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ON THE SECTION OF A LATTICE-COVERING OF BALLS

by A. BEZDEK

1. Introduction

A Thue [9] has already proved in 1910, that the incircles and circumcircles of the faces of $\{6,3\}$ constitute a densest packing and a thinnest covering of equal circles. The corresponding densities are $\pi/\sqrt{12} = 0{,}9069\ldots$ and $2\pi/\sqrt{27} = 1{,}2091\ldots$.

An analogous result for incongruent circles does not hold. The circles of radii $r_1, r_2, \ldots$ such that $\Sigma r_i^2 = \infty$ and $r_i \to 0$ can be arranged so that they fill (or cover) the plane with density 1 [2].

Under special conditions the density of a packing (of a covering) of incongruent circles is investigated in [3-8].

Let us recall the nice results of G.Blind [3] and G.Fejes Tóth [5] respectively: The density of a packing (of a covering) of the plane by circles of radii $r_1, r_2, \ldots$ such that, for any two indexes i and j, $r_i/r_j \geq 0{,}9\ldots$ can not be greater than $\pi/\sqrt{12}$ (less than $2\pi/\sqrt{27}$). The density of a packing of circles arising by intersecting a packing of equal balls by a plane is at most $\pi/\sqrt{12}$. An analogous result for covering does not hold: Intersecting a covering with equal balls by a plane we can obtain circlecoverings with density arbitrarily close to 1.

G. Fejes Tóth raised the problem [6] of finding the thinnest covering with circles arising by intersecting a lattice-covering of balls. Of course, if a circle may arise from two balls, the density is calculated as if it would be two circles.

2. Results

We shall prove the following

Theorem 2.1.

The density of a covering of circles arising by intersecting a lattice-covering of balls by a plane is at least $(2\pi/\sqrt{27})+\varepsilon$ where $\varepsilon = 0{,}0174\ldots$ (1). Equality holds only for case described in 2.2.

2.2.

We define a parallelepiped $T(x)$ for $0,85 < x < 1$. Denote the vertices of $T(x)$ by A_i $i = 1,\dots,8$ so that the segments A_iA_{i+1} should be edges $i = 1,\dots,7$, and the vertices $A_1A_2A_3A_4$ should be on a plane. Denote F the midpoint of the face $A_1A_2A_3A_4$. Let $T(x)$ be the parallelepiped having the properties i1-i4

i1 The face $A_1A_2A_3A_4$ is a rhombus

i2 $A_1A_3 = 2x$

i3 $A_5F = 1$ and A_5F is an altitude of $T(x)$

i4 The tetrahedrom $A_1A_3A_4A_5$ has a circumsphere of radius 1.

Fourther we shall denote a body (a disc) and it's volume (area) with the same symbol. It is easy to verify that

$$T(x) = x(\sqrt{3+2x^2-x^4} + \sqrt{3-2x^2-x^4}) \ .$$

Let

$$T(x_o) := \max\{T(x) \mid x \in (0.85\ ,\ 1)\} \ .$$

By virtue of 4.5/4°

$$x_o = \sqrt[4]{\frac{46-\sqrt{1792}}{6}} = 0.8842\dots \ .$$

Equality can occure in theorem 2.1. only when the centres of the balls form a point-lattice having a generating parallelepiped $T(x_o)$. Let as consider a plane which touches a ball of this covering and which is parallel to the face $A_1A_2A_3A_4$ of $T(x_o)$ (fig. 1/a).

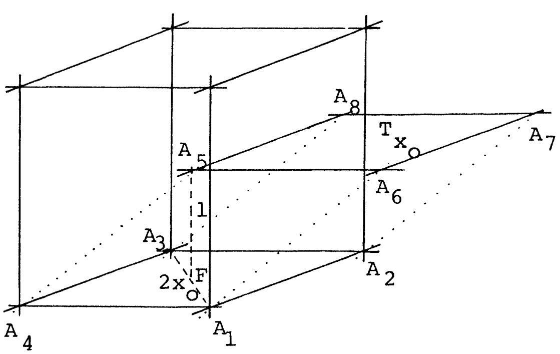

fig. 1/a

The density of the circles on this plane

$$d_o = \frac{\pi}{T(x_o)} = 1,2266888 \ldots \quad .$$

(1) Let

$$\varepsilon = d_o - \frac{2\pi}{\sqrt{27}} = 0,0174892\ldots \quad .$$

For comparing we show good coverings (fig. 1/b, 1/c). (The thinnest lattice sphere-covering is represented on fig. 1/b.)

