_{\pm,l} = \frac{1}{2}(\mathbb{I} \pm \mathcal{H}_l) , \quad Q_{\pm,r} = \frac{1}{2}(\mathbb{I} \pm \mathcal{H}_r)$$

are projection operators satisfying

$$\mathbb{I} = Q_{+,l} + Q_{-,l} = Q_{+,r} + Q_{-,r} ,$$

$$Q_{\pm,l}\, Q_{\mp,l} = Q_{\pm,r}\, Q_{\mp,r} = 0 .$$

Furthermore, it is easy to see that

$$Q_{\pm,\left\{\substack{l\\r}\right\}}\, \mathcal{C}(\Omega;\mathcal{A}) \cong \mathcal{C}_{\pm,\left\{\substack{l\\r}\right\}}(\Omega;\mathcal{A}) ,$$

so that $\Pi_{\pm,\left\{\substack{l\\r}\right\}}$ may be identified with $Q_{\pm,\left\{\substack{l\\r}\right\}} \circ \Pi$.
Notice that $Q_{\pm,l}$ and $Q_{\pm,r}$ in fact correspond to the upper (lower) boundary values in microfunction sense of the left and right monogenic representations of φ.
Notice also that, when φ is represented by an L_2-function f in $\mathbb{R}^m$, $\mathcal{H}_l\varphi$ is the microfunction associated to $\mathcal{H} f$, where $\mathcal{H}$ is the Hilbert-Riesz transform for L_2-functions introduced in [9]. So $\mathcal{H}_l$ and $\mathcal{H}_r$ are natural extensions of the Hilbert-Riesz transform of L_2-functions, with the supplementary advantage that they are defined locally. A similar extension of the Hilbert-Riesz transform of L_2-functions would not be possible in the hyperfunction setting since such a definition would depend on the choice of the representation of hyperfunctions as formal boundary values of monogenic functions.
The operators $\mathcal{H}_l$ and $\mathcal{H}_r$ admit decompositions of the form

$$\mathcal{H}_l = \mathcal{H}_l^+ + \mathcal{H}_l^- \quad \text{and} \quad \mathcal{H}_r = \mathcal{H}_r^+ + \mathcal{H}_r^- ,$$

where the operators $\mathcal{H}^+_{\left\{\substack{l\\r}\right\}}$ (resp. $\mathcal{H}^-_{\left\{\substack{l\\r}\right\}}$) are boundary (resp. coboundary) operators.

As to the construction of $\mathcal{H}^+_{\{l\atop r\}}$ and $\mathcal{H}^-_{\{l\atop r\}}$, we first decompose $\varphi \in \mathcal{C}(\Omega;\mathcal{A})$ into $\varphi = \sum_{j=0}^{m} \theta_j(\varphi)$.

Hence it is sufficient to consider the action of $\mathcal{H}_l$ and $\mathcal{H}_r$ on each $\theta_j(\varphi)$; $j = 0,\dots, m$, and by Theorem 3 we may restrict ourselves to the case where Ω is relatively compact.

In this case one may find $T_j \in \mathcal{Q}'_{(l)}(\bar\Omega;\mathcal{A})$ of the form $\sum_{\#A=j} T_{j,A}\, e_A$, $T_{j,A} \in \mathcal{Q}'(\bar\Omega)$, such that

$$\theta_j(\varphi) = T_j + \mathcal{Q}'_{(l)}(\partial\Omega;\mathcal{A}) + \mathcal{Q}_{(r)}(\Omega;\mathcal{A}) .$$

We now have that, when $\tilde\Omega = \mathbb{R}^{m+1} \setminus \partial\Omega$,

$$\hat T_j(x) = \frac{1}{\omega_{m+1}} \left\langle T_{j,\vec u}\, , \frac{\bar x + \vec u}{|\vec x + \vec u|^{m+1}} \right\rangle$$

is a representing left monogenic function in $\tilde\Omega\setminus\Omega$ of the hyperfunction $T_j + \mathcal{Q}'_{(l)}(\partial\Omega;\mathcal{A})$ and of $\theta_j(\varphi)$ itself.

Hence $\mathcal{H}_l(\theta_j(\varphi))$ is represented by

$$f(u) = \begin{cases} \hat T_j(u) , & u \in \tilde\Omega_+ \\ -\hat T_j(u) , & u \in \tilde\Omega_- . \end{cases}$$

Now let $\varepsilon > 0$ and put $\bar\Omega_\varepsilon = \bar\Omega + B_m(0,\varepsilon)$.

Then for every $g \in M_{(l)}(\bar\Omega_\varepsilon;\mathcal{A})$ there exists $\eta > 0$ such that

$$\langle A_\varepsilon(f) , g \rangle = \int_{\partial(\bar\Omega_\varepsilon \times [-\eta,\eta])} g(u)\, d\sigma_u\, f(u)$$

is defined.

Furthermore by Cauchy's theorem, the above integral does not depend on η , whence it defines an analytic functional $A_\varepsilon(f)$ in $\mathcal{Q}'_{(l)}(\bar\Omega_\varepsilon;\mathcal{A})$ and so a hyperfunction $S = A_\varepsilon(f) + \mathcal{Q}'_{(l)}(\bar\Omega_\varepsilon\setminus\Omega;\mathcal{A})$ in Ω .

<u>Lemma 1.</u> $\mathcal{H}_l(\theta_j(\varphi)) = S + \mathcal{Q}(\Omega;\mathcal{A})$.

<u>Proof.</u> In view of [8] it is sufficient to show that $f - \widehat{A_\varepsilon(f)}$ has an analytic boundary value in Ω .

To that end notice that in $\mathbb{R}^{m+1}_+$,

$$\begin{aligned} &(f - \widehat{A_\varepsilon(f)})(x) \\ &= \frac{1}{\omega_{m+1}} \int_{\partial(\bar\Omega_\varepsilon \times [-1,x_0/2])} \frac{\bar x - \bar u}{|x-u|^{m+1}}\, d\sigma_u \,(\hat T_j(u) - f(u)) \\ &= \frac{2}{\omega_{m+1}} \int_{\mathbb{R}^{m+1}_-\cap\, \partial(\bar\Omega_\varepsilon \times [-1,x_0/2])} \frac{\bar x - \bar u}{|x-u|^{m+1}}\, d\sigma_u\, \hat T_j(u) \end{aligned}$$

which immediately yields an extension of $(f - \widehat{A_\varepsilon(f)})|\, \mathbb{R}^{m+1}_+$ to $\mathbb{R}^{m+1} \setminus ((\bar\Omega_\varepsilon \times \{-1\}) \cup (\partial\bar\Omega_\varepsilon \times [-1 , 0]))$.

As a similar property holds for $(f - \hat{A}_\varepsilon(f))|\mathbb{R}^{m+1}_-$, $f - \hat{A}_\varepsilon(f)$ has an analytic boundary value in Ω . ∎

<u>Lemma 2.</u> S is of the form $S_{j-1} + S_{j+1}$, where $S_{j-1} = \theta_{j-1}(S_{j-1})$ and $S_{j+1} = \theta_{j+1}(S_{j+1})$.

<u>Proof.</u> Let $\delta > 0$. Then we put

$$f_\delta = \begin{cases} f(x+\delta), & x_0 > 0 \\ f(x-\delta), & x_0 < 0 \end{cases} .$$

Now it is easy to see that $A_\varepsilon(f_\delta) \to A_\varepsilon(f)$ in $\mathcal{O}'_{(1)}(\bar{\Omega}_\varepsilon ; \mathcal{A})$ if $\delta \to 0+$.

Hence we only have to show that $A_\varepsilon(f_\delta)$ takes values in $\mathcal{A}_{j-1} \oplus \mathcal{A}_{j+1}$.

But we have that

$$\begin{aligned} A_\varepsilon(f_\delta) &= \chi_{\bar{\Omega}_\varepsilon}(\vec{u}) \cdot (f(\vec{x}+\delta) - f(\vec{x}-\delta)) \\ &= \chi_{\bar{\Omega}_\varepsilon}(\vec{u}) \cdot (\hat{T}_j(\vec{u}+\delta) - \hat{T}_j(\vec{u}-\delta)) \\ &= \chi_{\bar{\Omega}_\varepsilon}(\vec{u}) \cdot \frac{-2}{\omega_{m+1}} \left\langle T_{j,\vec{x}} , \frac{\vec{x}-\vec{u}}{|\vec{x}-\vec{u}-\delta|^{m+1}} \right\rangle , \end{aligned}$$

and the right hand side clearly takes values in $\mathcal{A}_{j-1} \oplus \mathcal{A}_{j+1}$.

Notice that by Lemmas 1 and 2, $\mathcal{H}_l(\theta_j(\varphi))$ takes values in $\mathcal{A}_{j-1} \oplus \mathcal{A}_{j+1}$, which leads to

<u>Definition 6.</u> The operators $\mathcal{H}^+_l$ and $\mathcal{H}^-_l$ are given by

$$\mathcal{H}^+_l = \sum_{j=0}^{m-1} \theta_{j+1} \circ \mathcal{H}_l \circ \theta_j$$

and

$$\mathcal{H}^-_l = \sum_{j=1}^{m} \theta_{j-1} \circ \mathcal{H}_l \circ \theta_j .$$

In a similar way $\mathcal{H}^+_r$ and $\mathcal{H}^-_r$ may be introduced.

