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§ 0 Introduction

In 1950 Erdös and Rado proved the following result, known as 'Erdös-Rado-canonization-lemma':

Theorem [1] Let k,m be positive integers. Then there exists a positive integer n such that for every coloring $\Delta : [n]^k \to \omega$ of the k-element subsets of an n-element set with infinitely many colors there exists an m-element set $X \in [n]^m$ and there exists a $0-1$ sequence $I = (i_0,\ldots,i_{k-1}) \in 2^k$ such that two k-element subsets $A = \{a_0,\ldots,a_{k-1}\}_<$ and $B = \{b_0,\ldots,b_{k-1}\}_<$ of X are colored the same iff $a_\nu = b_\nu$ for every $\nu < k$ with $i_\nu = 1$.

Informally this means that A and B are colored the same iff they agree on the subsets given by the sequence I .
Obviously none of the 2^k many equivalence relations given by $0-1$ sequences $I \in 2^k$ may be omitted without violating the statement of the theorem. Thus, for fixed $I \in 2^k$, the subset $A \cdot I$ of A given by the sequence I is a characteristic data for A . Two k-element subsets A,B of X are colored the same iff they have the same characteristic data.
The Erdös-Rado-canonization-lemma shows that the only characteristic data (in this sense) are given by subsets. This generalizes the well-known theorem of Ramsey, which states that with respect to two-colorings necessarily $I = \emptyset$. In this paper we investigate analogous questions for the class of finite distributive lattices, thus generalizing the corresponding partition results, see [3] for an account on recent partition results for some classes of lattices.

As by the Stone representation theorem each distributive lattice may be embedded into a Boolean algebra (i.e. power-set lattice) it is convenient to consider first canonization results for Boolean algebras. Such a theorem has been proven in [4].
This paper is organized as follows:
In section 1 we show how Boolean algebras and subalgebras may be represented using certain 0-1 matrices. This representation is used in section 2 in order to state a canonization lemma for Boolean algebras. In section 3 this result is generalized to arbitrary finite distributive lattices which is the main result of this paper. The main theorem then is proved in section 4.

1. How to represent Boolean algebras

A $P(k)$ - subalgebra $\mathfrak{A}$ of $P(m)$ may be given e.g. by k mutually distinct nonempty sets $A_1^*,\ldots,A_k^*$ having pairwise the same intersection, i.e. $A_i^* \cap A_j^* = A_1^* \cap A_2^*$ for every $1 \leq i < j \leq k$. The sets $A_1^*,\ldots,A_k^*$ form the atoms while their common intersection $A_0 = A_1^* \cap A_2^* = A_1^* \cap\ldots\cap A_k^*$ is the minimum. More appropriate for our purposes is to represent $\mathfrak{A}$ by its minimal element A_0 and the 'directions' $A_1^*\setminus A_0,\ldots,A_k^*\setminus A_0$. Thus $\mathfrak{A}$ is uniquely determined from

(1.1) $(A_0,A_1,\ldots,A_k)$, where $A_i \cap A_j = \emptyset$ for every $0 \leq i < j \leq k$ and $A_1,\ldots,A_k$ are nonempty and $\min A_1 < \min A_2 < \ldots < \min A_k$.

The intended interpretation is that $A_i^* = A_0 \cup A_i$, $1 \leq i \leq k$, are the atoms of $\mathfrak{A}$. Also, because of the ascending minima condition, to each $P(k)$ - sublattice $\mathfrak{A}$ belongs precisely one $(k+1)$ - tuple $(A_0,\ldots,A_k)$ satisfying (1.1) . The tuple $(A_0,\ldots,A_k)$ can be represented by a $m\times(k+1)$ matrix with 0-1 entries, where the i.th column, $0 \leq i \leq k$, contains the characteristic function of A_i .

(1.2) Notation: "$\mathit{0}$" denotes the one-way infinite vector consisting of zero entries only, i.e. $\mathit{0} = (0,0,0,\ldots)$. For nonnegative integers i the expression '$e(i)$' denotes the one-way infinite vector with all entries zero except for the i.th entry, which is one, e.g.

$e(0) = (1,0,0,\ldots)$, $e(1) = (0,1,0,\ldots)$.

