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A CHARACTERIZATION OF A CLASS OF GRAPHS CONNECTED WITH THE HYSTERESIS PHENOMENA

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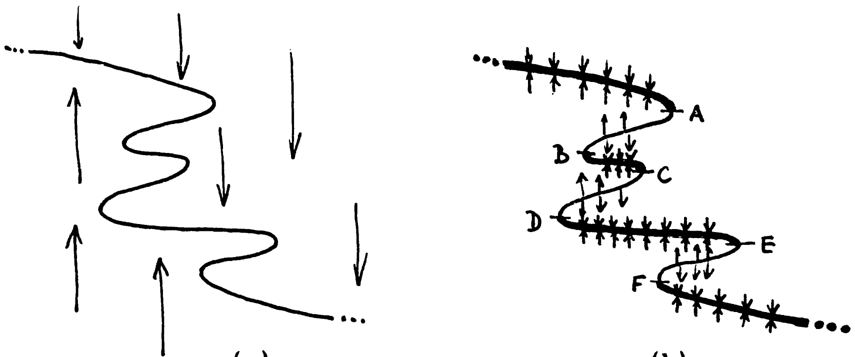
In the theory of dynamical sytems we are encountered with the following situation (see Fig.1(a)) : The system is governed by a field tending to opposite directions in two (almost complemental)



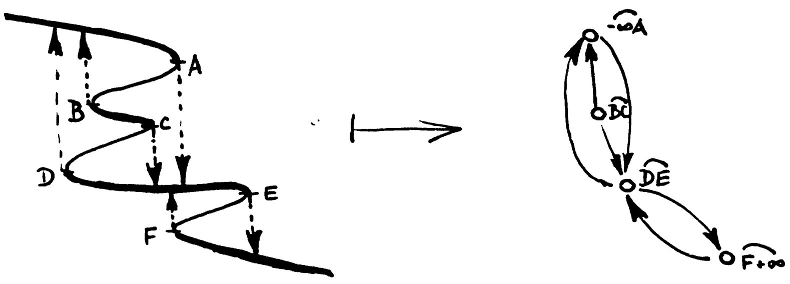
(a) (b)

Fig.1

areas of the plane. The borderline is then naturally divided into the parts of stable behavior (the segments $\widehat{-\infty A}$, $\widehat{BC}$, $\widehat{DE}$, $\widehat{F+\infty}$ in Fig.1(b)) and the unstable ones ($\widehat{AB}$, $\widehat{CD}$, $\widehat{EF}$ in Fig.1(b)).

If a point moves along a stable part which is connected (obeying some further unspecified forces), its state can be taken for qualitatively unchanged. If we, however, reach an edge, the field causes a jump into another state. Thus (see Fig.2(a)), e.g. at the edge B we jump from $\widehat{BC}$ to $\widehat{-\infty A}$, at the edge A from $\widehat{-\infty A}$ to $\widehat{DE}$, etc. We obtain an oriented graph of state transitions the nodes of which are the possible states (the maximal connected stable parts) and the oriented edges represent the possible jumps. M.Katětov and J.Šiška put the question as to how to characterize the finite oriented graphs thus obtained.

In this article the question is considered under two restrictions. One of them is limiting ourselves so far to the monotone case, in which the borderline never changes direction along the y axis (i.e. when it can be described by a formula $\{(x,y) \mid x=f(y), y \in J\}$ where J is an open interval and f is a continuous function



(a) (b)

Fig.2

tending to $-\infty$ at the one end of J and to $+\infty$ at the other end). This is an essential restriction. In Fig.3 an example of a borderline such that its graph can be obtained from no monotone one is

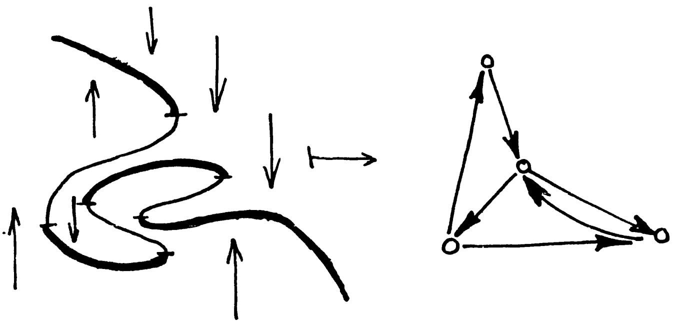


Fig.3

indicated. On the other hand, very often a non-monotone case can be replaced by a monotone one (see Fig.4).