By the Lemmas 1 and 2,

$$\begin{aligned} \mathcal{H}^+_l + \mathcal{H}^-_l &= \theta_1 \circ \mathcal{H}_l \circ \theta_0 + \sum_{j=1}^{m-1} (\theta_{j+1} + \theta_{j-1}) \circ \mathcal{H}_l \circ \theta_j \\ &\quad + \theta_{m-1} \circ \mathcal{H}_l \circ \theta_m = \sum_{j=0}^{m} \mathcal{H}_l \circ \theta_j = \mathcal{H}_l \end{aligned}$$

and analogously

$$\mathcal{H}^+_r + \mathcal{H}^-_r = \mathcal{H}_r .$$

Moreover

<u>Theorem 4.</u> $\mathcal{H}^{+2}_l = \mathcal{H}^{-2}_l = \mathcal{H}^{+2}_r = \mathcal{H}^{-2}_r = 0$

and

$$\mathbb{I} = \mathcal{H}^+_l \mathcal{H}^-_l + \mathcal{H}^-_l \mathcal{H}^+_l = \mathcal{H}^+_r \mathcal{H}^-_r + \mathcal{H}^-_r \mathcal{H}^+_r$$

<u>Proof.</u> We have that

$$\mathbb{1} = \mathcal{H}_l^2 = \mathcal{H}_l^{+2} + \mathcal{H}_l^{-2} + \mathcal{H}_l^+ \mathcal{H}_l^- + \mathcal{H}_l^- \mathcal{H}_l^+$$

or

$$(\mathbb{1} - \mathcal{H}_l^+ \mathcal{H}_l^- + \mathcal{H}_l^- \mathcal{H}_l^+) \circ \theta_j = \mathcal{H}_l^{+2} \circ \theta_j + \mathcal{H}_l^{-2} \circ \theta_j ,$$

which leads to

$$\begin{aligned}\mathcal{H}_l^{+2} &= \sum_{j=o}^{m-2} \theta_{j+2} \circ \mathcal{H}_l^{+2} \circ \theta_j \\ &= \sum_{j=o}^{m-2} \theta_{j+2}\, \theta_j - \theta_{j+2} \circ (\mathcal{H}_l^+ \mathcal{H}_l^- + \mathcal{H}_l^- \mathcal{H}_l^+) \circ \theta_j \\ &= 0 ,\end{aligned}$$

Similarly $\mathcal{H}_l^{-2} = 0$ and so $\mathbb{1} = \mathcal{H}_l^+ \mathcal{H}_l^- + \mathcal{H}_l^- \mathcal{H}_l^+$. ■

<u>Corollary.</u> Let $\Omega \subseteq \mathbb{R}^m$ be open. Then the following sequences are exact:

$$\mathcal{E}(\Omega ; \mathcal{A}) \xrightarrow{\mathcal{H}^+_{\{l,r\}}} \mathcal{E}(\Omega ; \mathcal{A}) \xrightarrow{\mathcal{H}^+_{\{l,r\}}} \mathcal{E}(\Omega ; \mathcal{A})$$

$$\mathcal{E}(\Omega ; \mathcal{A}) \xrightarrow{\mathcal{H}^-_{\{l,r\}}} \mathcal{E}(\Omega ; \mathcal{A}) \xrightarrow{\mathcal{H}^-_{\{l,r\}}} \mathcal{E}(\Omega ; \mathcal{A})$$

<u>Proof.</u> Clearly $\mathrm{im}\, \mathcal{H}^+_{\{l,r\}} \subseteq \ker \mathcal{H}^+_{\{l,r\}}$.

Let $\varphi \in \mathcal{E}(\Omega ; \mathcal{A})$ be such that $\mathcal{H}^+_{\{l,r\}} \varphi = 0$. Then by Theorem 4, $\varphi = \mathcal{H}^+_{\{l,r\}} \mathcal{H}^-_{\{l,r\}} \varphi$ and so φ belongs to $\mathrm{im}\, \mathcal{H}^+_{\{l,r\}}$.

Notice that $\mathcal{H}^+_{\{l,r\}}$ transforms j-vectors into (j+1)-vectors and $\mathcal{H}^-_{\{l,r\}}$ transforms j-vectors into (j-1)-vectors.

Hence we have long exact sequences

$$0 \to \theta_o(\mathcal{E}(\Omega ; \mathcal{A})) \xrightarrow{\mathcal{H}^+_{\{l,r\}}} \cdots \theta_m(\mathcal{E}(\Omega ; \mathcal{A})) \xrightarrow{\mathcal{H}^+_{\{l,r\}}} 0$$

$$0 \to \theta_m(\mathcal{E}(\Omega ; \mathcal{A})) \xrightarrow{\mathcal{H}^-_{\{l,r\}}} \cdots \theta_o(\mathcal{E}(\Omega ; \mathcal{A})) \xrightarrow{\mathcal{H}^-_{\{l,r\}}} 0 .$$

Furthermore, $\mathcal{H}_l$ and $\mathcal{H}_r$ are related to each other by

<u>Theorem 5.</u> We have that for each $i \in \mathbb{N}$, $0 \leq i \leq \frac{m}{2}$,

$$\mathcal{H}_l^+ \circ \theta_{2i} = \mathcal{H}_r^+ \circ \theta_{2i} , \quad \mathcal{H}_l^+ \circ \theta_{2i+1} = - \mathcal{H}_r^+ \circ \theta_{2i+1}$$

and

$$\mathcal{H}_l^- \circ \theta_{2i} = - \mathcal{H}_r^- \circ \theta_{2i} , \quad \mathcal{H}_l^- \circ \theta_{2i+1} = \mathcal{H}_r^- \circ \theta_{2i+1} .$$

<u>Proof.</u> Let us check the case where $\varphi \in \mathcal{E}(\Omega ; \mathcal{A})$ is a test-function. By a standard density argument this is sufficient in order to prove the theorem.

Let $\theta_{2i}(\varphi) = \varphi$.

Then

$$\mathcal{H}_l^+ \varphi(\vec{x}) = \frac{2}{\omega_{m+1}} \int_{\mathbb{R}^m} \theta_{2i+1} \left[\frac{\vec{x} - \vec{u}}{|\vec{x} - \vec{u}|^{m+1}} \varphi(\vec{u}) \right] d\vec{u}$$

$$= \frac{2}{\omega_{m+1}} \int_{\mathbb{R}^m} \theta_{2i+1}\left[\varphi(\vec{u}) \frac{\vec{x}-\vec{u}}{|\vec{x}-\vec{u}|^{m+1}} \right] d\vec{u}$$
$$= \mathcal{H}_r^+ \varphi(\vec{x})$$

while

$$\mathcal{H}_l^- \varphi(\vec{x}) = \frac{2}{\omega_{m+1}} \int_{\mathbb{R}^m} \theta_{2i-1}\left[\frac{\vec{x}-\vec{u}}{|\vec{x}-\vec{u}|^{m+1}} \varphi(\vec{u}) \right] d\vec{u}$$
$$= \frac{-2}{\omega_{m+1}} \int_{\mathbb{R}^m} \theta_{2i-1}\left[\varphi(\vec{u}) \frac{\vec{x}-\vec{u}}{|\vec{x}-\vec{u}|^{m+1}} \right] d\vec{u}$$
$$= - \mathcal{H}_r^- \varphi(\vec{x}) .$$

The case $\theta_{2i+1}(\varphi) = \varphi$ is similar. ∎

<u>Remarks.</u> (1) Theorem 5 yields the identities

$$\mathcal{H}_l^+ \mathcal{H}_l^- = \mathcal{H}_r^+ \mathcal{H}_r^- \quad \text{and} \quad \mathcal{H}_l^- \mathcal{H}_l^+ = \mathcal{H}_r^- \mathcal{H}_r^+ .$$

Furthermore,

$$\begin{aligned}
\mathcal{H}_l \mathcal{H}_r &= (\mathcal{H}_l^+ + \mathcal{H}_l^-) (\mathcal{H}_r^+ + \mathcal{H}_r^-) \\
&= \mathcal{H}_l^+ \mathcal{H}_r^- + \mathcal{H}_l^- \mathcal{H}_r^+ \\
&= (\mathcal{H}_l^- \mathcal{H}_l^+ - \mathcal{H}_l^+ \mathcal{H}_l^-) \circ \gamma \\
&= \mathcal{H}_l^+ \circ \gamma \circ \mathcal{H}_l^- - \mathcal{H}_l^- \circ \gamma \circ \mathcal{H}_l^+ \\
&= \mathcal{H}_r^+ \mathcal{H}_l^- + \mathcal{H}_r^- \mathcal{H}_l^+ \\
&= \mathcal{H}_r \mathcal{H}_l .
\end{aligned}$$