For technical reasons we first consider 'homogeneous' subalgebras, i.e. $\mathcal{P}(k)$ - subalgebras with mutually disjoint atoms:

(1.3) <u>Definition:</u> For nonnegative integers $k \leq m$ let $\mathbb{B}_0\binom{m}{k}$ consist of all mappings $A : m \to \{\mathcal{O}, e(0), \ldots, e(k-1)\}$ satisfying:

(1.3.1) for every $j < k$ there exists an $i < m$ such that $A(i) = e(j)$,

(1.3.2) $\mu^A(i) < \mu^A(j)$ for every $i < j < k$, where $\mu^A(i) = \min A^{-1}(e(i))$.

<u>Remark:</u> $\mathbb{B}_0\binom{m}{k}$ may be interpreted as the set of $m \times k$ matrices with zero-one entries satisfying:

- each row contains at most one non-zero entry,
- each column contains at least one non-zero entry,
- the columns are ordered according to the first occurences of 1 .

Namely $A \in \mathbb{B}_0\binom{m}{k}$ is the matrix consisting of rows $A(0),\ldots,A(m-1)$.

Using the usual multiplication of matrices a composition $\mathbb{B}_0\binom{n}{m} \times \mathbb{B}_0\binom{m}{k} \to \mathbb{B}_0\binom{n}{k}$ is defined by

(1.4)
$$(A \cdot B)(i) = \mathcal{O} \quad \text{if } A(i) = \mathcal{O}$$
$$= B(j) \quad \text{if } A(i) = e(j) \quad ,$$

where $A \in \mathbb{B}_0\binom{n}{m}$ and $B \in \mathbb{B}_0\binom{m}{k}$.

(1.5) <u>Definition:</u> $\mathbb{B}_1\binom{m}{k} = \{A \in \mathbb{B}_0\binom{1+m}{1+k} \mid A(0) = e(0)\}$.

One easily observes that $\mathbb{B}_1$ is closed under the composition defined in (1.4) , i.e. $A \in \mathbb{B}_1\binom{n}{m}$ and $B \in \mathbb{B}_1\binom{m}{k}$ imply that $A \cdot B \in \mathbb{B}_1\binom{n}{k}$.

(1.6) <u>Example:</u> Consider $A \in \mathbb{B}_1\binom{3}{2}$ which is given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} .$$

Interpret the i.th column of A , $i < 3$, as the characteristic function of a set $A_i \subseteq \{0,1,2\}$, but ignore the first row of A , i.e.

$$A_0 = \emptyset \quad , \quad A_1 = \{0,2\} \ , \ A_2 = \{1\} \quad .$$

According to (1.1) the three-tuple (A_0,A_1,A_2) , and thus the matrix A from which this is derived, determines a $P(2)$ - subalgebra of $P(3)$.

Following the pattern of example (1.6) one immediately observes that each $A \in \mathbb{B}_1\binom{m}{k}$ determines a $P(k)$ - subalgebra $\mathfrak{A}$ of $P(m)$ and vice versa to each $P(k)$ - subalgebra $\mathfrak{A}$ of $P(m)$ there corresponds precisely one such $A \in \mathbb{B}_1\binom{m}{k}$. Moreover for $A \in \mathbb{B}_1\binom{n}{m}$ and $B \in \mathbb{B}_1\binom{m}{k}$ the composite $A \cdot B \in \mathbb{B}_1\binom{n}{k}$ yields the $P(k)$ - subalgebra $\mathfrak{B}$ in the $P(m)$ - subalgebra $\mathfrak{A}$ in a $P(n)$ - algebra.

Let us mention that the following partition theorem for finite Boolean algebras has been established by Graham and Rothschild:

(1.7) <u>Theorem [2]</u> Let $k \leq m$ be nonnegative integers. Then there exists a positive integer n such that for every coloring $\Delta : \mathbb{B}_1\binom{n}{k} \to 2$ of the $P(k)$ - subalgebras of a $P(n)$ - algebra with colors 0 and 1 there exists a $P(m)$ - subalgebra $A \in \mathbb{B}_1\binom{n}{m}$ with all its $P(k)$ - subalgebras in the same color, i.e. $\Delta(A \cdot B) = \Delta(A \cdot C)$ for all $B,C \in \mathbb{B}_1\binom{m}{k}$.

2. Canonical equivalence relations for $\mathbb{B}_1\binom{m}{k}$

In this section we describe the canonical equivalence relations in $\mathbb{B}_1\binom{m}{k}$.

