CAUCHY-KOWALEWSKI EXTENSION THEOREMS AND REPRESENTATIONS OF ANALYTIC FUNCTIONALS ACTING OVER SPECIAL CLASSES OF REAL n-DIMENSIONAL SUBMANIFOLDS OF C^{n+1}

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INTRODUCTION

The study of holomorphic extension of real analytic functions defined on real hypersurfaces of complex manifolds has been developed by a number of authors [3, 7 and 8]. In this paper we utilise the invariance of the kernel of the differential operator $d+d^*$, under orthogonal transformations, to provide Cauchy-Kowalewski extensions for the elements of complex Clifford modules of real analytic functions defined on special classes of real n-dimensional submanifolds of C^{n+1}. Each of these extensions is a holomorphic function in (n+1)-complex variables and satisfies the operator $d'+d^{*\prime}$.

In the cases where $n=1 \mod 2$, the manifolds are compact, satisfy a further geometric restriction, we are able to use the generalized Cauchy integral formula established in [10] to construct a generalized Cauchy transform acting on the duals of the modules introduced here. Using this generalized Cauchy transform and the Cauchy-Kowalewski extensions obtained here, we are able to present an integral representation of the dual space acting on these Clifford modules.

The results obtained here generalize results obtained by Sommen [13] on representations of analytic functionals on the unit sphere in R^{n+1}, by means of solutions to generalized Cauchy-Riemann equations. Our methods make use of a number of results from Clifford analysis [4, 5, 11]. We begin by developing the necessary background on Clifford algebras, Clifford analysis and differential forms that we require to establish our main results.

PRELIMINARIES

For each positive integer n it is demonstrated in [9, Chap. 13] and [2, Part 1] that from the vector space R^{n+1}, with orthonormal basis $\{e_j\}_{j=1}^{n+1}$, it is possible to construct a 2^{n+1} dimensional, real, associative algebra A_{n+1}, containing the space R^{n+1} as a subspace. The algebra A_{n+1} has an identity e_o and the basis vectors $\{e_j\}_{j=1}^{n+1}$ of R^{n+1} satisfy the relation

$$e_j e_k + e_k e_j = 2\delta_{jk} e_o , \tag{1}$$

where δ_{jk} is the Kronecker delta, and $1 \leq j , k \leq n+1$.

The algebra has as basis elements the vectors

$$e_o, e_1, \ldots, e_{n+1}, \ldots, e_n e_{n+1}, \ldots, e_1 \ldots e_{n+1} . \tag{2}$$

The algebra A_{n+1} is called a Clifford algebra, but it is not the most general example of such an algebra. A general basis element of this algebra is denoted by $e_{j_1} \ldots e_{j_r}$ with $r \leq n+1$ and $j_1 < \ldots < j_r$. Also a general basis element of the algebra is written as

$$u = x_o e_o + x_1 e_1 + \ldots + x_{n+1} e_{n+1} + \ldots + x_{j_1 \ldots j_r} e_{j_1} \ldots e_{j_r} + \ldots \\ \ldots + x_{1 \ldots n} e_1 \ldots e_n ,$$

with $x_o, x_1, x_{n+1}, x_{j_1 \ldots j_r}, x_{1 \ldots n} \in R$.

We denote the subspace of A_{n+1} spanned by the vectors $\{e_j\}_{j=2}^{n+1}$ by R^n.

From expressions (1) and (2) it may be observed that the vector space A_{n+1} is canonically isomorphic to $\Lambda(R^{n+1})$, the alternating algebra generated from the vector space R^{n+1}.

We observe that each element

$$x = x_1 e_1 + \ldots + x_{n+1} e_{n+1} \subseteq R^{n+1} - \{o\} \subseteq A_{n+1}$$

has a multiplicative inverse

$$x^{-1} = \frac{x_1 e_1 + \ldots + x_{n+1} e_{n+1}}{x_1^2 + \ldots + x_{n+1}^2}$$

in the algebra A_{n+1}.

By considering the real symmetric tensor product of the algebra A_{n+1} with the complex field $A_{n+1} \otimes_R C$ we obtain the complex Clifford algebra $A_{n+1}(C)$ introduced in [9, Chap. 13]. Again this algebra is spanned by the basis elements (2). A general element Z of this algebra is denoted by

$$z_o e_o + z_1 e_1 + \ldots + z_n e_n + \ldots + z_{j_1 \ldots j_r} e_{j_1} \ldots e_{j_r} + \ldots + z_{1 \ldots n+1} e_1 \ldots e_{n+1} ,$$

where $z_0, z_1, z_n, z_{j_1 \ldots j_r}, z_{1 \ldots n+1} \in C$, and each $z_{j_1 \ldots j_r} = x_{j_1 \ldots j_r} + i y_{j_1 \ldots j_r}$, with $x_{j_1 \ldots j_r}$ and $y_{j_1 \ldots j_r} \in R$.