(2) The relations in Theorem 5 are equivalent with the following ones:

$$\mathcal{H}_l^+ \circ C_+ = \mathcal{H}_r^+ \circ C_+ , \quad \mathcal{H}_l^+ \circ C_- = - \mathcal{H}_r^+ \circ C_-$$
$$\mathcal{H}_l^- \circ C_+ = - \mathcal{H}_r^- \circ C_+ , \quad \mathcal{H}_l^- \circ C_- = \mathcal{H}_r^- \circ C_- .$$

Hence

$$\mathcal{H}_r = \mathcal{H}_l^+ \circ C_+ - \mathcal{H}_l^+ \circ C_- - \mathcal{H}_l^- \circ C_+ + \mathcal{H}_l^- \circ C_- .$$

From which we obtain that

$$\mathcal{H}_l^+ = \frac{1}{2} (\mathcal{H}_l + \mathcal{H}_r \circ \gamma)$$

and

$$\mathcal{H}_l^- = \frac{1}{2} (\mathcal{H}_l - \mathcal{H}_r \circ \gamma) .$$

(3) The following relations hold:

$$[\mathcal{H}_l , \mathcal{H}_r] = 0 ,$$
$$\{\mathcal{H}_l , \gamma\} = 0 , \quad \{\mathcal{H}_r , \gamma\} = 0 ,$$
$$\{\mathcal{H}_l^+ , \mathcal{H}_l^-\} = \mathbb{I}$$
$$[\mathcal{H}_l^- , \mathcal{H}_l^+] \circ \gamma = \mathcal{H}_l \mathcal{H}_r .$$

Furthermore it is easy to see that

$$\mathcal{H}_l^+ \circ \alpha = \beta \circ \mathcal{H}_l^+ , \quad \mathcal{H}_l^- \circ \alpha = - \beta \circ \mathcal{H}_l^- ,$$

$$\mathcal{H}_l^+ \circ \beta = - \alpha \circ \mathcal{H}_l^+ , \quad \mathcal{H}_l^- \circ \beta = \alpha \circ \mathcal{H}_l^- .$$

Analogous relations hold for $\mathcal{H}_r^+$ and $\mathcal{H}_r^-$.
Furthermore we have that

$$\mathcal{H}_{\{l \atop r\}} \circ \alpha = - \alpha \circ \mathcal{H}_{\{r \atop l\}} , \quad \mathcal{H}_{\{l \atop r\}} \circ \beta = \beta \circ \mathcal{H}_{\{r \atop l\}} .$$

(4) The group generated by $\{\alpha, \beta, \mathcal{H}_l, \mathcal{H}_r\}$ contains thirty-two elements which may be written in the form $\pm g_1 g_2$ with $g_1 \in \{\mathbb{1}, \mathcal{H}_l, \mathcal{H}_r, \mathcal{H}_l \mathcal{H}_r\}$ and $g_2 \in \{\mathbb{1}, \alpha, \beta, \gamma\}$.

In the following theorem we investigate a relation between the values of a microfunction and its singularities.

Theorem 6. We have that

(i) $\ker A_{\pm} \cap \ker (A_{\mp} \circ Q_{\{\pm\},\{l \atop r\}}) \subseteq \ker Q_{\{\pm\},\{l \atop r\}}$

(ii) $\ker B_{\pm} \cap \ker (B_{\mp} \circ Q_{\{\pm\},\{l \atop r\}}) \subseteq \ker Q_{\{\pm\},\{l \atop r\}}$

Proof. As (ii) is similar to (i), it suffices to prove (i).
Let $\varphi \in \mathcal{C}(\Omega ; \mathcal{A})$ and consider the case $\mathrm{Im}\, \varphi = 0$ only.
Then $\varphi = \sum_{j=0}^{\infty} (\theta_{4j}(\varphi) + \theta_{4j+1}(\varphi) + \theta_{4j+2}(\varphi) + \theta_{4j+3}(\varphi))$,
whereby $\theta_j(\varphi) = 0$ if $j > m$.
As $\varphi \in \ker A_{\pm}$, for every $l \in \mathbb{N}$, $\theta_{4l}(\varphi) = \theta_{4l+3}(\varphi) = 0$.
Furthermore,

$$\begin{aligned} & A_- Q_{+,l}\varphi \\ &= \frac{1}{2} \sum_{j=0}^{\infty} (\theta_{4j+1}(\varphi) + \mathcal{H}_l^- \theta_{4j+2}(\varphi) + \theta_{4j+2}(\varphi) \\ & \qquad + \mathcal{H}_l^+ \theta_{4j+1}(\varphi)) \end{aligned}$$

Hence, $A_- Q_{+,l}\varphi = 0$ implies that for every $j \in \mathbb{N}$,

$$\theta_{4j+1}(\varphi) = - \mathcal{H}_l^- \theta_{4j+2}(\varphi)$$
$$\theta_{4j+2}(\varphi) = - \mathcal{H}_l^+ \theta_{4j+1}(\varphi) .$$

So, $\varphi = (\mathbb{1} - \mathcal{H}_l^+) \sum_{j=0}^{\infty} \theta_{4j+1}(\varphi)$, and as $\mathcal{H}_l^- \theta_{4j+1}(\varphi) = 0$,
$\varphi = Q_{-,l} \sum_{j=0}^{\infty} \theta_{4j+1}(\varphi)$, whence $Q_{+,l}\varphi = 0$. ■

Notice that Theorem 6 may be generalized as follows.
Let $\{\varphi_1, \ldots, \varphi_k\}$ be a set of microfunctions satisfying

$$\varphi_1 = - \mathcal{H}_l^- \varphi_2 , \quad \varphi_2 = - \mathcal{H}_l^+ \varphi_1 - \mathcal{H}_l^- \varphi_3 , \ldots ,$$
$$\varphi_k = - \mathcal{H}_l^{\pm}{}_k \varphi_{k-1} .$$

Then $Q_{+,l}(\sum_{j=1}^{k} \varphi_j) = 0$.

2. Decomposition of singularities

In this section we study the singularities of $\mathcal{A}$-micros.

First we show that every $\mathcal{A}$-micro φ admits a decomposition of the form $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$, where $\varphi_q \in \mathcal{C}_q = \mathfrak{I}_q/\mathfrak{I}_o$ or $s(\varphi q) = s_q$, $q = 1, \dots, 4$.
To that end we introduce some new projection operators.
Put

$$\varepsilon_{\pm} = Q_{\pm,l}\, Q_{\pm,r} = Q_{\pm,r}\, Q_{\pm,l}$$

and

$$\mu_{\pm} = Q_{\pm,l}\, Q_{\mp,r} = Q_{\mp,r}\, Q_{\pm,l} .$$

Then $\mathbb{I} = \varepsilon_+ + \varepsilon_- + \mu_+ + \mu_-$, $\varepsilon_{\pm}^2 = \varepsilon_{\pm}$, $\mu_{\pm}^2 = \mu_{\pm}$,
$\varepsilon_{\pm}\varepsilon_{\mp} = 0$, $\mu_{\pm}\mu_{\mp} = 0$, $\varepsilon_{\pm}\mu_{\pm} = 0$ and $\varepsilon_{\pm}\mu_{\mp} = 0$.

Furthermore we have

Theorem 7. Every $\varphi \in \mathcal{C}_{o,m}$ admits a unique decomposition of the form $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$, with $\varphi_q \in \mathcal{C}_q$, where $\mathcal{C}_q = \mathfrak{I}_q/\mathfrak{I}_o$, $q = 1, \dots, 4$.

Proof. Let $\varphi \in \mathcal{C}_{o,m}$ and put

$$\varphi_1 = \varepsilon_+\varphi, \quad \varphi_2 = \varepsilon_-\varphi, \quad \varphi_3 = \mu_+\varphi \quad \text{and} \quad \varphi_4 = \mu_-\varphi ;$$

Then $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ and $\varphi_q \in \mathcal{C}_q$.
Indeed, take for example $\varphi_1 = \varepsilon_+\varphi$. Then $\varphi_1 = Q_{+,l}Q_{+,r}\varphi$ and $\varphi_1 = Q_{+,r}Q_{+,l}\varphi$ whence $Q_{-,l}\,\varphi_1 = Q_{-,r}\,\varphi_1 = 0$ or $s(\varphi_1) = s_1$.
On the other hand, suppose that φ admits a decomposition of the form $\varphi = \varphi_1' + \varphi_2' + \varphi_3' + \varphi_4'$ with $\varphi_q' \in \mathcal{C}_q$. Then
$\varphi_1 = \varepsilon_+\varphi_1' + \varepsilon_+\varphi_2' + \varepsilon_+\varphi_3' + \varepsilon_+\varphi_4'$.
But, as $s(\varphi_1') = s_1$, there exists a representing microfunction ψ for φ_1', defined in some neighbourhood of the origin, such that $\psi = \varepsilon_+\psi$. Hence $\varepsilon_+\varphi_1' = \varphi_1'$ in $\mathcal{C}_{o,m}$.
Analogously $\varepsilon_+\varphi_2' = \varepsilon_+\varphi_3' = \varepsilon_+\varphi_4' = 0$. ∎