A less important restriction is the second one : We consider the recurrent parts of the graph only (the subgraph consisting of the recurrent states, i.e. those into which one can always return, the transient states being left apart - this terminology is borrowed from Markov processes, see e.g.[2]; in graph terminology, this amounts to restricting ourselves to connected graphs). In the description of real processes, the recurrent states are those which play the main role. Besides, the procedure described is able to handle many transient cases, too; we just have not so far investi-

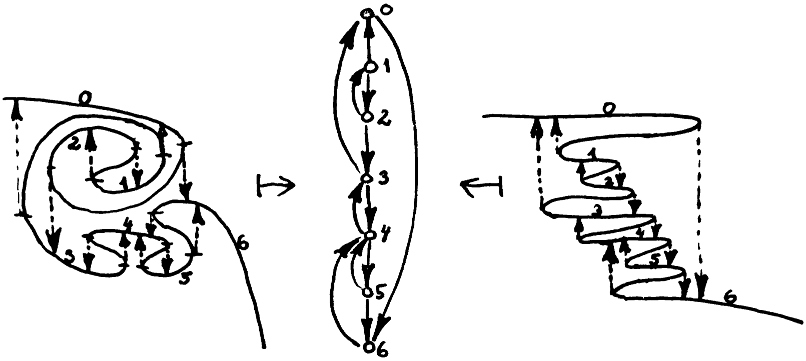


Fig.4

gated the extent of this.

The characterization is not given in a form of a compact system of necessary and sufficient conditions. Instead, we present a procedure which decides in polynomial time whether a given graph belongs to the class or not.

1. H-graphs and their description by means of couples of mappings

1.1. The monotone case can be represented as follows : We are given a system $\mathcal{X}$ of non-void open intervals

$$((a_i,b_i))_{i=0,\dots,n}$$

such that

(i) $a_0 = -\infty$, $b_n = +\infty$ and all the other a_i, b_j are finite,

(ii) $a_i > b_{i+1}$ for all $i < n$.

There is a jump "down" from i to j if j is smallest such that the half-line $\{(b_i, i+t) \mid t \geq 0\}$ meets the segment $(a_j,b_j)\times\{j\}$; similarly, there is a jump "up" from i to j if j is largest such that $\{(a_i,i-t) \mid t \geq 0\}$ meets $(a_j,b_j)\times\{j\}$ (see Fig.5).

1.2. From now on, the motivation being certainly quite lucid, we will stop speaking on borderlines, fields and jumps, and consider the situation in the form indicated in 1.1. Given a system $\mathcal{X} = ((a_i,b_i))_{i=0,\dots,n}$ with the described properties, consider the mappings

$$\varphi_{\mathcal{X}}: \{0,1,\dots,n-1\} \to \{0,1,\dots,n\},$$
$$\psi_{\mathcal{X}}: \{1,2,\dots,n\} \to \{0,1,\dots,n\}$$

defined by

$$\varphi_{\mathcal{X}}(i) = \min\{j \mid j>i,\ a_i<a_j\},$$
$$\psi_{\mathcal{X}}(i) = \max\{j \mid j<i,\ b_i>b_j\}.$$

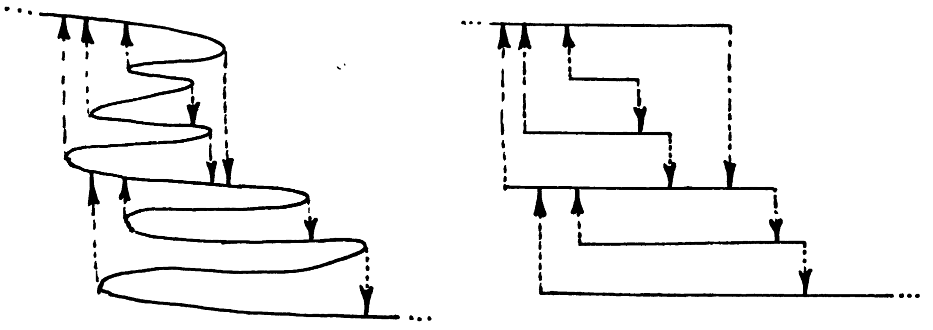


Fig.5

Define an oriented graph $G(\mathcal{X})$ as

$$(\{0,1,\dots,n\}, \varphi_{\mathcal{X}} \cup \psi_{\mathcal{X}}).$$

A graph (X,R) is said to be an <u>H-graph</u> if it is isomorphic with a $G(\mathcal{X})$.