We define the norm of the vector Z to be

$$(|z_0|^2 + \ldots + |z_{j_1 \ldots j_r}|^2 + \ldots + |z_{1 \ldots n+1}|^2)^{1/2} .$$

We denote the complex vector space spanned by the vectors $\{e_j\}_{j=1}^{n+1}$ by C^{n+1}. Unlike the real case, not every element of $C^{n+1} - \{o\}$ is invertible in the algebra $A_{n+1}(C)$. For example the vector $(e_1 + ie_2)$ is an element of the set $C^{n+1} - \{o\}$, and $(e_1 + ie_2)(e_1 + ie_2) = 0$. For each point $\underline{z}_0 \in C^{n+1}$ the set $S(\underline{z}_0) = \{\underline{z} \in C^{n+1} : (\underline{z} - \underline{z}_0)(\underline{z} - \underline{z}_0) = 0\}$ is called the singularity cone at $\underline{z}_0$. Each element of the set $C^{n+1} - S(o)$ is invertible in the algebra $A_{n+1}(C)$.

For each set $\chi \subseteq C^{n+1}$ we denote the set $\bigcup_{z \in \chi} S(\underline{z})$ by $S(\chi)$. For each pair of vectors $\underline{z} = z_1 e_1 + \ldots + z_{n+1} e_{n+1}$ and $\underline{z}' = z_1' e_1 + \ldots + z_{n+1}' e_{n+1}$ we define their Hermitian product to be

$$\langle \underline{z}, \underline{z}' \rangle = \sum_{j=1}^{n+1} z_j \bar{z}_j' .$$

Using these algebraic preliminaries we may now develop the differential calculus we require.

In [5] Delanghe introduces the generalized Cauchy-Riemann operator

$$\sum_{j=1}^{n+1} e_j \frac{\partial}{\partial x_j} . \tag{3}$$

This operator acts on pointwise differentiable functions defined on subdomains of R^{n+1}, and taking values in the algebra A_{n+1}. The operator (3) may also be described in terms of differential operators acting on differential forms.

Construction: Using the canonical isomorphism $\theta : A_{n+1} \longrightarrow \Lambda(R^{n+1})$ we may [6], for each domain $U \subseteq R^{n+1}$, define an inner product between smooth L^2 integrable forms $g, h : U \longrightarrow \Lambda(U)$. We define this inner product to be $\int_U \mathrm{Trace}\, \{\theta(\theta^{-1}(g) . \theta^{-1}(h))\} dx^{n+1}$.

<u>Definition 1</u> [6]: For $r \in N^+$, for each smooth $(r-1)$ form $\Phi : U \longrightarrow \Lambda^r(U)$ with compact support, and each smooth r form $g : U \longrightarrow \Lambda^{r-1}(U)$ we define the operator d^* to be the adjoint of the operator d arising in the inner product

$$\int_U \mathrm{Trace}\, \{\theta(\theta^{-1}(d\Phi) . \theta^{-1}(g))\} dx^{n+1} ,$$

where d is the usual de Rham cohomology boundary operator

$$\sum_{j=1}^{n+1} dx_j \frac{\partial}{\partial x_j} .$$

It may now easily be deduced that for each pointwise differentiable function $f : U \to A_{n+1}$ we have

$$\sum_{j=1}^{n+1} e_j \frac{\partial f}{\partial x_j} = \theta^{-1}((d+d^*)\theta(f)) . \tag{4}$$

Definition 2: We define $\ker_U(d+d^*)$ to be the set of pointwise differentiable forms $g : U \to \Lambda(U) \otimes_R C$ which satisfy the equation $(d+d^*)g(x) = 0$ for each $x \in U$.

The set $\ker_U(d+d^*)$ is a right module over the complex algebra $\Lambda(R^{n+1}) \otimes_R C$, of alternating tensors.