Put $\varepsilon = \varepsilon_+ + \varepsilon_-$ and $\mu = \mu_+ + \mu_-$; then

$$\varepsilon = \frac{1}{2}(\mathbb{I} + \mathcal{H}_l\,\mathcal{H}_r)$$

$$\mu = \frac{1}{2}(\mathbb{I} - \mathcal{H}_l\,\mathcal{H}_r)$$

and so $\mathcal{H}_l\,\mathcal{H}_r = \varepsilon - \mu$.
Notice that when $m = 1$, $\varepsilon = \mathbb{I}$ and $\mu = 0$.
ε will be called " electric projection operator " while μ will be called " magnetic projection operator ".
The reason for this will be explained in section 5.
Let $\varphi \in \mathcal{C}_{o,m}$. Then we associate with φ a 2×2-matrix

$p(\varphi)$ in the following way:

$$p_{11}(\varphi) = \begin{cases} 1, & \text{if } \varepsilon_+\varphi \neq 0 \\ 0, & \text{if } \varepsilon_+\varphi = 0 \end{cases}$$

$$p_{22}(\varphi) = \begin{cases} 1, & \text{if } \varepsilon_-\varphi \neq 0 \\ 0, & \text{if } \varepsilon_-\varphi = 0 \end{cases}$$

$$p_{12}(\varphi) = \begin{cases} 1, & \text{if } \mu_+\varphi \neq 0 \\ 0, & \text{if } \mu_+\varphi = 0 \end{cases}$$

$$p_{21}(\varphi) = \begin{cases} 1, & \text{if } \mu_-\varphi \neq 0 \\ 0, & \text{if } \mu_-\varphi = 0 . \end{cases}$$

The image of p consists of all matrices over $\mathbb{Z}_2$.
Furthermore, $p(\varphi) = p(\mathcal{K}_l\varphi) = p(\mathcal{K}_r\varphi) = p(\mathcal{K}_l\mathcal{K}_r\varphi)$, while the group $\mathcal{G} = \{\mathbb{I}, \alpha, \beta, \gamma\}$ acts as follows on the 2×2-matrices over $\mathbb{Z}_2$.
Let $g \in \mathcal{G}$; then we put $g(p)(\varphi) = p(g(\varphi))$. As $p(\varphi) = p(\psi)$ implies that $g(p)(\varphi) = g(p)(\psi)$, $\mathcal{G}$ may be considered as a group acting on the image of p .
The orbits under this group action are:

$$\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\},\ \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\},\ \left\{\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\};$$
$$\left\{\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix}\right\},\ \left\{\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right\},\ \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\},\ \left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}.$$

These orbits are respectively denoted by AB_0 , B_1 , A_1 , O , B_2 , A_2 , AB_1 , AB_2 , AB_3 . The orbit number $O(\varphi)$ of an $\mathcal{A}$-micro φ is the orbit to which $p(\varphi)$ belongs.
When the orbit number of φ equals B_1 or B_2 , then $p(\varphi)$ is invariant under β but not under α ; when $O(\varphi) \in \{A_1, A_2\}$, $p(\varphi)$ is invariant under α but not under β ; when $O(\varphi) = O$, $p(\ )$ is not invariant under α , β and γ and when $O(\varphi) \in \{AB_0, AB_1, AB_2, AB_3\}$, $p(\varphi)$ is invariant under $\mathcal{G}$.
These remarks lead immediately to the following theorem which extends a wellknown property of hyperfunctions on the real line, stating that when a hyperfunction F is real valued and $x \in \text{sing supp } F$, then $x \in SS_+F \cap SS_-F$.

<u>Theorem 8.</u> Let $\varphi \in \mathcal{E}_{o,m}$ be non zero. Then $A_\pm\varphi = 0$ implies that $O(\varphi) \in \{A_1, A_2, AB_1, AB_2, AB_3\}$, $B_\pm\varphi = 0$ implies that

$O(\varphi) \in \{B_1, B_2, AB_1, AB_2, AB_3\}$ and $C_\pm \varphi = 0$ implies that $O(\varphi) \in \{AB_1, AB_2, AB_3\}$.

Let $\Omega \subseteq \mathbb{R}^m$ be open and let $\tilde{\Omega} \subseteq \mathbb{R}^{m+1}$ be open such that Ω is relatively closed in $\tilde{\Omega}$. Then in [2] we have shown a Painlevé type theorem stating that when $f \in M_{(r)}(\tilde{\Omega}\setminus\Omega;\mathcal{A})$ is such that the boundary value of f for $x_0 \to 0+$ equals the boundary value of f for $x_0 \to 0-$, and this in the sense of continuous functions, then f extends to a monogenic function in $\tilde{\Omega}$. Moreover, by using [8] , this result may easily be extended to the case where $f(\vec{x} \pm 0)$ are hyperfunctions.
We now show a Painlevé type theorem which at the same time involves left and right monogenic functions and which makes use of the theory of $\mathcal{A}$-microfunctions.

Theorem 9. Let $f_l, g_l \in M_{(r)}(\tilde{\Omega}\setminus\Omega;\mathcal{A})$ and let $f_r, g_r \in M_{(l)}(\tilde{\Omega}\setminus\Omega;\mathcal{A})$. Then we have:
(i) If the boundary value F of $f_l|\tilde{\Omega}_+$ equals the boundary value of $f_r|\tilde{\Omega}_+$ and is also equal to the sum of the boundary values of $g_l|\tilde{\Omega}_-$ and $g_r|\tilde{\Omega}_-$, then F is analytic;
(ii) If the boundary value F of $f_l|\tilde{\Omega}_+$ equals the boundary value of $f_r|\tilde{\Omega}_-$ and is also equal to the sum of the boundary values of $g_l|\tilde{\Omega}_-$ and $g_r|\tilde{\Omega}_+$, then F is analytic.
Proof. We only show (i).
As F is the sum of the boundary values of $g_l|\tilde{\Omega}_-$ and $g_r|\tilde{\Omega}_-$, $p(\Pi(F)) \neq \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ since then $\varepsilon_+\Pi(F) = 0$.
On the other hand, if F is the boundary value of $f_l|\tilde{\Omega}_+$ and of $f_r|\tilde{\Omega}_+$, $\Pi(F) = \varepsilon_+\Pi(F)$.
Hence $\Pi(F) = 0$ or F is analytic. ■

3. Special microfunctions

In this section we introduce some microfunctions which we shall need in order to define some special micro-differential operators used in section 5 .

(i) The Dirac microfunctions δ , $\delta_\pm$, δ_-

The Dirac microfunction δ is represented by the Cauchy kernel $\frac{1}{\omega_{m+1}}\frac{\bar{x}}{|x|^{m+1}}$ which is left and right monogenic in $\mathbb{R}^{m+1}\setminus\{0\}$.

Hence $p(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Furthermore δ_+ and δ_- are defined by

$$\delta_\pm = Q_{\pm,l}\,\delta = Q_{\pm,r}\,\delta = \varepsilon_\pm \delta \ .$$

Clearly $\delta = \delta_+ + \delta_-$ and $\mathcal{H}_l \delta = \mathcal{H}_r \delta = \delta_+ - \delta_-$.

Moreover representing monogenic functions for $\delta_\pm$ are given by

$$\delta_\pm \longleftrightarrow \begin{cases} \frac{1}{\omega_{m+1}} \frac{\bar{x}}{|x|^{m+1}} , & x_0 \gtrless 0 \\ 0 , & x_0 \lessgtr 0 \end{cases} .$$

We therefore sometimes use the notations

$$\delta_\pm = \frac{1}{\omega_{m+1}} \frac{0 \mp \vec{x}}{|0 \mp \vec{x}|^{m+1}}$$

and

$$\mathcal{H}\delta = -\frac{2}{\omega_{m+1}} \frac{\vec{x}}{|\vec{x}|^{m+1}} \ .$$

(ii) The logarithmic microfunctions λ , λ_+ , λ_-

We construct generalizations to $\mathbb{R}^m$ of the logarithmic microfunction on the real line, called logarithmic microfunctions.
To that end, we first introduce the logarithmic functions $\Lambda^\pm(x)$ by

$$\Lambda^+(x) = \frac{1}{\omega_{m+1}} \int_{-\infty}^{0} \frac{\bar{x} - h}{|x-h|^{m+1}}\, dh$$

$$\Lambda^-(x) = -\frac{1}{\omega_{m+1}} \int_{0}^{+\infty} \frac{\bar{x} - h}{|x-h|^{m+1}}\, dh \ .$$

Another expression for $\Lambda^\pm(x)$ has been given in [4].
Notice that $\Lambda^+(x)$ is left and right monogenic in $\mathbb{R}^{m+1} \setminus \{x : \vec{x} = 0 \text{ and } x_0 \leq 0\}$ while $\Lambda^-(x)$ is left and right monogenic in $\mathbb{R}^{m+1} \setminus \{x : \vec{x} = 0 \text{ and } x_0 \geq 0\}$.