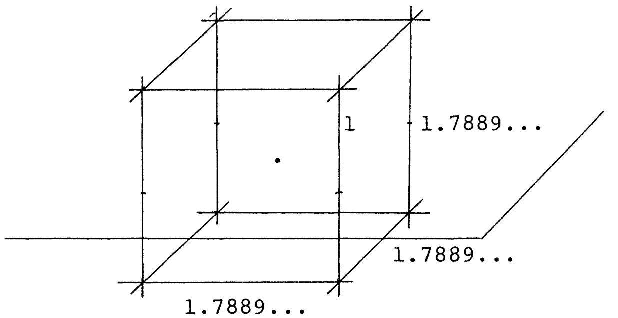

fig. 1/b

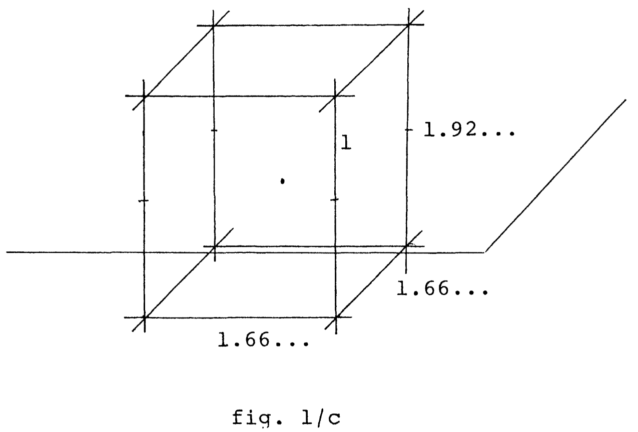

fig. 1/c

It will follow from Lemma 3.1. that in both cases the plane having the thinnest circle covering is parallel to the base face of the generating parallelepiped and touches a ball of this covering. The corresponding densities are $d_1 = 1,3414\ldots$ and $d_2 = 1,303\ldots$ respectively.

Corollary 2.3.

The thinnest covering of equal circles can not arise by intersecting a lattice-covering of balls.

3. An analogous problem for sphere-rows

Let us consider all the unit spheres, the centres of which lie on the coordinate axis e and have coordinates ℓa (where ℓ is an integer and a is a given positiv number)

We want to find from among all planes being perpendicular to axis e, that one which minimizes the total area of the intersections of the plane with the balls. The following Lemma states that the extremal plane touches a ball.

Lemma 3.1.

Denote $S_a(x)$ the sum of the intersections of the balls with the plane, beeing perpendicular to axis e and intersecting it in a point of coordinate x. Then

a/ If $\frac{1}{k+1} \leq a < \frac{2}{2k+1}$ (k > 0 integer), then

$$S_a(x) \geq (2k+1)[1-((k+1)a-1)^2]\pi - 2a^2 \frac{k(k+1)(2k+1)}{6}\pi = S_a(1)$$

b/ If $\frac{2}{2k+1} \leq a \leq \frac{1}{k}$ (k > 0 integer), then

$$S_a(x) \geq (2k+1)[1-(1-ka)^2]\pi - 2a^2 \frac{k(k+1)(2k+1)}{6}\pi = S_a(1)$$

c/ If $1 \leq a$ then $S_a(x) \geq (1-(a-1)^2)\pi = S_a(1)$

d/ If $a > 2$ then $S_a(x) \geq 0 = S_a(1)$

Proof.

The cases c/ and d/ are trivial.

Case a/ Considering that the function $S_a(x)$ is periodical we can restrict ourselves to the case $0 \leq x \leq a$

$$a < \frac{2}{2k+1} \iff 1-ka > \frac{a}{2} \iff (k+1)a - 1 < \frac{a}{2}$$

If $0 \leq x \leq (k+1)a - 1$ then the total area of the intersections of the plane with the balls having a centre of coordinate not greater than x is equal to

$$\pi \sum_{\ell=0}^{k} (1-(x+\ell a)^2) .$$

The total area of the intersections of the plane with the balls having a centre of coordinate greater than x is equal to

$$\pi \sum_{\ell=1}^{k} (1-(\ell a-x)^2)$$

The sum of these terms

$$S_a(x)=\pi(2k+1)(1-x^2)-a^2\pi\frac{k(k+1)(2k+1)}{3} =: f_1(x)$$

for $0 \leq x \leq (k+1)a - 1$.

On the other hand, for $(k+1)a - 1 \leq x \leq 1-ka$ we have

$$S_a(x)=f_1(x)+[1-((k+1)a-x)^2]\pi =: f_2(x)$$

Finally, for $1-ka < x < a$ we have

$$S_a(x) = f_1(a-x)$$

Since we can represent the functions f_1 and f_2 in the forms

$$f_1(x) = -cx^2+d \quad (c>0) \text{ and } \quad f_2(x) = c'(-x^2+ax)+d' \quad (c'>0)$$

$S_a(x)$ attains its minimum for $x = (k+1)a - 1$ and $x = 1-ka$ in the interval $(0,a)$. Therefor, in view of periodicity, we have

$$S_a(x) \geq \pi(2k+1)[1-((k+1)a-1)^2]-2\pi a^2 \frac{k(k+1)(2k+1)}{6} =$$

$$= S_a((k+1)a-1) = S_a(1)$$

<u>Case b/</u> We can restrict ourselves to the case $0 \leq x \leq a$ again.

$$a > \frac{2}{2k+1} \iff 1-ka < \frac{a}{2} \iff (k+1)a-1 \geq \frac{a}{2}$$

It is easy to see that for $0 \leq x \leq 1-ka$ we have $S_a(x) = f_1(x)$.
For $1-ka \leq x \leq (k+1)a-1$ we have $S_a(x) = f_1(x)-[1-(x+ka)^2]\pi$.
Finally, for $(k+1)a-1 \leq x \leq a$ $S_a(x) = f_1(x-a)$.
The proof can be finished similarly as in the case a. Since $f_1(x) = -cx^2+d$, $S_a(x)$ attains its minimum for $x = 1-ka$. Therefor, in view of periodicity, we have

$$S_a(x) \geq \pi(2k+1)[1-(1-ka)^2]-2\pi a^2 \frac{k(k+1)(2k+1)}{6} =$$

$$= S_a(1-ka) = S_a(1) \quad \circledast$$

4. Proof of Theorem 2.1.

Let us denote by Γ a lattice such that the unit spheres centred at the lattice-points should cover the space.