Definition 3: We define

$$\ker_U\left(\sum_{j=1}^{n+1} e_j \frac{\partial}{\partial x_j}\right) \tag{5}$$

to be the set of pointwise differentiable functions $f : U \to A_{n+1}(C)$ such that for each $x \in U$ we have $\sum_{j=1}^{n+1} e_j \frac{\partial f}{\partial x_j}(x) = 0$.

The set $\ker_U\left(\sum_{j=1}^{n+1} e_j \frac{\partial}{\partial z_j}\right)$ is a right module over the complex Clifford algebra $A_{n+1}(C)$.

It follows from equation (4) that the complex vector spaces $\ker_U(d+d^*)$ and (5) are equivalent.

The space $\ker_U(d+d^*)$ is independent of the choice of orthonormal basis in R^{n+1}. It thus follows that for each f in (5) and each orthonormal basis $\{e_j'\}_{j=1}^{n+1} \subseteq R^{n+1} \subseteq A_{n+1}(C)$ we have

$$\sum_{j=1}^{n+1} e_j' \frac{\partial}{\partial x_j'} f(x) = 0 .$$

We now proceed to give some examples of elements of the space (5).

Definition 4 [5]: Let us consider, for $2 \leq l \leq n+1$, the variables

$$s_l = x_l e_o - x_1 e_1 e_l ,$$

$$(s-a)_l = (x_l - a_l) e_o - (x_1 - a_1) e_1 e_l ,$$

for $a = a_1 e_1 + \dots + a_{n+1} e_{n+1}$. For each $(l_1, \dots, l_m) \in \{2, \dots, n+1\}^m$ we may construct the following homogeneous polynomials of degree m:

$$V_{l_1 \dots l_m}(x) = \sum_{\pi(l_1 \dots l_m)} s_{l_1} \dots s_{l_m} , \tag{6}$$

$$V_{l_1 \dots l_m}(x-a) = \sum_{\pi(l_1 \dots l_m)} (s-a)_{l_1} \dots (s-a)_{l_m} , \tag{7}$$

where the sum is taken over all permutations without repetition of the sequence $(l_1,\ldots,l_m)$.

In [5] it is established that for each domain $U\subseteq R^{n+1}$ the polynomials (6) and (7) are elements of the space $\ker_U(\sum_{j=1}^{n+1}e_j\frac{\partial}{\partial x_j})$. From [4] it may be established that for each element $f\in\ker_U(\sum_{j=1}^{n+1}e_j\frac{\partial}{\partial x_j})$ and each point $a\in U$ there is a subneighbourhood U_a, containing the point a, and there is a series

$$\sum_{m=o}^{\infty}\sum_{l_1\ldots l_m}V_{l_1\ldots l_m}(x-a)c_{l_1\ldots l_m,a}\quad, \tag{8}$$

with each $c_{l_1\ldots l_m,a}\in A_{n+1}(C)$, which converges uniformly on U_a to the function $f(x)$.

In [12] Sommen observes that for the case where $a=a_2e_2+\ldots+a_{n+1}e_{n+1}$ the series (8) restricted to the variable $x_2e_2+\ldots+x_{n+1}e_{n+1}$ becomes

$$\sum_{m=o}^{\infty}\sum_{l_1\ldots l_m}(x_{l_1}-a_{l_1})\ldots(x_{l_m}-a_{l_m})c_{l_1\ldots l_m,a}\quad.$$

Using this fact Sommen establishes [12]:

Theorem 1: For each domain $U'\subseteq R^n$ and each real analytic function

$$r : U'\longrightarrow A_{n+1}(C) \tag{9}$$

there is a domain $U_r\subseteq R^{n+1}$ and a unique function $f: U_r\longrightarrow A_{n+1}(C)$ such that:

i $\quad U'\subseteq U_r$,

ii $\quad f\in\ker_{U_r}(\sum_{j=1}^{n+1}e_j\frac{\partial}{\partial x_j})$,

iii $\quad f|_{U'}=r$. □

The function f is called the Cauchy-Kowalewski extension of the function r with respect to R^n.

In this paper we shall also consider the following type of functions:

Definition 5 [10]: For each subdomain $U(C)$ of C^{n+1} we say that a holomorphic function $f : U(C)\longrightarrow A_{n+1}(C)$ is complex left regular if for each $\underline{z}\in U(C)$ we have $\sum_{j=1}^{n+1}e_j\frac{\partial F}{\partial z_j}(\underline{z})=0$. A similar definition is given in [10] for complex right regular functions.

