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GEOMETRODYNAMICS OF WORMHOLES

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The original concept of wormhole arised in geometrodynamics founded by J.A. Wheeler and C.W. Misner [1].

Geometrodynamics regards general relativity as a dynamical theory of space-like 3-geometries. The description of space-time is nothing else but the description of the evolution of space-like hypersurfaces. Let σ be a space-like hypersurface in space-time. On this 3-dimensional manifold the metric $\overset{(\sigma)}{g}$ is positive definite. If we want to examine the evolution of this 3-geometry, the metric $\overset{(\sigma)}{g}$ and the speed of its change along the normal vector field, i.e., the Lie-derivative $L_N \overset{(\sigma)}{g} = -\overset{(\sigma)}{B}$ must be given at t=0, where $\overset{(\sigma)}{B}$ denotes the second fundamental form. Now, the following question arises: what conditions should be satisfied by $\overset{(\sigma)}{g}$ and $\overset{(\sigma)}{B}$, so that a space-time metric should exist, which at t=0 reduces to

$$ds^2 = -dt^2 + \overset{(\sigma)}{g}$$

$$\frac{\partial}{\partial t} \overset{(\sigma)}{g} = -\overset{(\sigma)}{B}$$

We can make use [2] of the Codazzi equation

$$\overset{(\sigma)}{\nabla}_Y \overset{(\sigma)}{B}(X_i, X_i) - \overset{(\sigma)}{\nabla}_{X_i} \overset{(\sigma)}{B}(X_i, Y) = \overset{(M)}{Ric}(Y, N)$$

$$\forall\, Y \in \Gamma(T\sigma)$$

and the Gauss equation

$$\overset{(\sigma)}{g}(V, \overset{(\sigma)}{R}(X,Y)Z) = \overset{(M)}{g}(V, \overset{(M)}{R}(X,Y)Z) + \\ + \overset{(\sigma)}{B}(X,V)\overset{(\sigma)}{B}(Y,Z) - \overset{(\sigma)}{B}(Y,V)\overset{(\sigma)}{B}(X,Z) \\ \forall\ X,Y,Z,V \in \Gamma(T\sigma).$$

Here $\overset{(\sigma)}{\nabla}$ denotes the covariant derivative on the hypersurface σ, $\{ X_i \}_{i=1,2,3}$ is basis in $\Gamma(T\sigma)$ and $\overset{(M)}{R}$ is the curvature of space--time. From the Einstein equations we obtain the constraint equations

$$\overset{(\sigma)}{\nabla}_Y \overset{(\sigma)}{B}(X_i, X_i) - \overset{(\sigma)}{\nabla}_{X_i} \overset{(\sigma)}{B}(X_i, Y) = 8\pi T(N, Y) \\ \forall\ Y \in \Gamma(T\sigma),$$

$$\frac{1}{2}\left(\overset{(\sigma)}{R} + 2\,\mathrm{Trace}\,\overset{(\sigma)}{B}\circ\overset{(\sigma)}{B} - (\mathrm{Trace}\,\overset{(\sigma)}{B})^2\right) = 8\pi\, T(N,N),$$

and the evolution equation

$$L_N \overset{(\sigma)}{B} = \overset{(\sigma)}{Ric} + 2\overset{(\sigma)}{B}\circ\overset{(\sigma)}{B} - (\mathrm{Trace}\,\overset{(\sigma)}{B})\overset{(\sigma)}{B} - 8\pi T \\ - 8\pi\, T(N,N)\overset{(\sigma)}{g}.$$

The motivations of geometrodynamics are the followings:

i. Constraint equations are more simple than the Einstein equations.

ii. Canonical quantization of space-time.

iii. The structure of space-like hypersurfaces may have a fancy physical meaning.

2.

Since $\overset{(3)}{M}$ is assumed to be initially at rest, the constraint equations reduce to

$$\overset{(3)}{R} = 0.$$

Misner [4] has shown that this constraint equation does have a solution in the $S^1 \times S^2$ topology:

$$^{(3)}ds^2 = \phi^4 (d\mu^2 + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

where

$$\phi = \sum_{n=-\infty}^{\infty} (\mathrm{ch}(\mu + 2n\mu_0) - \cos\vartheta)^{-1/2}$$

is a μ_0 -periodic solution of the Brill's wave-equations. This solution is asymptotically flat, and it looks like the Schwarzschild's one around the mouths of the "wormhole" /Fig. 1./. We can realize

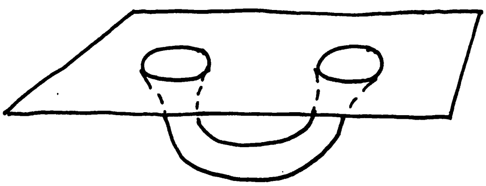

Fig. 1.

this solution as a topologycally "closed" Einstein-Rosen bridge /Fig. 2./. has been also shown that there is an initial solution

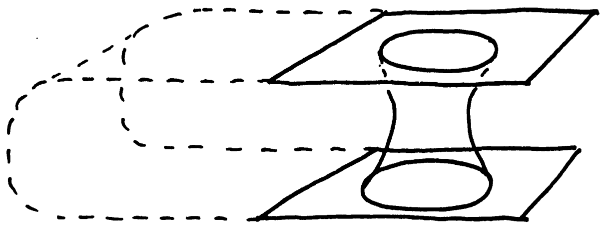

Fig. 2.

in wormhole topology of the Einstein-Maxwell equations. In that case the electromagnetic flux-lines are trapped by the wormhole, and the mouths look like a pair of charges, although we have a solution of the source-free field equations /Fig. 3./.