Lemma 4.1.

Any plane S satisfies at least one of the following propositions:

1./ There exist two lattice-points P_1,P_2 such that the distance-difference defined by $|d(P_1,S)-d(P_2,S)|$ is not greater than $\frac{1}{2}$. (Where $d(P,S)$ denotes the (algebraic) distance of the point P from the plane S.)

2./ The plane S is parallel to a face of a greating parallelepiped.

Proof.

Let us consider a pair of lattice-points P_1,P_2 of least distance-difference. If such a pair does not exist, the proposition 2./ is true. Let us denote the distance-difference $|d(P_1,S)-d(P_2,S)|$ by a. Since any translation transfering a lattice-point into another lieves the lattice invariant, for any lattice point P we have $d(P,S) = ka$ (k is an integer). Let us consider the plane S_1 containing P_1 and being parallel to S.

Let the vectors $\overrightarrow{P_1Q}$ and $\overrightarrow{P_1R}$ the basis vectors of the lattice $S_1 \cap \Gamma$. The parallelepiped generating by the points P_1,P_2,Q,R is empty, thus it is a fundamental parallelepiped. This means that proposition 1 is true. $\circledast$

We can get a lower bound for the density of circlesystem using the following method.

Divide the circles into groups and associate the groups with domains satisfying the properties i/-iii/

i/ The domains are disjunct.

ii/ The circles of a certain group and the domain belonging to this group can be covered by a circle of the fix radius $R < \infty$.

iii/ There exists a number $\mathcal{D}$ such that for any groups the total area of the circles belonging to the group divided by the area of the domain associated twith the group schould not less than $\mathcal{D}$.

* Then the density of the circle system is greater than or equal to $\mathcal{D}$.

Let P_1 and P_2 be lattice points such that no further lattice point schould be on the segment P_1P_2 and the distance-difference a of these points from S schould be greater than 0. Let

$$L = \{\ell \mid \text{the line } \ell \text{ is parallel to } P_1P_2 \text{ and } \ell \cap \Gamma \neq \emptyset\}$$

Let us consider the intersections of the plane S with the balls of the covering. We divide the circles of this covering into groups by the following property: Two circles are in the same group it the centres of the balls from which come these circles lie on the same line of L. We say that this line is the axis of the group. Let T be a fundamental parallelogram of the lattice $\bar{\Gamma} = \{S \cap \ell \mid \ell \in L\}$. Distinguish a vertex A of T. Finally, associate each group of circles with the parallelogram $B-A+T$ where B denotes the intersection of the axis of the group with the plane S.

Obviously the above construction satisfy the properties i/ and ii/. It follows from Lemma 3.1. that $S_a(1)/T$ correspounds for $\mathcal{D}$. Since $T \cdot a$ is the area of a foundamental parallelepiped of Γ and the density of any lattice-covering of the space by the balls is at least $5\sqrt{5}\pi/24$ [see Bambah [1]], we have

$$\frac{4\pi}{3Ta} \geq \frac{5\sqrt{5}\pi}{24}$$

If d denotes the density of the circle-covering arising on the plane S, in view of the proposition *, we obtain

$$d \geq \frac{5\sqrt{5}}{32} S_a(1)a \tag{2}$$

<u>Proposition 4.2.</u>

If $a \in (0;0,954)$ then

$$S(a) := \frac{5\sqrt{5}}{32} S_a(1)a > d_0$$

(d_0 is defined in 2.2.)

Proof.

We have already determined the value of $S_a(1)$ in the lemma 3.1. It is enough to consider the cases a/ and b/.

a/ If $\frac{1}{k+1} < a < \frac{2}{2k+1}$ then $S'(a)=\frac{5\sqrt{5}\pi}{32}(k+1)(2k+1)[4a-(4k+3)a^2]$

Then derivate is equal to zero for $a = 0$ and $a = \frac{4}{4k+3}$ and it is positive for $a = \frac{1}{k+1}$. Since

$$0 < \frac{1}{k+1} < \frac{4}{2k+1} < \frac{2}{2k+1} \quad ,$$

$S(a)$ attains its minimum for the boundary of the interval $[\frac{1}{k+1},\frac{2}{2k+1}]$. Thus

$$S(a) \geq \frac{5\sqrt{5}\pi}{32} \min\{S(\tfrac{1}{k+1});S(\tfrac{2}{2k+1})\} =$$

$$= \frac{5\sqrt{5}\pi}{32} \min \{\frac{(2k+1)(2k+3)}{3(k+1)^2} ; \frac{16k(k+1)}{3(2k+1)^2}\}$$

By same computation we obtain that this form is greater than $1.227 > d_o$ for $k \geqq 1$.

b/ If $\frac{2}{2k+1} \leqq a \leqq \frac{1}{k}$ then $S'(a) = \frac{5\sqrt{5}\pi}{32} k(2k+1)[4a-(4k+1)a^2]$.