Examples:

1. The holomorphic extension of the series (8) is a complex left

regular function. It follows that the holomorphic extension of the Cauchy-Kowalewski extension of the function (9) is a complex left regular function.

2. The function

$$G : C^{n+1} - S(o) \longrightarrow C^{n+1} \subseteq A_{n+1}(C) : G(\underline{z}) = \underline{z}(\underline{z},\underline{z})^{(n+1)/2} ,$$

defined for $n=1 \mod 2$, is a complex left regular function. Moreover, this function is a complex right regular function.

The class of complex left regular functions defined on an open set $U(C)$ is a right module over the algebra $A_{n+1}(C)$. We denote this module by $\Omega_r(U(C),A_{n+1}(C))$. The class of complex right regular functions defined on $U(C)$ is a left module over $A_{n+1}(C)$. We denote this module by $\Omega_l(U(C),A_{n+1}(C))$.

Using the complex isomorphism $\theta \otimes_R id : A_{n+1}(C) \longrightarrow \Lambda(R^{n+1}) \otimes_R C$, where id stands for the identity map, we observe that for each complex left regular function $F : U(C) \longrightarrow A_{n+1}(C)$ the holomorphic form $(\theta \otimes_R id)F : U(C) \longrightarrow \Lambda(R^{n+1}) \otimes_R C$ satisfies the equation $(d'+d^{*\prime})((\theta \otimes_R id)F) = 0$, where d' is the holomorphic extension $\sum_{j=1}^{n+1} dz_j \frac{\partial}{\partial z_j}$ of the operator d, and $d^{*\prime}$ is the holomorphic extension of the operator d^*.

We shall require the following classes of manifolds in our analysis.

Definition 6 [7]: A smooth, real $(n+1)$-dimensional submanifold, M, of C^{n+1} is said to be without complex structure if for each $\underline{z} \in M$ the tangent space $TM_{\underline{z}}$ is spanned by vectors $\{\underline{z}_j(\underline{z})\}_{j=1}^{n+1}$, where for each $\underline{z}_j(\underline{z})$ we have $i\underline{z}_j(\underline{z}) \notin TM_{\underline{z}}$. We shall refer to such manifolds as manifolds of type a.

Observation 1: If M is a manifold of type a then it follows from Definition 6 that for each $\underline{z} \in M$ the complex extension of the tangent space $TM_{\underline{z}}$ is isomorphic to the space C^{n+1}. If M is not a manifold of type a, then for each $\underline{z} \in M$ the complex extension of the tangent space $TM_{\underline{z}}$ is isomorphic to a proper complex subspace of C^{n+1}.

Definition 7: In the cases where $n=1 \mod 2$ a smooth, real, $(n+1)$-dimensional, compact submanifold, M, of C^{n+1}, with boundary, is called a manifold of type b if it is a manifold of type a, and for each $\underline{z} \in M$

i $\quad TM_{\underline{z}} \cap S(\underline{z}) = \{\underline{z}\}$,

ii $\quad M \cap S(\underline{z}) = \{\underline{z}\}$.

Definition 8: In the cases where $n=1 \mod 2$ a smooth, real $(n+1)$-dimensional, noncompact submanifold, M, of C^{n+1} is called a manifold of type c if each smooth, compact, (n+1)-dimensional submanifold of M is a manifold of type b.

An example of a manifold of type c is the real vector space $R^{n+1} \subseteq C^{n+1}$.

For each manifold, M, of type a, and each $\underline{z} \in M$ the vectors spanning the tangent space, $TM_{\underline{z}}$, are orthogonal with respect to the Hermitian structure of C^{n+1}. Thus, each manifold of type a is a Riemannian manifold, inheriting its Riemannian structure from the Hermitian structure of C^{n+1}. It follows [6] that for each manifold M of type a we can construct an adjoint, d^*, to the differential operator d. Thus, the operator $d+d^*$ is well defined over each manifold of type a. In fact, for $U_M(C) \subseteq \subseteq C^{n+1}$ a domain containing a manifold M of type a, and $H : U_M(C) \longrightarrow A_{n+1}(C)$ a holomorphic function, we have for each $\underline{z} \in M$

$$(d+d^*)((\theta \otimes_R id)H(\underline{z})) = (d'+d^{*\prime})((\theta \otimes_R id)H(\underline{z})) , \qquad (10)$$

where the operator $d+d^*$ is acting over the manifold M.

Definition 9: For M a connected manifold of type b we denote the component of $C^{n+1} - S(\partial M)$ containing the interior of M by $U(M)$.