Furthermore $\partial_{x_0} \Lambda^+(x) = \partial_{x_0} \Lambda^-(x) = -\frac{1}{\omega_{m+1}} \frac{\bar{x}}{|x|^{m+1}}$,

whence $D_0 \Lambda^+(x) = D_0 \Lambda^-(x) = \frac{1}{\omega_{m+1}} \frac{\bar{x}}{|x|^{m+1}}$.

The positive and negative microfunctions $\lambda_+(x)$ and $\lambda_-(x)$ are now determined by giving their representing monogenic functions:

$$\lambda_+(x) \longleftrightarrow \begin{cases} \Lambda^+(x) , & x_0 > 0 \\ 0 , & x_0 < 0 \end{cases}$$

$$\lambda_-(x) \longleftrightarrow \begin{cases} 0 , & x_0 > 0 \\ \Lambda^-(x) , & x_0 < 0 \end{cases} .$$

Clearly $\operatorname{supp} \lambda_+ = \operatorname{supp} \lambda_- = \{0\}$, while $\lambda_+ = \Pi(\Lambda^+(\vec{x}+0))$ and $\lambda_- = \Pi(-\Lambda^-(\vec{x}-0))$.

Notice also that $D_0 \lambda_{\pm} = \delta_{\pm}$.

Finally the logarithmic microfunction λ is given by $\lambda = \lambda_+ + \lambda_-$. It is represented by

$$\lambda \longleftrightarrow \begin{cases} \Lambda^+(x) \ , \ x_0 > 0 \\ \Lambda^-(x) \ , \ x_0 < 0 \end{cases}$$

and it satisfies the equation $D_0 \lambda = \delta$.

Hence

$$\lambda(\vec{x}) = \frac{1}{\omega_{m+1}} \int_{\mathbb{R}} \frac{-\vec{x} - h}{|\vec{x} - h|^{m+1}} \, dh = \frac{-1}{\omega_m} \cdot \frac{\vec{x}}{|\vec{x}|^m} .$$

Moreover a straightforward calculation yields

$$\lambda_{\pm}(\vec{x}) = \frac{-1}{2\,\omega_m} \frac{\vec{x}}{|x|^m} \pm \frac{1}{(m-1)\,\omega_{m+1}} \frac{1}{|\vec{x}|^{m-1}} .$$

This formula should be compared with the formula for $\ln x$, $x \in \mathbb{R}$, observing that $-\frac{1}{(m-1)\,\omega_{m+1}} \cdot \frac{1}{|x|^{m-1}}$ is the kernel of Δ_{m+1} while $-\frac{1}{\omega_m} \frac{\vec{x}}{|\vec{x}|^m}$ is the kernel of D_0 .

Notice also that $\mathcal{T}_l \lambda = \mathcal{T}_r \lambda = \lambda_+ - \lambda_- = \frac{2}{(m-1)\,\omega_{m+1}} \cdot \frac{1}{|\vec{x}|^{m-1}}$

whence $\mathcal{H}_l^+ \lambda = \mathcal{H}_r^+ \lambda = 0$ and so

$$\lambda = \mathcal{H}_l^+\left(\frac{2}{(m-1)\,\omega_{m+1}} \cdot \frac{1}{|\vec{x}|^{m-1}}\right) = \mathcal{H}_r^+\left(\frac{2}{(m-1)\,\omega_{m+1}} \cdot \frac{1}{|\vec{x}|^{m-1}}\right).$$

4. Convolution operators

In this section we mainly deal with convolution operators associated with microfunctions φ supported by the origin. The case where the microfunctions have compact support is quite similar. The convolution operators associated with microfunctions supported by the origin will be called micro-differential operators. They contain all usual differential operators as well as the inverse operators of elliptic differential operators.

We first define the convolution operators associated with special microfunctions.

(i) The case $(\mathbb{I} - \varepsilon_{\pm})\varphi = 0$

Let $S \subset \mathbb{R}^{m+1}$ be closed such that $\mathring{S} \subset \mathbb{R}^{m+1}_-$ and $\partial S \cap \mathbb{R}^m = \{0\}$ and let f be a function in $\mathbb{R}^{m+1} \setminus S$ such that $Df = fD = 0$ in $\mathbb{R}^{m+1} \setminus S$.

Moreover, let φ be the microfunction represented by

$$h = \begin{cases} f \ , \ x_0 > 0 \\ 0 \ , \ x_0 < 0 \ . \end{cases}$$

Then clearly $\mathrm{supp}\,\varphi = \{0\}$ and $(\mathbb{1} - \mathcal{E}_+)\varphi = 0$.
Let $\psi \in \mathcal{C}(\Omega;\mathcal{A})$ be of the form $\psi = Q_{+,1}\psi$. Then we define a micro-differential operator $\mathcal{D}(\varphi)$ acting on ψ at the left side as follows.
Let $\psi = g + M_{(r)+}(\mathbb{R}^{m+1}\setminus \partial\Omega, \Omega)$, $g \in M_{(r)}(\mathbb{R}^{m+1}_+;\mathcal{A})$ and let $\vec{x} \in \Omega$ and $\rho > 0$ be such that $B_m(\vec{x},\rho) \subseteq \Omega$.
Then for $\varepsilon > 0$ sufficiently small

$$S_{\varepsilon,\vec{x},\rho} = \{y \in \mathbb{R}^{m+1} : y - u \in S \text{ for some } u \text{ with } 0 \leq u_0 \leq \varepsilon \text{ and } \vec{u} \in \partial B_m(\vec{x},\rho)\}$$

is closed and does not contain $\vec{x}$.
Put

$$g_{\varepsilon,\vec{x},\rho}(x) = \int_{\Sigma_\varepsilon} f(x-u)\, d\sigma_u\, g(u) \quad , \; x_0 > \varepsilon \;,$$

where $\Sigma_\varepsilon = \{\vec{y} + \varepsilon : \vec{y} \in B_m(\vec{x},\rho)\}$ and $d\sigma_u = d\vec{u}$.
Then by Cauchy's theorem, for $x_0 > \varepsilon$,

$$g_{\varepsilon,\vec{x},\rho}(x) = \int_{\Sigma_{\varepsilon'} \cup (\,]\varepsilon',\varepsilon[\, \times\, \partial B_m(\vec{x},\rho))} f(x-u)\, d\sigma_u\, g(u) .$$

Now it is easily seen that $g_{\varepsilon,\vec{x},\rho}$ has a left monogenic extension to $\Omega_\varepsilon = (\mathbb{R}_+ \times \mathring{B}_m(\vec{x},\rho)) \setminus S_{\varepsilon,\vec{x},\rho}$ and so, as $\vec{x} \notin S_{\varepsilon,\vec{x},\rho}$, it determines a microfunction $\psi_{\varepsilon,\vec{x},\rho}(\vec{u})$ in the neighbourhood $\omega_{\varepsilon,\vec{x},\rho} = \mathring{B}_m(\vec{x},\rho) \setminus S_{\varepsilon,\vec{x},\rho}$ of $\vec{x}$.
Furthermore, when ε', ρ' are such that $\omega_{\varepsilon',\vec{x},\rho'}$ is still a neighbourhood of $\vec{x}$, $g_{\varepsilon',\vec{x},\rho'} - g_{\varepsilon,\vec{x},\rho}$ admits a left monogenic extension about each point of $\omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon',\vec{x},\rho'}$, whence

$$\psi_{\varepsilon,\vec{x},\rho} \,|\, \omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon',\vec{x},\rho'} = \psi_{\varepsilon',\vec{x},\rho'} \,|\, \omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon',\vec{x},\rho'} .$$

Moreover, when $\vec{y} \in \Omega$ is such that $\omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon,\vec{y},\rho} \neq \emptyset$, $g_{\varepsilon,\vec{x},\rho} - g_{\varepsilon,\vec{y},\rho}$ admits a left monogenic extension about each point of $\omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon,\vec{y},\rho}$, whence $\psi_{\varepsilon,\vec{x},\rho}$ coincides with $\psi_{\varepsilon,\vec{y},\rho}$ in $\omega_{\varepsilon,\vec{x},\rho} \cap \omega_{\varepsilon,\vec{y},\rho}$.
Hence, in view of the sheaf properties of $\mathcal{C}(\Omega;\mathcal{A})$, there exists a unique microfunction ψ_f in Ω such that
$\psi_{\varepsilon,\vec{x},\rho} = \psi_f \,|\, \omega_{\varepsilon,\vec{x},\rho}$.
Furthermore, when $f' \in M_{(r)+}(\mathbb{R}^{m+1},\mathbb{R}^m) \cap M_{(1)+}(\mathbb{R}^{m+1},\mathbb{R}^m)$, $\psi_{f+f'} = \psi_f$. Hence ψ_f depends only on φ and ψ and so we may define the action of the micro-differential operator $\mathcal{D}(\varphi)$ on ψ by

$$\mathcal{D}(\varphi)\psi = \psi_f .$$

Notice that the definition of $\mathcal{D}(\varphi)\psi$ is completely local and that $\mathrm{supp}\,\mathcal{D}(\varphi)\psi \subseteq \mathrm{supp}\,\psi$, for every $\psi \in \mathcal{C}(\Omega;\mathcal{A})$ for

which $\psi = Q_{+,l}\psi$.
By using Cauchy's theorem, one can globalize the definition of $\partial(\varphi)\psi$ as follows.
Let K_1 , K_2 be compact in Ω such that ∂K_2 is a smooth surface and $K_1 \subseteq \mathring{K}_2$.
Then for some $\varepsilon > 0$, $S_{\varepsilon,K_2} \cap K_1 = 0$, where

$$S_{\varepsilon,K_2} = \{ y \in \mathbb{R}^{m+1} : y - u \in S \text{ for some } u \text{ with } 0 \le u_0 \le \varepsilon \text{ and } \vec{u} \in \partial K_2 \} .$$