The derivate is equal to zero for $a = 0$ and $a = \frac{4}{4k+1}$ and it is positive for $a = \frac{2}{2k+1}$. Since

$$0 < \frac{2}{2k+1} < \frac{4}{2k+1} < \frac{1}{k} \; ,$$

$S(a)$ attains its minimum for the boundary of the interval $[\frac{2}{2k+1},\frac{1}{k}]$. Thus

$$S(a) \geq \frac{5\sqrt{5}\pi}{32} \min \{S(\tfrac{2}{2k+1}),S(\tfrac{1}{k})\} =$$

$$= \frac{5\sqrt{5}\pi}{32} \min \{ \frac{k(k+1)16}{3(2k+1)^2} , \frac{(2k+1)(2k-1)}{3k^2} \}$$

By same computation we obtain that this form is greater than $1.227 > d_o$ for $k \geqq 1$. ◉

According to Lemmas 4.1. and 4.2. we obtain

Corollary 4.3.

A circle-covering with density greater than or equal to d_o may occur on the plane S if and only if S is parallel to a face (it will be denoted by T in further again) of a fundamental parallelepiped and the altitude (a) belonging to T is greater than 0.954. Obviously a is less than 2, otherwise the balls not cover the space.

Proposition 4.4.

If $a \in [1,2]$ then $d \geqq d_o$. Equality holds only for case described in 2.2.

Proof.

We can restrict ourself to the plane S mentioned in the Corollary 4.3. It follows from the Lemma 3.1. that $d > S_a(1)/T$ and there is a plane parallel to S on which the density is equal to $\frac{S_a(1)}{T} = \frac{2a-a^2}{T}$. So we have to minimize the value $\frac{2a-a^2}{T}$. For technical reasons we will do it in another form.

Proposition 4.5.

Let Γ be a lattice such that 1/ the spheres of radii r centred at the lattice points should cover the space 2/ it should have a fundamental parallelepiped of a face T and of an altitude a belonging to T such that $r \leqq a \leqq 2r$. The value $\frac{2ar-a^2}{T}$ attains its minimum for the lattice described in 2.2. Apart from the similarity this is the only extremal lattice.

Proof.

Denote G the system of spheres of radii r centred at the lattice points of Γ. Consider in G a particular sphere K. Since the system G is homogeneous, the spheres of G cover the space if and only if

(3) the surface of K is covered by the rest.

We divide the spheres of G into layers: Two spheres are in the same layer if and only if their centres are in the same plane parallel to T. A layer is called layer Ψ if the centres of its spheres lie on the plane Ψ. Denote the layer containing the sphere K by layer S. Let layer S′ be the layer next to layer S.

Denoting, finally, the part of K lying between the planes S and S' by K^+, since $r \leq a \leq 2r$ we have:
The statement (3) is true if and only if

(4) K^+ is covered by the spheres of layers S and S'.

Consider the Dirichlet-tiling $\mathcal{D}$ of the plane S, generating by the centres of spheres of the layer S. The cells are congruent, centro-symmetrical hexagons inscribed in a circle. (In the sence, that a rectangle is a degenerated hexagon.) Let R be the circumradius of the cells. Obviously $R \leq r$, otherwise the spheres not cover K^+. Denote D the cell belonging to the centre O of the sphere K. Denote P the part of K^+, the projection of which (onto plane S) lie in D. Obviously the part $K^+ \setminus P$ is covered by the six spheres of layer S, the centres of which have Dirichlet cells neighbouring to cell D. Thus the statement (4) is true if and only if

(5) $R < r$ and P is covered by the spheres of layer S'.

Let us project the centres of the layer S' onto the plane S. Consider the Dirichlet-tiling $\mathcal{D}'$ generating by the projections. The tiling $\mathcal{D}'$ consist of translates of the disc D. It is easy too see, that 1/ the cell D contains at least one edge of $\mathcal{D}'$ 2/ the four cells $D_1,\dots,D_4$ of $\mathcal{D}'$, which have common point with this edge, cover the cell D.

Denote O_i $(i=1,2,3,4)$ the centre of the sphere of layer S', the projection of which is associated with the cell D_i. The parallelogram $O_1O_2O_3O_4$ is a fundamental parallelogram of the centres of layer S'. We have proved the statement (5) is true if and only if

(6) P is covered by the spheres centred at the points O_1, O_2, O_3, O_4.

We shall need a Lemma

<u>Lemma</u>

Let $\widehat{AB}$ an arc on the intersection of K^+ and of a plane perpendicular to S. If a sphere K' of the layer S' contains A and B, then it contains the whole arc $\widehat{AB}$.

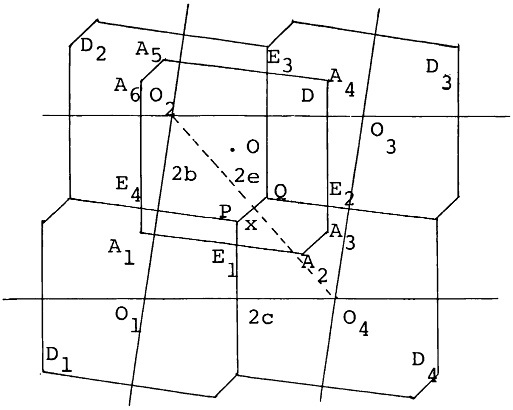

fig. 2

The proof of this Lemma is trivial.

Corollary.

An arc polygon consisting of arcs similar to $\widehat{AB}$ is covered by the sphere K_i if and only if the sphere K_i covers the vertices of the polygon.

Marking:

We denote the point of P, projection of which is X onto S, by X'.

For symmetrical reasons Q' (and P' resp.) is covered if and only if A_4' (and A_1' resp.) is covered so, the statement (6) is true if and only if

(7) the spheres K_1, K_2, K_3, K_4 cover the points A_i' and E_j' $i=1,\dots,6;$ $j=1,\dots,4.$

The paragraphs 1^o–4^o will contain processes diminishing the value of $\frac{2ar-a^2}{T}$. Continuing this processes we obtain the extremal lattice.