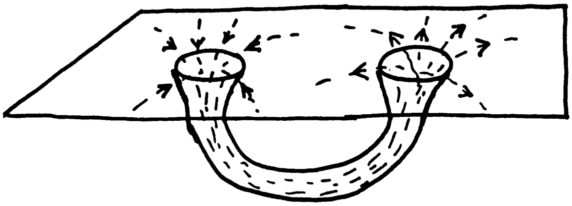

Fig. 3.

It has been shown that the space-like hypersurfaces may have more complicated topology, namely the knot wormholes:

$$S^1 \times K^2$$

where K^2 is 2-dimensional knot, and the links of wormholes:

$$S^1 \times L_\mu(S^2)$$

where $L_\mu(S^2)$ is 2-dimensional link [3].

3.

It has been shown that one can construct a Yang-Mills-charged wormhole within the framework of generalized Kaluza-Klein theory.

The Kaluza-Klein model is a unified geometrical description of gravitation and a Yang-Mills field in an /r+4/ - dimensional space-time assumed to be a principal fibre bundle: H/M,G,π/ [5], where

$(M, \overset{(M)}{g})$ is the space-time,

$(G, \overset{(G)}{g})$ is a compact semi-simple Lie-group with an invariant metric: $\overset{(G)}{g}_{ab} = f^c_{ad} f^d_{cb}$.

A connection is given on the bundle H, and the coefficients of it are identified with Yang-Mills pontentials. There is also given a pseudo-Riemannian metric on H which satisfies the following compactibility conditions:

i. $$\overset{(H)}{g}\big|_{vTH} = \Psi^*(\overset{(G)}{g})$$

ii. $$\overset{(H)}{g}\big|_{hTH} = \pi^*(\overset{(M)}{g})$$

iii. $$vTH \perp hTH ,$$

where vTH and hTH are the vertical and the horizontal distributions, $\pi: H \to M$ is the bundle projection, Ψ is the trivialization map. Further the indexes A,B,... run from 1 to r+4, a,b,... from 5 to r+4, and μ, ν,... from 1 to 4. The connection coefficients are defined as

$$v\, e_\mu = A^a_\mu\, e_a ,$$

where $v e_\mu$ is the vertical part of the basis vector e_μ. If we choose the basis which is the natural one on the basis manifold and which is the left-invariant one on the fibre, the matrix of the metric $\overset{(H)}{g}$ is

$$\overset{(H)}{g}_{AB} = \begin{pmatrix} \overset{(M)}{g}_{\mu\nu} + \overset{(G)}{g}_{ab} A^a_\mu A^b_\nu & \overset{(G)}{g}_{ab} A^a_\mu \\ \overset{(G)}{g}_{ab} A^a_\nu & \overset{(G)}{g}_{ab} \end{pmatrix}.$$

The scalar curvature is

$$\overset{(H)}{R} = \overset{(M)}{R} + \overset{(G)}{R} - \frac{1}{4} \overset{(G)}{g}_{ab} \overset{(M)}{g}^{\mu\kappa} g^{\nu\beta} F^a_{\mu\nu} F^b_{\kappa\beta} .$$

The unified action integral has the form:

$$\overset{(H)}{S} = \int \sqrt{-\overset{(H)}{g}}\, \overset{(H)}{R}\, d^4x\, d^rG .$$

It has been shown [4] that beyond the usual constraint equations a third constraint equation should be satisfied:

$$L_Z\, \overset{(\Sigma)}{B}(V, W) = 0,$$

for any vector fields V and W and for any fundamental field Z on the /r+3/-dimensional "space-like" hypersurface Σ.

It has been shown that these constraint equations do have solutions in the case when the topology of Σ is the Kaluza-Klein generalization of the wormhole, i.e., it is a fibre bundle: $\Sigma \xrightarrow{G} S^1 \times S^2$. A possible generalization of the "wormhole" concept in mathematical-physics is the following: Let θ be a p-form on a manifold M. θ is a conversation law for an exterior differential system $\mathcal{E}$ /i.e., for a collection of differential forms, which is closed under the exterior product and exterior derivation/ if the following condition is satisfied:

$$d\theta \in \mathcal{E} .$$

If we have integral submanifold of the system $\mathcal{E}$, such that its Betti-number $\beta_p \neq 0$, than $\int_C \theta$ can be different from zero, and this situation looks like as if there were "sources" of this conserved quantity.

4.

There are several ways to quantize general relativity. The most important of them is the "superspace" quantization of geometrodynamics. The superspace is the space where the 3-geometries are moving. Mathematically it is the following coset-space:

$$S(M) := \mathrm{Riem}(M) / \mathrm{Diff}(M)$$

where Riem /M/ is the set of Riemannian metrics of the 3-dimensional manifold M and Diff/M/ is the set of diffeomorfizms $M \to M$. The superspace is not a manifold but A.E. Fischer [6] has shown a partition into manifolds of geometries which have the same type of symmetries. Geometries with lower symmetries lie in the boundary of submanifold of more symmetric geometries. The metrizeability of the superspace has been also shown and it can be extended to be a manifold. A submanifold of homogeneous geometries of a given symmetry-type has finite dimension. The metric of the superspace introduced by Peres /1962/ [7] is

$$G_{ijkl} = \frac{1}{2} g^{-1/2} \left(g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl} \right).$$

The dynamics can be summed up in the Hamilton-Jacobi equation:

$$G_{ijkl} \frac{\delta S}{\delta g_{ij}} \frac{\delta S}{\delta g_{kl}} = g^{1/2}\, {}^{(3)}R$$

$$\partial_j \left(\frac{\delta S}{\delta g_{ij}} \right) = 0 .$$

Thus we have the Einstein-Schrödinger equation [8] :

$$\left(G_{ijkl} \frac{\delta^2}{\delta g_{ij}\, \delta g_{kl}} - g^{1/2}\, {}^{(3)}R \right) \psi = 0 .$$

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