In this section, we will discuss an easy characterization of H-graphs (X,R) using explicitely an ordering on X and the decomposition of R into the two mappings. In the following one we will show that the ordering and the decomposition can be, in essence, reconstructed from R itself.

<u>1.3.</u> Let L, E be linearly ordered sets, let L be finite and 0 resp. T be its first resp. last element. For a mapping (not necessarily monotone)

$$\alpha : L \setminus \{T\} \to E$$

define

$$\tilde{\alpha} : L \setminus \{T\} \to E$$

by putting $\tilde{\alpha}(x) = \min\{y \mid y>x \ \&\ \alpha(y)>\alpha(x)\}$.

<u>1.4.</u> <u>Lemma</u> : (a) $x<\tilde{\alpha}(x)$

(b) $x<y<\tilde{\alpha}(x) \Rightarrow \tilde{\alpha}(y) \leq \tilde{\alpha}(x)$.

<u>Proof</u> : (a) is trivial.

(b) : Let $x<y<\tilde{\alpha}(y)$. If $\tilde{\alpha}(y)>\tilde{\alpha}(x)$, we would have in particular $\alpha(y) \geq \alpha(\tilde{\alpha}(x))$ and consequently $\alpha(y)>\alpha(x)$ in contradiction with the minimality of $\tilde{\alpha}(x)$. □

1.5. Lemma : $\tilde{\tilde{\alpha}} = \tilde{\alpha}$.

Proof : For $x < y < \tilde{\alpha}(x)$ we have $\tilde{\alpha}(y) \leq \tilde{\alpha}(x)$ so that

$$\tilde{\tilde{\alpha}}(x) \geq \tilde{\alpha}(x) .$$

On the other hand, if we have $x < y$ and $\alpha(x) < \alpha(y)$, we have obviously $\tilde{\alpha}(x) \leq y < \tilde{\alpha}(y)$, and hence

$$M_1 = \{y \mid x < y \ \& \ \alpha(x) < \alpha(y)\} \subset \{y \mid x < y \ \& \ \tilde{\alpha}(x) < \tilde{\alpha}(y)\} = M_2 .$$

Thus,

$$\tilde{\tilde{\alpha}}(x) = \min M_2 \leq \min M_1 = \tilde{\alpha}(x) . \ \square$$

1.6. Observation : In the notation of 1.2, if we put $\beta(i) = b_i$, we have $\varphi = \tilde{\beta}$; if we put $\alpha(i) = a_i$ and reverse the ordering of $\{0,1,\dots,n\}$, we obtain $\psi = \tilde{\alpha}$.

1.7. Proposition : A graph (X,R) is an H-graph iff there exists a linear ordering $<$ on X and mappings

$$\varphi : X \setminus \{T\} \to X$$
$$\psi : X \setminus \{0\} \to X$$

(0 resp. T is the first resp. last element in $<$) such that

(1) $\psi(x) < x < \varphi(x)$,

(2) $x < y < \varphi(x) \Rightarrow \varphi(y) \leq \varphi(x)$,

$x > y > \psi(x) \Rightarrow \psi(y) \geq \psi(x)$,

(3) $R = \varphi \cup \psi$.

Proof : Without loss of generality we may assume that $X = \{0,1,\dots,n\}$ and $<$ is the usual ordering of this set. If $R = \varphi \cup \psi$ is the decomposition from the definition, we have (1)-(3) by 1.6 and 1.4.

On the other hand, let us have φ and ψ with the prescribed properties. Put $b_i = \varphi(i)$ for $i < n$, $b_n = +\infty$, $a_0 = -\infty$, $a_i = \psi(i)-n$ for $i > 0$. According to the observation in 1.6 and Lemma 1.5 we see that the mappings described in 1.2 coincide with the original φ and ψ . $\square$

1.8. Remark : In the definition of the intervals (a_i, b_i) in the second part of the preceding proof we have $b_i = b_j$ for $i < j < \varphi(i)$ (and similarly with a_i). If we wish to avoid this, it suffices to modify the construction to $b_i' = b_i - \frac{i}{n}$, $a_i' = a_i - \frac{i}{n}$.

2. Reconstruction of the decomposition $R = \varphi \cup \psi$

2.1. If (X,R) is an H-graph and $(<, \varphi, \psi)$ the data from the construction of R, the same R is obtained from $(>, \psi, \varphi)$. We immediately identify the subset $\{0,T\} \subset X$: these are the only two points with out-degree one.

Thus, the problem of unicity of the data goes as follows : Given an H-graph (X,R) and fixing one of the exceptional points as the first one in the ordering we seek, can we reconstruct the data $<$, φ and ψ ? We are going to show that for connected graphs the answer is positive.