1° Remove all the neighbouring layers in the direction perpendicular to S through the same distance. Preserving the property of covering we can do it as far as the points $A_i'; E_j'$ ($i=1,\ldots,6$; $j=1,\ldots,4$) are covered by the spheres K_ℓ $\ell=1,\ldots,4$.

(8) Let
$$O_1O_2 = 2b$$
$$O_1O_4 = 2c$$
$$O_2O_4 = 2e$$
$$PQ = 2x$$
and $PO_1 = QO_3 = R$

The distance between the point E_i' and the plane S (the plane S' resp.) cannot be greater than $\sqrt{r^2-b^2}$ (than $\sqrt{r^2-c^2}$ resp.). Thus

$$\sqrt{r^2-b^2} + \sqrt{r^2-c^2} \geq a$$

Since $\min\{\overline{OP},\overline{OQ}\} \geq x$, the distances $d(P,S)$ and $d(Q,S)$ are less than or equal to $\sqrt{r^2-x^2}$. Thus

$$\sqrt{r^2-x^2} + \sqrt{r^2-R^2} \geq a$$

That is
$$a \leq \min\{\sqrt{r^2-x^2} + \sqrt{r^2-R^2}\ ,\ \sqrt{r^2-b^2} + \sqrt{r^2-c^2}\}$$

Denote b', c' the two shorter sides of the triangle $O_1O_4O_2$. (b' and c' correspond to b and c not necessary.) Let x' be the distance between P and the third side of the triangle $O_1O_4O_2$. In any case

$$\min\{\sqrt{r^2-x^2} + \sqrt{r^2-R^2}\ ,\ \sqrt{r^2-b^2} + \sqrt{r^2-c^2}\} \leq$$
$$\leq \min\{\sqrt{r^2-x'^2} + \sqrt{r^2-R^2}\ ,\ \sqrt{r^2-b'^2} + \sqrt{r^2-c'^2}\} =: a(T,r)$$

Let us consider the layer S and the parallelogram-lattice generated by the sides b', c' as vectors. From now on T denote the parallelogram $O_1O_2O_3O_4$ of the lattice. We shall use the markings of the fig.2. We may suppose that the angle at the vertex O_1 is not abtuse. We give a new sphere-lattice. Put on top of the layer S a second layer (S') of the same kind so that S and S' will have a distance $a(T,r)$ and the projections of the centres of the layer S onto S correspond to the midpoints of the parallelograms. The translation which transfers the first layer into the second, transfers the second into the third and repeated translations of the same kind in both directions produce a sphere-lattice which is a covering. Since $(2ar-r^2)/T$ have decreased, we can restrict our consideration to such coverings. More exactly we have not to deal with the property of covering, but to find the parallelogram $T = O_1O_2O_3O_4$ so that

$$d(T,r) := \frac{2a(T,r)r - a^2(T,r)}{T}$$

should be minimal.

2^o We may assume that T is a rhombus.

Proof.

Otherwise let we move the point O_1 parallel to the side e as far as it will fall on the normal bisector of e.

While moving $a(T,r)$ decreases. Since T do not change and x,R decrease, it is enough to prove that the function

$$f(y) := \sqrt{C - \frac{(d-y)^2+h^2}{4}} + \sqrt{C - \frac{(d+y)^2+h^2}{4}}$$

is decreasing - where y denotes the distance between O_1 and the normal bisector of e; h denotes the altitude of $O_1O_4O_2\Delta$ belonging to the vertex O_1; finally, C is a suitable constant.

$$f'(y) = \frac{1}{2}\,\frac{d-y}{\sqrt{C^*-(d-y)^2}} + \frac{1}{2}\,\frac{-(d+y)}{\sqrt{C^*-(d+y)^2}}$$

After some computation we obtain that the inequality $f'(y) < 0$ is equivalent to $(d-y)^2 < (d+y)^2$ which is true.

3^o We may assume that

$$a(T,r) = r = \sqrt{r^2-R^2} + \sqrt{r^2-x^2}$$

Proof.

We shall need

i/ if r decreases, then $d(T,r)$ decreases

ii/ if r decreases, then $\frac{a(T,r)}{r}$ decreases.

By some computation we obtain

$$d_r(T,r) = \frac{2a(T,r)}{Tpq}(p-r)(q-r)$$

and

$$\left(\frac{a(T,r)}{r}\right)_r = \frac{a(T,r)}{pq}\,\frac{r^2-pq}{r^2}$$

where the pair (p,q) is equal to one of the pairs

$$(\sqrt{r^2-R^2}\ ,\ \sqrt{r^2-x^2})\ ,\ (\sqrt{r^2-b^2}\ ,\ \sqrt{r^2-c^2}).$$

Since $p,q \leq r$ (and one of them is less than r), the derivates are greater than 0. In view of ii/ we may decrease r so that one of the equalities comes true

(9) $$r = \sqrt{r^2-R^2} + \sqrt{r^2-x^2}$$

(10) $$r = 2\sqrt{r^2-b^2}$$

If (10) comes true first, then let us increase the side e leaving the sides $b = c$ unchanged. Thus $d(T,r)$ decreases. It is impossible that T become to rectangle before (9) become true. Otherwise it would be true the inequality

$$2\sqrt{r^2-b^2} < r + \sqrt{r^2-2b^2}\ ,$$

from which would follow $b^4 < 0$.