(2.1) <u>Definition:</u> Let k be a nonnegative integer. A family $(\xi_i,\rho_i,F_i)_{i\leq\ell}$ is a "k-canonical family" iff

(2.1.1) $\ell \leq k$ is a nonnegative integer,

(2.1.2) $1 \leq \xi_0 < \xi_1 < \ldots < \xi_{\ell-1} < \xi_\ell = 1+k$ are positive integers,

(2.1.3) $\rho_i \leq \xi_i$ are nonnegative integers, (2.1.4) $F_i \in \mathbb{B}_0\binom{\xi_i}{\rho_i}$,

(2.1.5) for every $G \in \mathbb{B}_0\binom{\rho_{i+1}}{1}$ there exists $H \in \mathbb{B}_0\binom{\rho_i}{1}$ such that $(F_{i+1} \cdot G)(\xi) = (F_i \cdot H)(\xi)$ for every $\xi < \rho_i$, where $i < \ell$.

(2.1.6) $F_{i+1}(\xi_i) \neq e(\rho_i)$ for every $i < \ell$.

(2.2) Example: (i) there exist precisely two 0-canonical families, viz. $(1,0,(\mathcal{O}))$ and $(1,1,(e(0)))$.

(ii) there exist precisely 10 1-canonical families; viz. $(2,2,(e(0),e(1)))$; $(2,1,(\mathcal{O},e(0)))$, $(2,1,(e(0),\mathcal{O}))$, $(2,1,(e(0),e(0)))$, $(2,0,(\mathcal{O},\mathcal{O}))$,
these being the 5 1-canonical families with $\ell = 0$, and
$((1,0,(\mathcal{O})) , (2,0,(\mathcal{O},\mathcal{O})))$, $((1,1,(e(0))) , (2,1,(e(0),\mathcal{O})))$,
$((1,1,(e(0))) , (2,1,(e(0),e(0))))$, $((1,1,(e(0))) , (2,1,(\mathcal{O},e(0))))$,
$((1,1,(e(0))) , (2,0,(\mathcal{O},\mathcal{O})))$,
these being the 5 1-canonical families with $\ell = 1$.

(2.3) Notation: Let $A \in \mathbb{B}_1\binom{m}{k}$ and ξ with $1 \leq \xi \leq k+1$ be a positive integer. Then $A^\xi \in \mathbb{B}_1\binom{\mu^A(\xi)}{\xi-1}$, where $\mu^A(k+1) = m$, is the restriction of A to $\mu^A(\xi)$, i.e. $A^\xi(i) = A(i)$ for every $i < \mu^A(\xi)$.

(2.4) Theorem [4] Let $k \leq m$ be nonnegative integers. Then there exists a nonnegative integer n such that for every coloring $\Delta : \mathbb{B}_1\binom{n}{k} \to \omega$ of the $P(k)$ - subalgebras of a $P(n)$ - algebra with an arbitrary number of colors there exists a $P(m)$ - subalgebra $A \in \mathbb{B}_1\binom{n}{m}$ and a k-canonical family $(\xi_i,\rho_i,F_i)_{i \leq \ell}$ such that two $P(k)$ - subalgebras $B,C \in \mathbb{B}_1\binom{m}{k}$ of A are colored the same (i.e. $\Delta(A \cdot B) = \Delta(A \cdot C)$) iff $B^{\xi_i} \cdot F_i = C^{\xi_i} \cdot F_i$ for every $i \leq \ell$.

This result is best possible, as the partitions given by k-canonical families are hereditary under subobjects, viz.

(2.5) Theorem [4] Let k be a nonnegative integer and let $(\xi_i,\rho_i,F_i)_{i \leq \ell}$ be a k-canonical family.

Let $\Delta : \mathbb{B}_1\binom{n}{k} \to \omega$ be a coloring such that $\Delta(B) = \Delta(C)$ iff $B^{\xi_i} \cdot F_i = C^{\xi_i} \cdot F_i$ for every $i \leq \ell$.
Then for every $A \in \mathbb{B}_1\binom{n}{m}$ and $B,C \in \mathbb{B}_1\binom{m}{k}$ it follows that $\Delta(A \cdot B) = \Delta(A \cdot C)$ iff $B^{\xi_i} \cdot F_i = C^{\xi_i} \cdot F_i$ for every $i \leq \ell$.