In the sequel, (X,R) is an H-graph and the notation $<$, φ, ψ, 0 and T is as in 1.2 and 1.3.

2.2. Lemma : Let $0=x_0Rx_1R\dots Rx_r=T$ be a squence of distinct elements. Then $x_i\varphi x_{i+1}$ for all $i=0,1,\dots,r-1$.

Proof : Obviously $x_0\varphi x_1$. Let i be the first index such that $x_i\psi x_{i+1}$. Since the x_j are distinct, we have $x_j<x_{i+1}<x_{j+1}$ for some $j<i$. We will show that then, however, $x_{i+k}<x_i$ for all $k\geq 1$ (and hence we cannot have $x_r=T$). The statement already holds for $k=1$. Suppose we know that $x_{i+k}<x_i$. Then $x_j<x_{i+k}<x_{j+1}$ for some $j<i$. Thus, $x_j<x_{i+k}<\varphi x_j$, hence $x_{i+k+1}\leq\varphi x_{i+k}\leq\varphi x_j=x_{j+1}$. Since $x_{i+k+1}\neq x_{j+1}$, we conclude that $x_{i+k+1}<x_{j+1}\leq x_i$. $\square$

2.3. Lemma : Let $x_1\varphi x_2Rx_3R\dots Rx_r$ be a sequence of distinct elements. Let $x_1=\psi y$ and $x_r=\varphi^p y$ $(p\geq 0)$ and let for all $i<r$ and $j\geq 0$ hold $x_i\neq\varphi^j y$. Then $x_{r-1}<y$ and

$$x_i\varphi x_{i+1} \quad \text{for all } i=1,\dots,r-1 .$$

Similarly, if $x_1\psi x_2R\dots Rx_r$ is a sequence of distinct elements, $x_1=\varphi(y)$, $x_r=\psi^p y$ $(p\geq 0)$ and $x_i\neq\psi^j y$ for all $i<r$ and $j\geq 0$, then $x_{r-1}>y$ and

$$x_i\psi x_{i+1} \quad \text{for all } i=1,\dots,r-1 .$$

Proof : First, we will show that

(*) if $x_1\varphi x_2\varphi\dots\varphi x_i$ for some $i<r$, we have $x_i<y$.

Indeed, we have $x_1<y$; let us know for some $k<i$ that $x_k<y$. If we had $x_{k+1}=\varphi x_k>y$, there would have been $z=\varphi^j y<x_{k+1}<\varphi^{j+1}y$ for some $j\geq 0$ (since x_{k+1} is still unequal to a $\varphi^j y$). Then, however, $x_k<z<\varphi x_k<\varphi z$ contradicting the property of φ. Thus, $x_{k+1}\leq y$ and the equality is not yet possible, hence $x_{k+1}<y$.

Now, let there be an i such that $x_i\psi x_{i+1}$, take the first of such indices. By (*) we have $x_i<y$. We will show that then $x_{i+k}<y$ for all k, in contradiction with $x_r=\varphi^p y\geq y$.

We have $x_{i+1}=\psi x_i<x_i<y$. Moreover, since

$$\psi y=x_1<x_i<y$$

(here we use the fact that $x_1\varphi x_2$), we have

$$x_1<\psi x_i=x_{i+1}<x_i$$

and hence there is a $j<i$ such that

$$x_j<x_{i+1}<x_{j+1} .$$

Thus, $x_{i+2}\leq\varphi x_{i+1}\leq x_{j+1}$ and $x_{i+2}\neq x_{j+1}$ by the assumption of distinctness. If $x_{i+2}=\varphi x_{i+1}$, the inequality $x_{i+2}>x_1$ is obvious; if $x_{i+2}=\psi x_{i+1}$ we conclude $x_{i+2}>x_1$ from $x_1=\psi y<x_{i+1}<y$

(and the distinctness of the elements).

Let us have already proved that there is a $j<i$ such that $x_j<x_{i+k}<x_{j+1}$. Then $\varphi x_{i+k}\leq \varphi x_j = x_{j+1}$ and $x_{i+k+1}\neq x_{j+1}$ so that $x_{i+k+1}<x_{j+1}$; if $x_{i+k+1}=\psi x_{i+k}$ we see that $x_1=\psi y\leq x_{i+k+1}$ from $\psi y<x_{i+k}<y$. $\square$

2.4. Theorem : Let (X,R) be a connected H-graph, 0 one of the two points with out-degree 1. Then there exists exactly one ordering $<$ on X and exactly one couple of mappings

$$\varphi: X\setminus\{0\}\to X\ ,\quad \psi: X\setminus\{T\}\to X$$

such that

(1) $\forall x\quad \psi(x)<x<\varphi(x)$,

(2) $x<y<\varphi(x)\Rightarrow \varphi(y)\leq\varphi(x)$,

$\psi(x)<y<x\Rightarrow \psi(x)\leq\psi(y)$,

(3) 0 is the first element in $(X,<)$,

(4) $R=\varphi\cup\psi$.