In this case,

$$g_{\varepsilon,K_2}(x) = \int_{K_2+\varepsilon} f(x-u)\, d\sigma_u\, g(u)$$

represents $\partial(\varphi)\psi$ in a neighbourhood of K_1 .
Consider now $\psi \in \mathcal{C}(\Omega;\mathcal{A})$ such that $\psi = Q_{+,r}\psi$ and let $g \in M_{(l)}(\mathbb{R}^{m+1};\mathcal{A})$ represent ψ . Then in a similar way, we may define $\psi\,\partial(\varphi)$, starting from

$$g_{\varepsilon,\vec{x},\rho}(x) = \int_{B_m(\vec{x},\rho)+\varepsilon} g(u)\, d\sigma_u\, f(x-u)$$

as a local right monogenic representation for $\psi\,\partial(\varphi)$.

Definition 7. Let $\psi \in \mathcal{C}(\Omega;\mathcal{A})$. Then the left and right action of the micro-differential operator $\partial(\varphi)$ on ψ are respectively given by

$$\partial(\varphi)\psi = \partial(\varphi)\,(Q_{+,l}\psi)$$

and

$$\psi\,\partial(\varphi) = (Q_{+,r}\psi)\,\partial(\varphi) .$$

Of course $Q_{-,l}(\partial(\varphi)\psi) = Q_{-,r}(\psi\,\partial(\varphi)) = 0$.
Moreover we have the representation formulae

Theorem 10. For every $\psi \in \mathcal{C}(\Omega;\mathcal{A})$,

$$\partial(\delta_+)\psi = Q_{+,l}\psi \quad\text{and}\quad \psi\,\partial(\delta_+) = Q_{+,r}\psi .$$

Proof. Let $g \in M_{(r)}(\mathbb{R}^{m+1}_+;\mathcal{A})$ represent $Q_{+,l}\psi$.
Then locally $\partial(\delta_+)\psi$ is represented by

$$\frac{1}{\omega_{m+1}} \int_{K+\varepsilon} \frac{\bar{x}-\bar{u}}{|x-u|^{m+1}}\, d\vec{u}\; g(u) \; , \; x_0 > 0 \; , \; K \subseteq \Omega \text{ compact.}$$

As for $0 < x_0 < 1$

$$\frac{1}{\omega_{m+1}} \int_{(K\times[\varepsilon,1])\setminus(K+\varepsilon)} \frac{\bar{u}-\bar{x}}{|u-x|^{m+1}}\, d\sigma_u\; g(u)$$

represents the microfunction 0 , $\partial(\delta_+)\psi$ is also represented by

$$\frac{1}{\omega_{m+1}} \int_{K\times[\varepsilon,1]} \frac{\bar{u}-\bar{x}}{|u-x|^{m+1}} d\sigma_u\, g(u) = g(x)$$

and so $\gamma(\delta_+)\psi = Q_{+,1}\psi$. ∎

Theorem 11. For every $\psi \in \mathcal{E}(\Omega;\mathcal{A})$,

$$\gamma(\lambda_+)D_o\psi = D_o\,\gamma(\lambda_+)\psi = Q_{+,1}\psi$$

and

$$\psi\,\gamma(\lambda_+)D_o = \psi D_o\,\gamma(\lambda_+) = Q_{+,r}\psi \ .$$

Proof. Let $g \in M_{(r)}(\mathbb{R}^{m+1}_+;\mathcal{A})$ represent $Q_{+,1}\psi$.
Then $Q_{+,1}D_o g = D_o Q_{+,1} g$ is represented by $-\frac{\partial}{\partial x_o} g$.
Hence, $\gamma(\lambda_+)D_o\psi$ admits the local representation in $\overset{\circ}{K}$:

$$\int_{K+\varepsilon} \Lambda^+(x-u)\, d\sigma_u \left(-\frac{\partial}{\partial u_o} g\right)$$

$$= \lim_{\eta\to 0} \frac{1}{\eta}\Big[\int_{K+\varepsilon}(\Lambda^+(x-(u+\eta)) - \Lambda^+(x-u))\, d\sigma_u\, g(u)$$

$$+ \int_{\partial K\times[\varepsilon,\varepsilon+\eta]} \Lambda^+(x-u)\, d\sigma_u\, g(u-\eta)\Big)$$

$$= -\int_{K+\varepsilon} \frac{\partial}{\partial x_o}\Lambda^+(x-u)\, d\sigma_u\, g(u) + \int_{\partial K}\Lambda^+(x-u)\, dS_u\, g(u),$$

dS_u being the oriented surface element on ∂K .
As $\int_{\partial K}\Lambda^+(x-u)\, dS_u\, g(u)$ represents the zero microfunction in $\overset{\circ}{K}$, $\gamma(\lambda_+)D_o\psi|\overset{\circ}{K}$ is also represented by

$$-\int_{K+\varepsilon} \frac{\partial}{\partial x_o}\Lambda^+(x-u)\, d\sigma_u\, g(u)$$

$$= \frac{1}{\omega_{m+1}} \int_{K+\varepsilon} \frac{\bar{x}-\bar{u}}{|x-u|^{m+1}} d\sigma_u\, g(u)$$

and so, by Theorem 10 , $\gamma(\lambda_+)D_o\psi = Q_{+,1}\psi$.
On the other hand, $D_o\,\gamma(\lambda_+)\psi$ is locally represented by

$$\int_{K+\varepsilon} D_o\Lambda^+(x-u)\, d\sigma_u\, g(u)$$

$$= \frac{1}{\omega_{m+1}} \int_{K+\varepsilon} \frac{\bar{x}-\bar{u}}{|x-u|^{m+1}} d\sigma_u\, g(u)$$

whence $D_o\,\gamma(\lambda_+)\psi = Q_{+,1}\psi$. ∎

Until now we investigated the case where $(\mathbb{1}-\varepsilon_+)\varphi = 0$.
We now consider the case where $(\mathbb{1}-\varepsilon_-)\varphi = 0$.
Let $S \subset \mathbb{R}^{m+1}$ be closed such that $\overset{\circ}{S} \subseteq \mathbb{R}^{m+1}_+$ and $\partial S \cap \mathbb{R}^m = \{0\}$, let f be a function in $\mathbb{R}^{m+1}\setminus S$ with $Df = fD = 0$ in $\mathbb{R}^{m+1}\setminus S$ and let the microfunction φ be represented by

$$h = \begin{cases} 0 \ , & x_o > 0 \\ f \ , & x_o < 0 \ . \end{cases}$$

Notice that supp $\varphi = \{0\}$ and $\varphi = \varepsilon_- \varphi$.
Let $\psi \in \mathcal{E}(\Omega; \mathcal{A})$ be such that $\psi = Q_{-,l}\psi$ and let

$$h' = \begin{cases} 0, & \text{if } x_0 > 0 \\ g, & \text{if } x_0 < 0, \end{cases}$$

with $g \in M_{(r)}(\mathbb{R}^{m+1}_-; \mathcal{A})$, represent ψ.
Then in order to define $\partial(\varphi)\psi$ we use the local representation

$$g_{\varepsilon,\vec{x},\rho}(x) = \int_{\Sigma_{-\varepsilon}} f(x-u)\, d\sigma_u\, g(u),$$

where $\Sigma_{-\varepsilon} = B_m(\vec{x}, \rho) - \varepsilon$ and $d\sigma_u = -d\vec{u}$.
In a similar way $\psi\partial(\varphi)$ may be introduced, where $\psi = Q_{-,r}\psi$.
Furthermore, for a general $\psi \in \mathcal{E}(\Omega; \mathcal{A})$ we put
$\partial(\varphi)\psi = \partial(\varphi)\, Q_{-,l}\psi$ and $\psi\partial(\varphi) = (Q_{-,r}\psi)\,\partial(\varphi)$
and in analogy with the Theorems 10 and 11 we have

<u>Theorem 12.</u> For every $\psi \in \mathcal{E}(\Omega; \mathcal{A})$

$$\partial(\delta_-)\psi = Q_{-,l}\psi \quad \text{and} \quad \psi\partial(\delta_-) = Q_{-,r}\psi .$$

Moreover

$$\partial(\lambda_-)\, D_0\psi = D_0\,\partial(\lambda_-)\psi = Q_{-,l}\psi$$

and

$$\psi\, D_0\,\partial(\lambda_-) = \psi\partial(\lambda_-)\, D_0 = Q_{-,r}\psi .$$

Notice that $Q_{+,l}\partial(\varphi)\psi = Q_{+,r}(\psi\partial(\varphi)) = 0$.