3. Canonical equivalence relations for finite distributive lattices

(3.1) Notation: $\mathbb{D}$ denotes the class of finite distributive lattices. The elements of $\mathbb{D}$ are denoted by capital letters $A,B,C,\ldots$. The expression "$\mathbb{D}\binom{A}{B}$" denotes the set of B-sublattices of A. In particular if $A \cong P(m)$ and $B \cong P(k)$ we use the representation from section 2 and by abuse of language $\mathbb{D}\binom{A}{B} = \mathbb{B}_1\binom{m}{k}$.

The following well-known observations enable us to determine canonical equivalence relations for finite distributive lattices.

(3.2) Observation: For every $M \in \mathbb{D}$ there exists a nonnegative integer n such that M may be embedded into $P(n)$, i.e. $\mathbb{D}\binom{P(n)}{M} \neq \emptyset$.

The smallest such n is called the "rank of M" and is abbreviated as $\text{rk}\, M$, also $\text{rk}\, M$ is the length of a maximal chain in M.

(3.3) Observation: Let $\hat{M} \in \mathbb{D}\binom{P(n)}{M}$ be an M-sublattice of $P(n)$. Then there exists precisely one $P(\text{rk}\, M)$-sublattice $A \in \mathbb{D}\binom{P(n)}{P(\text{rk}\, M)}$ containing $\hat{M}$.

Let us denote this $P(\text{rk}\, M)$-subalgebra, which envelops $\hat{M}$, by $\text{Env}\, \hat{M}$.

The last observation makes it possible to associate a certain number, viz. $\text{typ}\, \hat{M}$, to each M-sublattice $\hat{M} \in \mathbb{D}\binom{P(n)}{M}$.
Consider $\hat{M} \in \mathbb{D}\binom{\text{Env}\, \hat{M}}{M}$. Of course, $\hat{M}$ determines a subset of $\text{Env}\, \hat{M}$. Using e.g. the lexicographic ordering yields a total ordering on $\mathbb{D}\binom{\text{Env}\, \hat{M}}{M}$, say $\mathbb{D}\binom{\text{Env}\, \hat{M}}{M} = \{\hat{M}_0,\ldots,\hat{M}_{x-1}\}$, where the M-sublattices are enumerated monotonously.

(3.4) <u>Notation:</u> typ $\hat{M} = \nu$ iff $\hat{M} = \hat{M}_\nu$.

(3.5) <u>Example:</u> Let M be the three-element chain. Then rk $M = 2$. Consider a three-element chain $X \subseteq Y \subseteq Z$ in $P(n)$. Then Env $(X \subseteq Y \subseteq Z)$ has atoms Y and $X \cup (Z \setminus Y)$ (see diagram 1) . Thus typ $(X \subseteq Y \subseteq Z) = 0$ iff $\min Y \setminus X < \min Z \setminus Y$ and typ $(X \subseteq Y \subseteq Z) = 1$ otherwise.

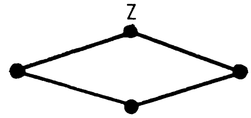

diagram 1

These observations can be used in order to show that finite Boolean algebras are the only finite distributive lattices which have the partition property, see [3].

In order to give a precise formulation of the canonical partition theorem for finite distributive lattices let us adopt the following convention:

(3.6) <u>Convention:</u> Let $A \in \mathbb{B}_1\binom{n}{m}$ be a $P(m)$ - sublattice of $P(n)$ and let $\hat{M} \in \mathbb{D}\binom{P(m)}{M}$ be an M - sublattice of $P(m)$, then $A \cdot \hat{M} \in \binom{P(n)}{M}$ denotes the corresponding M - sublattice of A .