4^o We want to minimize $d(T,r)$ among all (T,r) which satisfy 2^o and 3^o.

This exercise agrees with the extremal-exercise mentioned in 2.2. The inequality $e > b$ corresponds to $0.85 \leqq x \leqq 1$. By some computation we obtain

$$T(x) = x(\sqrt{3+2x^2-x^4} + \sqrt{3-2x^2-x^4})$$

Thus

$$T'(x) = \sqrt{3-2x^2-x^4}(3+4x^2-3x^4)+\sqrt{3+2x^2-x^4}(3-4x^2-3x^4)$$

The equation $T'(x) = 0$, in view of $3-4x^2-3x^4 < 0$ for $x \in (0.85,1)$, is equivalent to the equation $27x^2-46x^6+3x^{10} = 0$.

Solving this equation for $x \in (0.85,1)$ we have

$$x_o = \sqrt[4]{\frac{46-\sqrt{46^2-324}}{6}} = 0.884237\ldots .$$

Since $T'(x)$ is negative in the interval $(x_o,1)$ and it is positiv in $(0.85,x_o)$, the function $T(x)$ attains its minimum for x_o. Then the minimal-density $\pi/T(x_o) = 1.2266888$. ◉

5. Let us return to the Corollary 4.3. According to the propositions 4.2. and 4.4. we have only to deal with the case $a \in (0.954,1)$.

Let Π a plane containing the face Π of a fundamental parallelepiped of Γ.

Denote $\bar{\Gamma}$ the lattice $\Pi \cap \Gamma$.

Denote Π_1 and Π_2 to Π parallel planes such that the distance between Π and Π_1 is $\frac{a}{2}$ and the distance between Π and Π_2 is a. We call the coverings arising on Π, Π_1 and Π_2 basic-, middle- and tangential-covering.

1./ basic-covering. It contains circles of radii 1 and $\sqrt{1-a^2}$. The centres of the congruent circles form a lattice congruent with $\bar{\Gamma}$. Reflecting one of them in the midpoint of a fundamental-parallelepiped of it we get the other lattice.

2./ middle-covering. It contains circles of radii $\sqrt{1-\frac{a^2}{2}}$. The centres of the circles form two translated lattices congruent to $\bar{\Gamma}$.

3./ tangential-covering. It contains circles of radii $\sqrt{1-(1-a)^2}$ and $\sqrt{1-(2a-1)^2}$. The centres of the congruent circles form a lattice congruent with $\bar{\Gamma}$.

(11) We suppose that the lattice $\bar{\Gamma}$ satisfy the inequality

$$\frac{S_a(1)}{T} < d_o \quad .$$

<u>Proposition 5.1.</u>

Any lattice-vector of $\bar{\Gamma}$ has a length greater than $2\sqrt{1-a^2}$

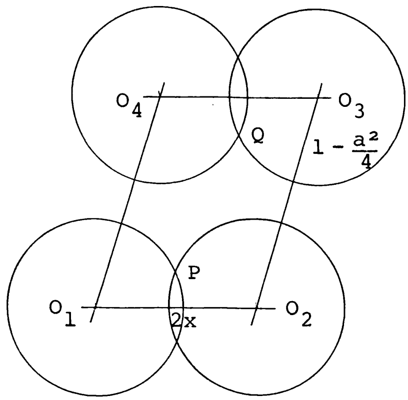

fig. 3.

Proof.

Let us consider a middle-covering (fig.3.). Suppose, on the contrary, that $O_1O_2 = 2x$ and $0 < x \leqq 1-a^2$. Then

$$T \leqq 4x\left(\sqrt{1-\frac{a^2}{4}} + \sqrt{1-\frac{a^2}{4}-x^2}\right) =: f(x)$$

In view of

$$f'(x) = 4\left(\sqrt{1-\frac{a^2}{4}} + \sqrt{1-\frac{a^2}{4}-x^2} - \frac{x^2}{\sqrt{1-\frac{a^2}{4}-x^2}}\right.$$

the roots of the equation $f'(x) = 0$ are 0 , $\pm \frac{\sqrt{3}}{2}\sqrt{1-\frac{a^2}{4}}$. Since they don't lie in the interval $(0,\sqrt{1-a^2})$ we obtain $f'(x) > 0$ there. Thus $f(x)$ attains its minimum for $x = \sqrt{1-a^2}$ in the interval $(0,\sqrt{1-a^2})$. Thus

$$T \leqq 2\sqrt{1-a^2}\left(2\sqrt{1-\frac{a^2}{4}} + \sqrt{3}a\right).$$

We show that

$$\frac{S_a(1)}{T} > \frac{(6-5a)a\Pi}{2\sqrt{1-a^2}\left(2\sqrt{1-\frac{a^2}{4}} + \sqrt{3}a\right)} > d_o ,$$

which contradicts the supposition (11).

It is enough to prove, in view of $0.95 < a$, that

$$(6-5a)a > \frac{2d_o}{\Pi}\sqrt{1-a^2}\left(2\sqrt{1-\frac{0.95^2}{4}} + \sqrt{3}\right) .$$

After transfering, we have

$$25a^4 - 60a^3 + 43.42a^2 - 7.42 > 0$$

which is true in the interval $(0.954,1)$. ◉

Proposition 5.2.

A small circle of a basic-covering can have common points at most with two other small circles.

Proof.