In [11] we establish that $U(M)$ is an open subset of C^{n+1}.

Using Definitions 7 and 9 we establish the following generalization of the classical Cauchy integral formula [1, Chap. 4].

Theorem 2 [11, 14]: Suppose $F : U(C) \longrightarrow A_{n+1}(C)$ is a complex left regular function, and suppose $M \subseteq U(C)$, is a connected manifold of type b, then for each point $\underline{z}_0$ in $U(M) \cap U(C)$ we have

$$F(\underline{z}_0) = \frac{1}{w_n} \int_{\partial M} G(\underline{z}-\underline{z}_0) D\underline{z} F(\underline{z}) ,$$

where w_n is the surface area of the unit sphere lying in R^{n+1} and $D\underline{z}$ is the complex n-form

$$\sum_{j=1}^{n+1} (-1)^{j+1} e_j dz_1 \wedge \ldots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \ldots \wedge dz_{n+1} . \qquad \square$$

CAUCHY-KOWALEWSKI EXTENSIONS OVER MANIFOLDS OF TYPE a

All manifolds of type a considered in this section will be real analytic, Riemannian manifolds.

Definition 10: Suppose $M \subseteq C^{n+1}$ is a manifold of type a, without boundary, and M' is a real analytic, (n+1)-dimensional, Riemannian submanifold of M, with boundary. Then the manifold M' is called a manifold of type d.

Any type b real analytic submanifold of a real analytic manifold of type c is an example of a manifold of type d.

We denote the set of real analytic, $A_{n+1}(C)$ valued functions defined over $\partial M'$ by

$$\mathcal{A}(\partial M', A_{n+1}(C)) . \tag{11}$$

The set (11) is a right $A_{n+1}(C)$ module. For each element of this module we may deduce the following extension theorem.

Theorem 3 (A Cauchy-Kowalewski Extension Theorem): Suppose M' is a manifold of type d lying in a type a manifold, M, without boundary. Suppose also the function g is an element of the module $\mathcal{A}(\partial M', A_{n+1}(C))$. Then there is a domain $U_g(C) \subseteq C^{n+1}$ containing the manifold $\partial M'$, and there is a complex left regular function $f : U_g(C) \to A_{n+1}(C)$ such that $F|_{\partial M'} = g$.

Proof: As the manifolds M and M' are real analytic and Riemannian there exist real analytic chart maps

$$\{\Psi_m : U_m \subseteq R^{n+1} \to M\}_{m=1}^{\infty} , \tag{12}$$

such that each chart, Ψ_m, preserves the Riemannian structure of the manifold M, and for

$$R_+^{n+1} = \{x = x_1e_1 + \ldots + x_{n+1}e_{n+1} \in R^{n+1} : x_1 \geq 0\} ,$$
$$R_-^{n+1} = \{x = x_1e_1 + \ldots + x_{n+1}e_{n+1} \in R^{n+1} : x_1 \leq 0\}$$

we have for each $m \in N^+$

$$\Psi_m : U_m \cap R_+^{n+1} \to M' ,$$
$$\Psi_m : U_m \cap R_-^{n+1} \to (M - M') \cup \partial M' .$$

We shall restrict our attention to the subset $\{\Psi_p : U_p \to M , U_p \quad R^n \neq \Phi\}$ of the set (12). It may be observed that the set of maps $\{\Psi_p : U_p \cap R^n \to M\}$ is a set of real analytic charts for the manifold $\partial M'$. We shall denote each chart map $\Psi_p : U_p \cap R^n \to \partial M'$ by μ_p.

Suppose now that g is an element of the set $\mathcal{A}(\partial M', A_{n+1}(C))$. Then it follows from Theorem 1 that for each real analytic function $g(\mu_p) : U_p \cap R^n \to A_{n+1}(C)$ there is an open set $U_{p,g} \subseteq U_p$ containing the set $U_p \cap R^n$, and there is a function $f_{p,g} : U_{p,g} \to A_{n+1}(C)$ satisfying the conditions

i $\qquad f_{p,g} \in \ker_{U_{p,g}} \left(\sum_{j=1}^{n+1} e_j \frac{\partial}{\partial x_j}\right) ,$

ii $\qquad f_{p,g}|_{U_p \cap R^n} = g(\mu_p) .$

As the kernel space of the operator $d + d^*$, acting over a Riemannian manifold, is invariant under diffeomorphisms which preserve the Riemannian structure of the manifold we have that the real analytic form