<u>(ii) The case $(\mathbb{I} - \mu_\pm)\varphi = 0$</u>

Let $S \subset \mathbb{R}^{m+1}$ be closed such that $\mathring{S} \subset \mathbb{R}^{m+1}_-$ and $\partial S \cap \mathbb{R}^m = \{0\}$ and let f be a function in $\mathbb{R}^{m+1} \setminus S$ satisfying $\bar{D} f = f D = 0$.
Then the microfunction φ, admitting the left monogenic representation

$$h_l = \begin{cases} 0, & x_0 > 0 \\ -f(-\bar{x}), & x_0 < 0 \end{cases}$$

and the right monogenic representation

$$h_r = \begin{cases} f, & x_0 > 0 \\ 0, & x_0 < 0, \end{cases}$$

is supported by the origin and satisfies $\varphi = \mu_- \varphi$.
Let $\psi \in \mathcal{E}(\Omega; \mathcal{A})$ admit the left monogenic representation g. Then a local representation for $\partial(\varphi)\psi = \partial(\varphi)(Q_{+,l}\psi)$ is given by

$$g_{\varepsilon,\vec{x},\rho}(x) = - \int_{\Sigma_\varepsilon} f(-(\bar{x}+u))\, d\sigma_u\, g(u) \,.$$

Notice that $g_{\varepsilon,\vec{x},\rho}$ is left monogenic in $\mathring{B}_m(\vec{x},\rho') \times]-\infty, 0[$, for some ρ' with $0 < \rho' \leq \rho$.

Notice also that now $\mathcal{D}(\varphi)\psi = Q_{-,1}(\mathcal{D}(\varphi)\psi)$.

On the other hand, let g be a right monogenic representation of ψ and put $f' = -f(-\bar{x})$. Then $Df' = f'\bar{D} = 0$ in $S_0(\mathbb{R}^m \setminus S)$ and in analogy with the previous case we define

$\psi\mathcal{D}(\varphi) = (Q_{-,r}\psi)\mathcal{D}(\varphi)$ by means of its local representation

$$\bar{g}_{\varepsilon,\vec{x},\rho}(x) = - \int_{\Sigma_{-\varepsilon}} g(u)\, d\sigma_u\, f'(-(\bar{x}+u))$$
$$= \int_{\Sigma_{-\varepsilon}} g(u)\, d\sigma_u\, f(\bar{u}+x) \,, \quad x_0 > 0 \,,$$

and we obtain that $\psi\mathcal{D}(\varphi) = Q_{+,r}(\psi\mathcal{D}(\varphi))$.

Finally, let f be a function in $S_0(\mathbb{R}^{m+1} \setminus S)$ satisfying $\bar{D}f = fD = 0$ and let φ be the microfunction admitting the left monogenic representation

$$h_l = \begin{cases} -f(-\bar{x}) \,, & x_0 > 0 \\ 0 \,, & x_0 < 0 \end{cases}$$

and the right monogenic representation

$$h_r = \begin{cases} 0 \,, & x_0 > 0 \\ f \,, & x_0 < 0 \,. \end{cases}$$

Then $\varphi = \mu_+\varphi$ and $\operatorname{supp}\varphi = \{0\}$.

Furthermore, let ψ admit the left monogenic representation g; then $\mathcal{D}(\varphi)\psi = \mathcal{D}(\varphi)(Q_{-,l}\psi)$ is represented by

$$g_{\varepsilon,\vec{x},\rho}(x) = - \int_{\Sigma_{-\varepsilon}} f(-(\bar{x}+u))\, d\sigma_u\, g(u) \,, \quad x_0 > 0$$

and $\mathcal{D}(\varphi)\psi = Q_{+,l}(\mathcal{D}(\varphi)\psi)$.

On the other hand, when ψ admits the right monogenic representation g, then $\psi\mathcal{D}(\varphi) = (Q_{+,r}\psi)\mathcal{D}(\varphi)$ is represented by

$$g_{\varepsilon,\vec{x},\rho}(x) = \int_{\Sigma_\varepsilon} g(u)\, d\sigma_u\, f(\bar{u}+x)$$

and it satisfies $\psi\mathcal{D}(\varphi) = Q_{-,r}(\psi\mathcal{D}(\varphi))$.

5. Electromagnetism and $\mathcal{A}$-microfunctions

In the previous section, the microfunction φ occurring in the definition of $\mathcal{D}(\varphi)\psi$ and $\psi\mathcal{D}(\varphi)$ was always represented by a function f satisfying either $Df = fD = 0$ or $\bar{D}f = fD = 0$ in some suitable domain of $\mathbb{R}^{m+1}$. At first sight these

conditions upon φ seem to be very restrictive. However we shall show that they allow us to define $\mathcal{V}(\varphi)\psi$ and $\psi\mathcal{V}(\varphi)$ for a general $\varphi \in \mathcal{E}(\mathbb{R}^m;\mathcal{A})$ with supp $\varphi = \{0\}$.
Let $\psi \in \mathcal{E}(\Omega;\mathcal{A})$; then it follows from Theorems 11 and 12 that $D_0\mathcal{V}(\lambda)\psi = \mathcal{V}(\lambda)D_0\psi = \psi D_0\mathcal{V}(\lambda) = \psi\mathcal{V}(\lambda)D_0 = \psi$.
Hence on $\mathcal{E}(\Omega;\mathcal{A})$, $\mathcal{V}(\lambda)D_0 = D_0\mathcal{V}(\lambda) = \mathbb{I}$ or

$$\mathcal{V}(\lambda) = D_0^{-1}.$$

Furthermore put

$$\{D_0,\psi\} = D_0\psi + \psi D_0,$$

$$[D_0,\psi] = D_0\psi - \psi D_0,$$

$$\mathcal{E}_\varepsilon(\Omega;\mathcal{A}) = \{\psi \in \mathcal{E}(\Omega;\mathcal{A}) : [D_0,\psi] = 0\}$$

and

$$\mathcal{E}_\mu(\Omega;\mathcal{A}) = \{\psi \in \mathcal{E}(\Omega;\mathcal{A}) : \{D_0,\psi\} = 0\}.$$

Notice that, if ψ is 1-vector valued, then

$$[D_0,\psi] = 2\,\mathrm{curl}\,\psi$$

and that, if ψ is bivector valued, then

$$\{D_0,\psi\} = 2\,\mathrm{div}\,\psi.$$

For this reason the elements of $\mathcal{E}_\varepsilon(\Omega;\mathcal{A})$ and $\mathcal{E}_\mu(\Omega;\mathcal{A})$ will respectively be called $\mathcal{A}$-microfunctions of electric and of magnetic type.
Notice that

$$[D_0,\cdot]\circ\{D_0,\cdot\} = \{D_0,\cdot\}\circ[D_0,\cdot] = 0$$

and that for every $\psi \in \mathcal{E}(\Omega;\mathcal{A})$,

$$\psi = \tfrac{1}{2}[D_0,\cdot]\circ D_0^{-1}\psi + \tfrac{1}{2}\{D_0,\cdot\}\circ D_0^{-1}\psi.$$

This leads to

<u>Theorem 13.</u> Let $\psi \in \mathcal{E}(\Omega;\mathcal{A})$. Then there exist unique $\psi_\varepsilon \in \mathcal{E}_\varepsilon(\Omega;\mathcal{A})$ and $\psi_\mu \in \mathcal{E}_\mu(\Omega;\mathcal{A})$ such that $\psi = \psi_\varepsilon + \psi_\mu$.

<u>Theorem 14.</u> The following sequences are exact:

$$0 \longrightarrow \mathcal{E}_\varepsilon(\Omega;\mathcal{A}) \xrightarrow{\{D_0,\cdot\}} \mathcal{E}(\Omega;\mathcal{A}) \xrightarrow{[D_0,\cdot]} \mathcal{E}_\mu(\Omega;\mathcal{A}) \rightarrow 0$$

$$0 \longrightarrow \mathcal{E}_\mu(\Omega;\mathcal{A}) \xrightarrow{[D_0,\cdot]} \mathcal{E}(\Omega;\mathcal{A}) \xrightarrow{\{D_0,\cdot\}} \mathcal{E}_\varepsilon(\Omega;\mathcal{A}) \rightarrow 0.$$

We now come to the main theorem of this section, which gives the connection between the operators ε, μ and the electric and magnetic field operators $[D_0,\cdot]$ and $\{D_0,\cdot\}$.
First notice that

$$[\beta, \{D_0, .\}] = 0 \ , \quad \{\alpha, \{D_0, .\}\} = 0 \ ,$$
$$[\alpha, [D_0, .]] = 0 \ , \quad \{\beta, [D_0, .]\} = 0 \ ,$$
$$\{\gamma, [D_0, .]\} = 0 \ , \quad \{\gamma, \{D_0, .\}\} = 0 \ ,$$
$$[\mathcal{X}_{\{r\}}^{1}, [D_0, .]] = 0 \ , \quad [\mathcal{X}_{\{r\}}^{1}, \{D_0, .\}] = 0 \ .$$

Theorem 15. (i) If $\psi \in \mathcal{C}_\varepsilon(\Omega ; \mathcal{A})$, then there exists a representation f of ψ in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$ such that $Df = fD = 0$.