Suppose, on the contrary, that there exist three small circles centered at the points A, B, C having common points with a chosen small circle (fig.4.). According to proposition 5.1. these three

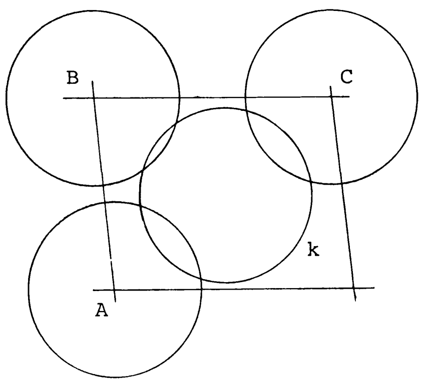

fig. 4.

circles come from spheres of the same layer. The points A, B, C cannot lie on a line. Thus $\frac{1}{2}T$ equals the area of the triangle ABC, which is less than the area of the regular triangle of circumradius $2\sqrt{1-a^2}$. Thus $\frac{1}{2}T < \sqrt{3}\cdot 3(1-a^2)$. It is easy to verify for $a \in (0.954,1)$ that

$$\frac{S_a(1)}{T} > \frac{(6-5a)a\pi}{2\sqrt{3}\cdot 3(1-a^2)} > d_o ,$$

which contradicts the supposition (11). ◉

Remark 5.3.

The proposition 5.2. says, that the region covered by the small circles of a basic-covering can not be connected. If each small circle has common points with two others, the circles form connected rows. If each small circle has common point with 1 or 0 others, then the covered region consist of bounded domains. It follows, that the unit circles of a basic-covering can not be disjunkt. Thus:

Proposition 5.4.

The lattice $\bar{\Gamma}$ has lattice-vectors of length ≤ 2.

Proposition 5.5.

The lattice-vectors of $\bar{\Gamma}$ of length ≤ 2 cannot divide the plane into stripes.

Proof.

Supposing the opposite, it follows that the small circles of a tangential-covering are not disjunkt. Denote $2x$ the side of T, which is parallel to the stripes. Thus we have

$$2x < 2\sqrt{1-(2a-1)^2} < 0.84$$

and similarly, the altitude of T belonging to $2x$ is less than $\sqrt{1-x^2} + 0.84 + 1$. Thus

$$T < 2x(\sqrt{1-x^2} + 0.84 + 1) < 0.84 \cdot 2.84 < 2,39$$

which yields the inequalities

$$\frac{(6-5a)a\pi}{T} > \frac{\pi}{2.39} > d_o$$

which contradicts the supposition (11). ◉

Proposition 5.6.

The lattice-vectors of $\bar{\Gamma}$ of length ≤ 2 cannot divide the plane into parallelograms.

Proof.

Supposing the opposite, consider a connected region H not covered by the unit circles of a basic covering. H is not empty and it is centro-symmetrical. We show that

H is covered by at most two small circles

Otherwise, the small circles form connected chains and thus T has a side of length $2x < 4\cdot 0.3 = 1.2$. Consider a tangential-covering We may suppose that the small circles are disjunkt, otherwise we finish the proof as at the proposition 5.5. Thus the altitude of T belonging to the side $2x$ is less than

$$2(\sqrt{4a-4a^2} + \sqrt{1-x^2}) .$$

Thus

$$T < 4x(\sqrt{4a-4a^2} + \sqrt{1-x^2}) =: h(x) .$$

Since

$$h'(x) = 4\,\frac{\sqrt{4a-4a^2}\sqrt{1-x^2}+1-2x^2}{\sqrt{1-x^2}} > 0$$

for $0 < 2x < 1.2$

we have

$$T < 4\cdot 0.6(\sqrt{4a-4a^2} + 0.8) .$$

We show the inequality,

$$\frac{(6-5a)a\pi}{2.4\cdot(\sqrt{4a-4a^2} + 0.8)} > d_o$$

which contradicts the supposition (11).

Since the functions $\sqrt{4a-4a^2}$ and $(6-5a)a$ are decreasing in the interval $(0.954;1)$, it is enough to find numbers $0.954 = a_o < < a_1 < \dots < a_n = 1$ such that

$$\frac{(6-5a_i)a_i\pi}{2.4(\sqrt{4a_{i-1}-4a_{i-1}^2} + 0.8)} > d_o \quad \text{for } i=1,\dots,n .$$

It can be verify, that the sequence 0.954, 0.96, 0.968, 0.976, 0.985, 1 is good.

We show that H cannot be covered only by one circle.

We give upper bound for the area of the parallelogram $T = O_1O_2O_3O_4$ which is covered by the circles centred at O_i and by a circle of radius $\sqrt{1-a^2}$. Since H is centro-symmetrical, transfering the small circle so that the centre of H should be the centre of it, we obtain a covering again. Move the side O_3O_4 parallel to the side O_1O_2 as far as T becomes a rectangle. Since the distances of the opposit vertices of H decrease the rectangle is covered by the circles centred at the vertices and by the circle of radius $\sqrt{1-a^2}$ centred at the intersection of diagonals. The same is true for the square the side of which equals the longer side of the rectangle. Thus we have

$$T < 2+2\sqrt{1-a^4}$$

We show

$$h(a) := \frac{(6-5a)a\pi}{2+2\sqrt{1-a^4}} > d_o$$

which contradicts the supposition (11).