$$D\Psi_p^{-1}(\theta\otimes_R id)(f_{p,g}(\Psi_p^{-1})) : \Psi_p(U_{p,g}) \to \Lambda(R^{n+1})\otimes_R C \qquad (13)$$

satisfies the equation

$$(d+d^*)D\Psi_p\{(\theta\otimes_R id)(f_{p,g}(\Psi_p^{-1}))\} = 0 , \qquad (14)$$

where $D\Psi_p$ is the complex vector bundle transform

$$D\Psi_p : \Lambda(U_p)\otimes_R C \to \Lambda(\Psi_p(U_p))\otimes_R C$$

induced by the diffeomorphism Ψ_p .

As the form (13) is a real analytic form (over an (n+1)-dimensional manifold without complex structure it follows from Observation 1 that there is an open set $U_{p,g}(C)\subseteq C^{n+1}$ containing the set $\Psi_p(U_{p,g})$, and there is a complex left regular function $F_{p,g} : U_{p,g}(C)\to A_{n+1}(C)$ which satisfies the condition

$$F_{p,g}\big|_{\Psi_p(U_{p,g})} = \{(\theta\otimes_R id)^{-1}D\Psi_p^{-1}(\theta\otimes_R id)\}(f_{p,g}(\Psi_p^{-1})) .$$

It now follows from equations (4), (10) and (14) that each function $F_{p,g}$ is an element of the right module $\Omega_r(U_{p,g}(C),A_{n+1}(C))$.

If for some p_i and $p_j\in N^+$ we have that

$$\psi_{p_i}(U_{p_i}\cap R^n)\cap \psi_{p_j}(U_{p_j}\cap R^n) \neq \Phi$$

then it follows from the uniqueness of the Cauchy-Kowalewski extensions $f_{p_i,g}$ and $f_{p_j,g}$, and the invariance of the operator $d+d^*$ under the chart maps $\{\Psi_p\}$, that the function

$$f_{p_i,g}\big|_{U_{p_i,g}\cap\Psi_{p_i}^{-1}(\Psi_{p_j}U_{p_j,g})}$$

is identical to the function

$$(\theta\otimes_R id)^{-1}D\Psi_{p_i}^{-1}D\Psi_{p_j}(\theta\otimes_R id)f_{p_j,g}(\Psi_{p_j}^{-1}(\Psi_{p_i}\big|_{U_{p_i,g}\cap\Psi_{p_i}^{-1}(\Psi_{p_j}U_{p_j,g})})).$$

Thus on the open set $U_{p_i,g}(C)\cap U_{p_j,g}(C)$ the functions $F_{p_i,g}$ and $F_{p_j,g}$ are identical. On placing $U_g(C) = \bigcup_p U_{p,g}(C)$ we may now construct a complex left regular function $F_g: U_g(C)\to A_{n+1}(C)$ by placing $F_g\big|_{U_{p,g}(C)} = F_{p,g}$ for each $p\in N^+$.

The function F_g satisfies the condition $F_g\big|_{\partial M'} = g$. □

We call the function F_g , constructed in Theorem 3, the Cauchy-Kowalewski extension of the function g .

REPRESENTATIONS OF ANALYTIC FUNCTIONALS OVER CLASSES OF TYPE d MANIFOLDS

We begin by introducing, for the case where $\partial M'$ is compact, the dual to the right $A_{n+1}(C)$ module $\mathcal{A}(\partial M', A_{n+1}(C))$.

Definition 11: For M' the compact boundary of a manifold of type d we call a map

$$T : \mathcal{A}(\partial M', A_{n+1}(C)) \rightarrow A_{n+1}(C)$$

a bounded, right $A_{n+1}(C)$ linear, analytic functional over $\partial M'$ if

i for each $g, h \in \mathcal{A}(\partial M', A_{n+1}(C))$ and $a \in A_{n+1}(C)$ we have

$$T(ga+h) = T(g)a + T(h) ,$$

ii there exists a positive real number $C(T)$ such that for each $g \in \mathcal{A}(\partial M', A_{n+1}(C))$ we have

$$|T(g)| \leq C(T) \sup_{\underline{z} \in \partial M'} |g(\underline{z})| .$$

Definition 12: The set of bounded, right $A_{n+1}(C)$ linear analytic functionals over $\partial M'$ is called the dual space of $\mathcal{A}(\partial M', A_{n+1}(C))$.