(ii) If $\psi \in \mathcal{C}_\mu(\Omega ; \mathcal{A})$, then there exists a function f in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$, satisfying $Df = f\overline{D} = 0$, such that f $(-f(-\overline{x}))$ is a left (right) monogenic representation of ψ.

Proof. We only show (i) since (ii) is similar.

Let $\psi \in \mathcal{C}_\varepsilon(\Omega ; \mathcal{A})$. Then $Q_{+,1}\psi \in \mathcal{C}_\varepsilon(\Omega ; \mathcal{A})$. Moreover, let $g \in M_{(r)}(\mathbb{R}^{m+1}_+ ; \mathcal{A})$ be such that

$$g' = \begin{cases} g \ , & x_0 > 0 \\ 0 \ , & x_0 < 0 \end{cases}$$

represents $Q_{+,1}\psi$.

Then by assumption $D_0 g - g D_0 \in M_{(r)+}(\mathbb{R}^{m+1} ; \mathbb{R}^m)$.

Take $h \in M_{(r)}(\mathbb{R}^{m+1}_+ ; \mathcal{A})$ with $-\frac{\partial}{\partial x_0} h = g$; then

$$g = D_0 h = \frac{1}{2}[D_0, h] + \frac{1}{2}\{D_0, h\} \ .$$

But, as $[D_0, g] = -\frac{1}{2}[D_0, h]D$ belongs to $M_{(r)+}(\mathbb{R}^{m+1} ; \mathbb{R}^m)$, there exists a right monogenic function l in $\mathbb{R}^{m+1}_+$ and an analytic function k on $\overline{\mathbb{R}}^{m+1}_+$ such that

$$\frac{1}{2}[D_0, h] = l + k \ .$$

As $\Delta k = 0$ in $\overline{\mathbb{R}}^{m+1}_+$ we may find functions k_1 and k_2 such that $k_1\overline{D} = 0$, $k_2 D = 0$ and $k = k_1 + k_2$.

Hence $l + k_2 = \frac{1}{2}[D_0, h] - k_1$ is simultaneously a solution of $fD = 0$ and $f\overline{D} = 0$, whence $\frac{\partial}{\partial x_0}\left(\frac{1}{2}[D_0, h] - k_1\right) = 0$.

But this means that $\frac{1}{2}[D_0, h]$ is extendable to a left monogenic function on $\overline{\mathbb{R}}^{m+1}_+$ or $\frac{1}{2}[D_0, h] \in M_{(r)+}(\mathbb{R}^{m+1} ; \mathbb{R}^m)$.

So $Q_{+,1}\psi$ admits the representation

$$f = \begin{cases} \frac{1}{2}\{D_0, h\} \ , & x_0 > 0 \\ 0 \ , & x_0 < 0 \end{cases}$$

with $Df = fD = 0$ in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$. ∎

Corollaries. (i) $\mathcal{E} = \frac{1}{2}\{D_o, \cdot\} \circ D_o^{-1}$

(ii) $\mu = \frac{1}{2}[D_o, \cdot] \circ D_o^{-1}$

(iii) $\mathcal{H}_r \mathcal{H}_l \psi = D_o^{-1} \psi D_o \quad , \quad \psi \in \mathcal{C}(\Omega; \mathcal{A})$.

Notice too that when $\psi = \mathcal{E}\psi$, ψ admits a representing function which is left and right monogenic, while, when $\psi = \mu\psi$, ψ admits a representing left monogenic function f which also satisfies $f\bar{D} = 0$.

We now come to the general definition of a micro-differential operator (convolution operator with support at the origin) .

Let $\varphi \in \mathcal{C}(\mathbb{R}^m; \mathcal{A})$ be such that $\operatorname{supp}\varphi = \{0\}$ and let $\psi \in \mathcal{C}(\Omega; \mathcal{A})$.

Then using the representations of Theorem 15 we may put

$$\partial(\varphi)\psi = \partial(\mathcal{E}_+\varphi)\, Q_{+,l}\psi + \partial(\mathcal{E}_-\varphi)\, Q_{-,l}\psi$$
$$+ \partial(\mu_+\varphi)\, Q_{-,l}\psi + \partial(\mu_-\varphi)\, Q_{+,l}\psi$$

and

$$\psi\partial(\varphi) = (Q_{+,r}\psi)\partial(\mathcal{E}_+\varphi) + (Q_{-,r}\psi)\partial(\mathcal{E}_-\varphi)$$
$$+ (Q_{+,r}\psi)\partial(\mu_+\varphi) + (Q_{-,r}\psi)\partial(\mu_-\varphi) .$$

Notice that

$$\partial(\varphi)\psi = \partial(Q_{+,r}\varphi)\, Q_{+,l}\psi + \partial(Q_{-,r}\varphi)\, Q_{-,l}\psi$$

and

$$\psi\partial(\varphi) = (Q_{+,r}\psi)\partial(Q_{+,l}\varphi) + (Q_{-,r}\psi)\partial(Q_{-,l}\varphi) .$$

To conclude, suppose now that $\psi \in \mathcal{C}_{o,m}$ is such that $\psi = \mathcal{E}\psi$ and $(\mathbb{I} \pm \alpha)\psi = 0$. Then by Theorem 8 either $\psi = 0$ or $\mathcal{E}_+\psi \neq 0$ and $\mathcal{E}_-\psi \neq 0$.

Similarly, when $\psi = \mu\psi$ and $(\mathbb{I} \pm \beta)\psi = 0$ we have that $\mu_+\psi = 0$ if and only if $\mu_-\psi = 0$.

These properties have the following meaning in electromagnetic field theory.

Let $m = 2$, $D_o = \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3$,

$\tilde{D} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \mathcal{E}_2 + \frac{\partial}{\partial x_3} \mathcal{E}_3$ with $\mathcal{E}_j = \bar{e}_1 e_j$, $j = 2, 3$,

and let f be a bivector solution of $D_o f = 0 = f D_o$ in $\mathbb{R}^3 \setminus \mathbb{R}^2$. Then $e_1 f$ is a solution of $\tilde{D} g = 0$ with values in the Clifford algebra generated by $\mathcal{E}_2$, $\mathcal{E}_3$. Hence f is a solution of $\tilde{D} f = f \bar{\tilde{D}} = 0$ in $\mathbb{R}^3 \setminus \mathbb{R}^2$.

Now $f = f_{12}\, e_1 e_2 + f_{13}\, e_1 e_3 + f_{23}\, e_2 e_3$

$$= - f_{12}\mathcal{E}_2 - f_{13}\mathcal{E}_3 + f_{23}\mathcal{E}_2\mathcal{E}_3$$

so that the microfunction $\psi = f(\vec{x}+0) - f(\vec{x}-0)$ has the form $\psi = \psi_2 \varepsilon_2 + \psi_3 \varepsilon_3 + \psi_{23} \varepsilon_2 \varepsilon_3$.
But we also have that in hyperfunction sense

$$D_0 f = e_1 \tilde{D} f = e_1 \delta_{x_1} \otimes \psi$$

$$= \delta_{x_1} \otimes (\psi_2 e_2 + \psi_3 e_3 + \psi_{23} e_1 e_2 e_3) ,$$

whence f may be interpreted as a magnetic field generated by a current $\psi_2 e_2 + \psi_3 e_3$ and a magnetic charge $\psi_{23} e_1 e_2 e_3$ in $\mathbb{R}^2$.
If we assume that magnetic charge does not exist, $\psi_{23} = 0$.
But then $\psi = \beta \psi$ and so $\mu_+ \psi = 0$ if and only if $\mu_- \psi = 0$.
So the singularities of $f(\vec{x}+0)$ coincide with the singularities of $f(\vec{x}-0)$.
If we assume the existence of magnetic charge, one can imagine the magnetic field

$$f = \begin{cases} 0 \ , \ x_1 < 0 \\ -\frac{1}{\omega_3} \cdot \frac{\vec{x}}{|x|^3} \cdot e_1 e_2 e_3 \ , \ x_1 > 0 \end{cases}$$

the corresponding microfunction of which equals

$$\psi = + \frac{1}{2} \delta \ \varepsilon_2 \varepsilon_3 + \frac{x_2 \varepsilon_3 - x_3 \varepsilon_2}{\omega_3 |\vec{x}|^3} .$$

We clearly have a magnetic charge $\frac{1}{2} \delta \varepsilon_2 \varepsilon_3$ at the origin and $0 \in SS_+ \psi$ while $SS_- \psi = \emptyset$.
So the existence of classical magnetic poles is related to the existence of magnetic fields having singularities at one side of an analytic surface and not at the other side.

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