It is sufficient to prove, in view of $\sqrt{1-a^4}-\sqrt{4a-4a^2} < 0$ for $a \in (0.954,1]$, the inequality

$$\frac{(6-5a)a\,\pi}{2+2\sqrt{4a-4a^2}} > d_o \tag{12}$$

It is eneugh to find numbers $0.954 = a_o < \dots < a_n = 1$ such that for $i=1,\dots,n$

$$\frac{(6-5a_i)a_i\ \pi}{2+2\sqrt{4a_{i-1}-4a_{i-1}^2}} > d_o$$

It can be verify, that the sequence 0.954, 0.96, 0.97, 0.98, 1 is good.

We have obtained that H must be covered exactly by two circles (k_1,k_2) of radii $\sqrt{1-a^2} =: r_1(a)$. Denote Q_1, Q_2 the centres of them.

Translate a tangential-covering so that the centres of the great circles should correspond to the centres of the unit circles of the basic-covering.

Then the circle of radius $\sqrt{1-(2a-1)^2} =: r_2(a)$ centred at Q_1 (or Q_2). Since Q_1Q_2 is symmetrical to the centre of H both pairs of opposit vertices of H have one vertex nearer Q_1 (or Q_2) than $r_1(a)$. Thus a circle of radius

$$\frac{r_1(a) + r_2(a)}{2} =: r(a)$$

covers H. Using the process of previous case we have

$$T < 2+2r(a)\sqrt{2-r^2(a)} =: t(a)$$

We show that for $a \in (0.954,1)$

$$\frac{(6-5a)a}{t(a)} > d_o$$

which contradicts (11).

Since $r(a)$ is a decreasing function in the interval $(0.954,1]$ and $r(0.954) = 0.3593$; $r(1) = 0$, further the function $2+2x\sqrt{2-x^2}$ increases in the interval $(0,1)$, the function $t(a)$ decreases in the interval $(0.954,1)$. Thus it is enough to find numbers $0.954 = a_o < \dots < a_n = 1$ such that

$$\frac{(6-5a_i)a_i\pi}{t(a_{i-1})} > d_o \qquad \text{for } i=1,\dots,n$$

It can be verify, that the sequence 0.954, 0.956, 0.958, 0.96, 0.962, 0.965, 0.97, 0.975, 0.98, 0.988, 1 is good. ◉

Remark 5.7.

According to foregoing, if the lattice $\bar{\Gamma}$ satisfy the inequality

$$\frac{(6-5a)a\pi}{T} \leq d_o$$

then the lattice-vectors of length ≤ 2 divide the plane into congruent triangles.

Proposition 5.8.

The small circles of a tangential-covering are disjunct.

Proof.

Suppose, on the contrary, that the triangles have a side $< 2\sqrt{1-(2a-1)^2}$. Thus $T < 4\sqrt{1-(2a-1)^2}$ and so we have for $a \in (0.954,1)$ that

$$\frac{(6-5a)a\pi}{T} > \frac{\pi}{4\sqrt{1-(2\cdot 0.954-1)^2}} = 1.87 > d_o$$

contradicts for (11). ◉

<u>Proposition 5.9.</u>

The great circles of a tangential-covering cover the plane.

<u>Proof.</u>

Suppose the opposite. According to proposition 5.8. each small circle have to cover the gaps of two neighbouring triangles. Consider the union of these triangles. This parallelogram has a longer diagonal $< 2(1+\sqrt{1-(2a-1)^2}$ thus $T < 2+2\sqrt{4a-4a^2}$

According to (12) we have

$$\frac{(6-5a)a\pi}{T} > d_o \quad \text{which contradicts (11).} \qquad ◉$$

<u>Proposition 5.10.</u>

The proposition 4.4. is true for $a \in (0.954,1)$.

<u>Proof.</u>

The proof of 4.4. must be modified. Suppose that (11) is true. We have to take the following amplifications.

- Now it is not trivial that K^+ (see (4)) is covered by the spheres of layer S and S'. According to 5.9. the spheres of a layer under S may be left out of consideration. In case a sphere over the layer S' intersects K^+, according to lemma after (6) it is enough to prove that it cannot intersect the boundary of P. In that case $\bar{\Gamma}$ should have a lattice-vector of length $< 2\sqrt{1-(2a-1)^2} < 0.84$. Thus $T < 2$ and
$$\frac{(6-5a)a\pi}{T} > \frac{\pi}{2} > d_o \qquad \text{which contradicts (11).}$$
- For 1^o we need that the function $d(a)$ is decreasing, which is trivial.
- In 3^o we have
$$d(T,r) = \frac{(6r-5a(T,r))a(T,r)}{T}$$
where the definition of $a(T,r)$ is unchanged. Instead of (i) and (ii) we have to prove:
(i)' if r increases then $d(T,r)$ decreases
(ii)' if r increases then $\frac{a(T,r)}{T}$ increases

(ii)′ is equivalent to (ii). The proof of (i)′: The derivate

$$d_r(T,r) = \frac{a(T,r)}{pq}(6r^2+6pq-10r(p+q))$$

where the pair (p,q) is equal to one of the pairs

$$(\sqrt{r^2-R^2},\sqrt{r^2-x^2}) \quad (\sqrt{r^2-b^2},\sqrt{r^2-c^2})$$

Since $0.918r < a(T,r) < r$, we have $pq < 0.25r^2$.
Thus

$$d_r(T,r) < c(6r^2+1.5r^2-9.18r^2) < 0$$

The rest of the proof is true replacing the words "we decrease r" by "we increase r". ◉

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