We denote this space by

$$\mathcal{A}^*(\partial M', A_{n+1}(C)) . \tag{15}$$

For each $T_1, T_2 \in \mathcal{A}^*(\partial M', A_{n+1}(C))$, each $a \in A_{n+1}$ and each $g \in \mathcal{A}(\partial M', A_{n+1}(C))$ we have $(aT_1+T_2)(g) = a(T_1(g)) + T_2(g)$. It follows that the dual space (15) is a left $A_{n+1}(C)$ module.

For a special class of manifolds M' of type d, with compact boundary, we can transform the dual space (15) into a space of complex right regular functions. We now introduce this special class of manifolds.

Definition 13: A type d manifold, M', with compact boundary, is called a manifold of type e if for each $\underline{z} \in \partial M'$ we have

$$\partial M' \cap S(\underline{z}) = \{\underline{z}\} .$$

For each manifold, M', of type e we may introduce the following transform on the dual space (15):

Definition 14: For M' a manifold of type e and T an element of the module $\mathcal{A}^*(\partial M', A_{n+1}(C))$ we call the transform

$$TG : C^{n+1} - S(\partial M') \rightarrow A_{n+1}(C) : TG(\underline{z}) = T(G(\underline{z}-\underline{z}_0)) ,$$

where the complex vector $\underline{z}_0$ varies over the manifold $\partial M'$, the G-transform over $\partial M'$ of the functional T.

The G-transform is a generalization of a transform introduced by Sommen [13] and [4, Chap. 4], in his study of representations of analytic functionals over the unit sphere in R^{n+1}.

Theorem 4: For each manifold M' of type e, and each element T of the module $\mathcal{A}^*(\partial M', A_{n+1}(C))$ the G-transform, TG, defines a complex right regular function on the open set $C^{n+1} - S(\partial M')$.

Proof: For each point $\underline{z} \in C^{n+1}$ we consider the spaces

$$\chi(\underline{z}_1) = (C^{n+1} - S(\partial M')) \cap (R^{n+1} + \underline{z}_1),$$

$$Y(\underline{z}_1) = (C^{n+1} - S(\partial M')) \cap (iR^{n+1} + \underline{z}_1).$$

Suppose $\Phi : \chi(z_1) \rightarrow A_{n+1}(C)$ is an $A_{n+1}(C)$ valued test function. Then it may be observed that the integral

$$\int_{\chi(\underline{z}_1)} G(\underline{z}-\underline{z}_0)\Phi(\underline{z})dx^{n+1},$$

where dx^{n+1} is the Lebesgue measure of $\chi(\underline{z}_1)$, gives a well defined real analytic function on the manifold $\partial M'$. As T is a bounded analytic functional it follows that the transform, TG, restricted to the set $\chi(\underline{z}_1)$, is a well defined $A_{n+1}(C)$ valued distribution. Similar arguments reveal that the transform, TG, restricted to the set $Y(\underline{z}_1)$ is also a well defined $A_{n+1}(C)$ valued distribution. We shall call these distributions $TG\chi_{\underline{z}_1}$ and $TGY_{\underline{z}_1}$ respectively.

As the integral

$$\int_{\chi(\underline{z}_1)} G(\underline{z}-\underline{z}_0)\sum_{j=1}^{n+1} e_j \frac{\partial \Phi}{\partial x_j}(\underline{z})dx^{n+1}$$

vanishes it may be deduced from [4, Chap. 3] that the distribution $TG\chi_{\underline{z}_1}$ is a real analytic function $TG\chi_{\underline{z}_1} : \chi(\underline{z}_1) \rightarrow A_{n+1}(C)$ which satisfies the equation

$$\sum_{j=1}^{n+1} \frac{\partial TG\chi_{\underline{z}_1}}{\partial x_j} e_j = 0. \qquad (16)$$

Similar considerations reveal that the distribution $TGY_{\underline{z}_1}$ is a real analytic function $TGY_{\underline{z}_1} : Y(\underline{z}_1) \rightarrow A_{n+1}(C)$ which satisfies the equation

$$\sum_{j=1}^{n+1} \frac{\partial TGY_{\underline{z}_1}}{\partial y_j} e_j = 0.$$

