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Open cover of a metric space admits l_∞-partition of unity

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OPEN COVER OF A METRIC SPACE ADMITS ℓ_∞-PARTITION OF UNITY

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The result stated in the title has been already proved independently by Z.Frolík and J.Pelant. Our aim is to give an easy proof. The proof is a slight modification of Isbell's proof of the fact that every uniform cover admits an ℓ_∞-partition of unity.

We recall that a partition of unity $\{ f_a \mid a \in A \}$ is an ℓ_∞-partition if the pseudometric

$$\rho (x , y) = \sup \{ | f_a(x) - f_a(y)| \mid a \in A \}$$

is uniformly continuous.

<u>Theorem</u> : let $\{ U_a \mid a \in A \}$ be an open cover of a metric space (X , d) . Then there exists an ℓ_∞-partition of unity subordinated to $\{ U_a \mid a \in A \}$.

Proof: it is enough to construct a partition of unity consisting of Lipschitz functions, since every Lipschitz function can be replaced by the finite sum of Lipschitz functions with Lipschitz constant $\leq$ 1. Partitions of unity consisting of functions with Lipschitz constant $\leq$ 1 are obviously ℓ_∞ - partitions.

We may and shall assume that our open cover contains the empty set $\emptyset$. Take some well-order $\precsim$ on A, such that $\emptyset$ has the largest index. Take the lexicographic order $\leq$ on $\omega \times A$. For our purpose define $\sup \emptyset = 0$. Define

$$g_{n,a}(x) = \min \{ 1, \max \{ \sup \{ g_{m,b}(x) \mid (m,b) < (n,a) \} , \sup \{ nd(x,X - U_b) \mid b \prec a \} \} \} .$$

It is easy to see that :

(i) $g_{n,a}$ are non-negative Lipschitz functions with Lipschitz constant n,

(ii) $g_{n,a}(x)$ increase monotonically to 1, in fact for each $x \in X$ there exist m and b such that $U_{\frac{1}{m}}(x) \subset U_b$, hence $g_{n,a}(x) = 1$ for all $(n,a) \geq (m,b)$,

(iii) for limit indexes $g_{n,a}(x) = \sup \{ g_{m,b}(x) \mid (m,b) < (n,a) \}$.

Now, define

$$f_{n,a}(x) = g_{(n,a)^+}(x) - g_{n,a}(x)$$

for $U_a \neq \emptyset$ ($(n,a)^+ = \min \{ (m,b) \mid (m,b) > (n,a) \}$).
It is clear that

1) $f_{n,a}$ are Lipschitz functions

2) $f_{n,a}(x) > 0$ only for finitely many n's and for x U_a .

Hence, $\{ f_{n,a} \mid n \in \omega, U_a \neq \emptyset \}$ is the partition of unity subordinated to the cover $\{ U_a \mid a \in A \}$ and it consits of Lipschitz functions.

Remark. It is obvious that the partition constructed above (and hence the corresponding ℓ_∞-partition) is point-finite (i.e. the cover $\{ \operatorname{coz} f_{n,a} \}$ is point-finite), provided $\{ U_a \mid a \in A \}$ is point-finite.

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