Proof : Put

$A_0 = \{\varphi^k 0\mid k\geq 0,\ \varphi^k 0 \text{ defined}\}$,

for $i\geq 0$, $A_{2i+1} = \{\psi^k x\mid x\in A_{2i}\ ,\ k\geq 0\ ,\ \psi^k x \text{ defined}\}$,

for $i>0$, $A_{2i} = \{\varphi^k x\mid x\in A_{2i+1}\ ,\ k\geq 0\ ,\ \varphi^k x \text{ defined}\}$.

The proof will be done by subsequent identifying and placing the points of A_i on the basis of the knowledge on the sets A_j for $j<i$.

First of all we deal with the points of A_0. According to Lemma 2.2 it suffices to find an unrepetitive squence $0Rx_1R\ldots Rx_{r-1}RT$.

The steps from 2i to 2i+1 and from 2i-1 to 2i are quite analogous (differring in the interchange of φ and ψ only). Thus, we will discuss just the former one.

Describe A_{2i} as $\{y_0,y_1,\ldots,y_r\}$ with

$$0=y_0<y_1<\ldots<y_r=T\ .$$

Consider those y_j which were not yet in A_{2i-1} , denote them by

$$y_{t_0}<y_{t_1}<\ldots<y_{t_m}\ .$$

Of the points of A_{2i+1}, it suffices to identify and place the $\psi^k y_{t_j}$ since the other $\psi^k y_j$ have been already dealt with.

Each y_{t_j} has to have the form φy_s for a y_s (otherwise it would not be new in A_{2i}). We have

$$y_s\leq y_{t_j-1}<y_{t_j}=\varphi(y_s)$$

which immediately yields

$$(*)\qquad y_{t_j}=\varphi y_{t_j-1}\ .$$

Now, we will identify and place the $\psi^k y_j$ inductively by j. For

$j=0$ there is $0 = \psi^0 0$ only. Let $\psi^k y_j$ be dealt with for $j<q$. We can assume $q=t_p$ for some p. By $(*)$ we have $y_q = \varphi y_{q-1}$.

Since y_q is in A_{2i} we already know the φy_q. This identifies also the ψy_q, namely as the target of the remaining arrow starting in y_q. Find an unrepetitive sequence

$$y_q = x_0 \psi x_1 R x_2 \dots x_{r-1} R x_r$$

such that $x_r = \psi^s y_{q-1}$ and $x_j \neq \psi^a y_{q-1}$ for all $j<r$ and $a \geq 1$. Such a sequence necessarily exists since for some $a,b \geq 0$ one has $\psi^a y_q = 0 = \psi^b y_{q-1}$. By 2.3 we see that $x_j = \psi^j y_q$ for $j \leq r$ and

$$y_{q-1} < x_{r-1} < x_{r-2} < \dots < x_1 < y_q .$$

The values $\psi^{r+j} y_q$ are the already identified $\psi^{s+j} y_{q-1}$. □

2.5. Conclusions : The time required for finding a path between given two points in a connected graph (X,R) with bounded out-degrees expands only linearly with the expanding X (see, e.g., [1]). Thus, the described procedure does not need more than quadratically many steps (in fact, less, since the longer the search for a concrete path has been, the more points are dealt with at the stage).

Now, given an (X,R) we can decide whether it is an H-graph simply by running on it the procedure from the proof of 2.4 (having previously checked whether the out-degrees are always two with exactly two exceptions of points with out-degree one). It is an H-graph iff the procedure works.

3. A remark on the general case

Let us for a moment return to the general (not necessarily monotone) case. A graph (X,R) of state transitions obviously satisfies the condition

(1) two of the nodes have out-degree one, all the other ones have out-degree two.

This necessary condition is obviously not sufficient. One can see easily that such a graph satisfies, moreover,

(2) (X,R) can be drawn in the plane (without intersecting edges) so that the two exceptional points can be connected with the infinity.

I do not know an example of a graph satisfying (1)&(2) which is not a graph of state transitions.

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