It follows that the G-transform of the functional T is a real analytic function in the variables $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$, on the open set $C^{n+1} - S(\partial M')$. As the function $G(\underline{z})$ is holomorphic it may be observed that the integrals

$$\int_{\chi(\underline{z}_1)} \frac{\partial}{\partial x_j} G(\underline{z}-\underline{z}_0)\Phi(\underline{z})dx^{n+1}, \quad \int_{\chi(\underline{z}_1)} - \frac{i\partial}{\partial y_j} G(\underline{z}-\underline{z}_0)\Phi(\underline{z})dx^{n+1}$$

are equivalent for each j, $1 \leq j \leq n+1$. It follows from the classical Cauchy-Riemann equations [1, Chap. 1] that the G-transform

$$TG : C^{n+1} - S(\partial M') \to A_{n+1}(C) \tag{17}$$

is a holomorphic function in the variables $z_1, \dots, z_{n+1}$. Moreover, it may now be observed from equation (16) that the function (17) is a complex right regular function. $\square$

In fact the G-transform, TG, given in Theorem 4 is the following type of complex right regular function.

Definition 15: From M' a manifold of type e we say that a complex right regular function $F : C^{n+1} - S(\partial M') \to A_{n+1}(C)$ is complex right regular at infinity with respect to $\partial M'$ if for each unbounded, continuous function $s : (0,+\infty) \to C^{n+1} - S(\partial M')$, which is not asymptotic to the set $S(\partial M')$, we have

$$\lim_{t \to \infty} F(s(t)) = 0 .$$

We denote the set of complex right regular functions at infinity with respect to $\partial M'$ by

$$\tilde{\Omega}_1(C^{n+1} - S(\partial M'), A_{n+1}(C)) . \tag{18}$$

It may easily be deduced that the set (18) is a left $A_{n+1}(C)$ module, and the set of G-transforms over $\partial M'$ is a submodule of the module (18). In fact, by using similar arguments to those used in [4, Sec. 28] we may obtain the following isomorphism.

Theorem 5: For M' a real analytic manifold of type b, lying in a real analytic manifold of type e, the left $A_{n+1}(C)$ modules $\mathcal{A}^*(\partial M', A_{n+1}(C))$ and $\tilde{\Omega}_1(\partial M', A_{n+1}(C))$ are isomorphic. $\square$

In the cases where M' is a manifold of type b we can use the G-transform to give an integral representation of an analytic functional acting on an element of the set $\mathcal{A}(\partial M', A_{n+1}(C))$.

Theorem 6: For M' a real analytic manifold of type b, lying in a manifold M of type c, for T an element of the module $\mathcal{A}^*(\partial M', A_{n+1}(C))$ and for g an element of the module $\mathcal{A}(\partial M', A_{n+1}(C))$ there exist a manifold M_g, of type b, a complex right regular function $F_T : C^{n+1} - S(\partial M') \to A_{n+1}(C)$ and a complex left regular function $F_g : U_g \subseteq C^{n+1} \to A_{n+1}(C)$ such that

$$T(g) = \int_{\partial M_g} F_T(\underline{z}) D\underline{z} F_g(\underline{z}) .$$

Proof: For each $g \in \mathcal{A}(\partial M', A_{n+1}(C))$ we take the Cauchy-Kowalewski extension $F_g : U_g(C) \to A_{n+1}(C)$ constructed in Theorem 3.

As the manifold M' is a submanifold of a manifold M of type c, there exists a manifold $M_g \subseteq M$, of type b which satisfies the conditions i $\qquad M_g \subseteq U_g(C)$,

ii $\qquad \partial M' \subseteq M_g$,

iii $\qquad \partial M' \cap \partial M_g = \Phi$.

Thus, for each vector $\underline{z}_o \in \partial M'$ we have from the generalized Cauchy integral formula, given in Theorem 2,

$$g(\underline{z}_o) = \frac{1}{w_n} \int_{\partial M_g} G(\underline{z}-\underline{z}_o) D\underline{z} F_g(\underline{z}) .$$

Thus, for T an element of $\mathcal{A}^*(\partial M', A_{n+1}(C))$ we have

$$T(g) = T\left(\int_{\partial M_g} G(\underline{z}-\underline{z}_o) D\underline{z} F_g(\underline{z})\right) .$$

From Fubinni's theorem we deduce

$$T(g) = \int_{\partial M_g} TG(\underline{z}) D\underline{z} F_g(\underline{z}) . \tag{19}$$

On placing the function $TG(\underline{z}) = F_T(\underline{z})$ we obtain our result. $\square$

The integral (19) generalizes an integral representation obtained by Sommen [13], and [3, Sec. 28], for analytic functionals acting on analytic functions over the unit sphere in R^{n+1} .

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