Now the main result of this paper can be stated in the following way:

(3.7) <u>Theorem:</u> Let $M \in \mathbb{D}$ be a finite distributive lattice and say $|\mathbb{D}\binom{P(\text{rk } M)}{M}| = \chi$. Then for every integer m there exists a positive integer n such that for every coloring $\Delta : \mathbb{D}\binom{P(n)}{M} \to \omega$ of the M - sublattices of $P(n)$ with arbitrary many colors there exists a $P(m)$ - subalgebra $A \in \mathbb{B}_1\binom{n}{m}$, there exists an equivalence relation π on $\{0,\ldots,\chi-1\}$ and for each $\nu < \chi$ there exists a (rk M) - canonical family $(\xi_i^\nu, \rho_i^\nu, F_i^\nu)_{i \leq \ell^\nu}$ such that each two M - sublattices $\hat{M}, \hat{M}' \in \mathbb{D}\binom{P(m)}{M}$ of A are colored the same (i.e. $\Delta(A \cdot \hat{M}) = \Delta(A \cdot \hat{M}')$) iff

$\alpha =$ typ $\hat{M}$ and $\beta =$ typ $\hat{M}'$ satisfy

$\alpha \approx \beta \pmod{\pi}$ and

$$((\mathrm{Env}\ \hat{M})^{\xi_i^\alpha} \cdot F_i^\alpha)_{i\leq \ell^\alpha} = ((\mathrm{Env}\ \hat{M}')^{\xi_i^\beta} \cdot F_i^\beta)_{i\leq \ell^\beta} \quad .$$

Informally this result may be stated in the following way:

If M is a Boolean algebra, then the canonical equivalence relations are given by k-canonical families $(\xi_i,\rho_i,F_i)_{i\leq \ell}$ as stated in theorem 2.5 .

Here the simplest case are k-canonical families (ξ_0,ρ_0,F_0) , i.e. $\ell = 0$. Recall that $\xi_0 = 1+k$ by (2.1.2) . Then two $P(k)$ - subalgebras B and C of a $P(m)$, i.e. $B,C \in \mathbb{B}_1\binom{m}{k}$ are equivalent iff $B \cdot F_0 = C \cdot F_0$. But as $F_0 \in \mathbb{B}_0\binom{1+k}{\rho_0}$ this means that B and C are equivalent iff they have the same (homogenous) F_0 - subalgebra, where also B and C are interpreted as homogeneous $P(1+k)$ - subalgebras of $P(1+m)$, i.e. $B,C \in \mathbb{B}_0\binom{1+m}{1+k}$. Compare (1.5) and the example (1.6).

The next simplest case is represented by k-canonical families $(\xi_i,\rho_i,F_i)_{i\leq 1}$, i.e. $\ell = 1$. A necessary condition for $B,C \in \mathbb{B}_1\binom{m}{k}$ to be equivalent then is that $\mu^B(\xi_0) = \mu^C(\xi_0) = \nu$, recall that again $\xi_1 = 1+k$. By definition (1.3) the first ν rows of B and C , i.e.

$$B^{\xi_0} = (B(0),\ldots,B(\nu-1)) \quad \text{and}$$
$$C^{\xi_0} = (C(0),\ldots,C(\nu-1))$$

represent $P(\xi_0-1)$ - subalgebras of $P(\nu)$, viz. B^{ξ_0} , $C^{\xi_0} \in B_0\binom{1+\nu}{\xi_0}$

The next necessary condition for $B,C \in \mathbb{B}_1\binom{m}{k}$ to be equivalent modulo $(\xi_i,\rho_i,F_i)_{i\leq 1}$ then is that B^{ξ_0} and C^{ξ_0} are equivalent modulo (ξ_0,ρ_0,F_0) , viz. they have to have the same F_0 - subalgebra. Finally the third necessary condition is that B and C have the same F_1 - subalgebra.

All these three necessary conditions put together yield a sufficient condition for the equivalence modulo $(\xi_i,\rho_i,F_i)_{i\leq 1}$.

Observe that the subspaces F_0 and F_1 are linked by (2.1.5) and (2.1.6) . Generally speaking $B,C \in \mathbb{B}_1\binom{m}{k}$ are equivalent modulo $(\xi_i,\rho_i,F_i)_{i\leq \ell}$ iff $\mu^A(\xi_i) = \mu^B(\xi_i)$ for every $i \leq \ell$ and the initial rows B^{ξ_i} resp. C^{ξ_i} - interpreted as elements of $\mathbb{B}_0\binom{1+\mu^B(\xi_i)}{\xi_i}$ - have the same F_i - subalgebras.

Thus the sequence $(B^{\xi_i} \cdot F_i)_{i\leq \ell}$ of these subalgebras gives the characteristic data of B with respect to $(\xi_i,\rho_i,F_i)_{i\leq \ell}$ and two $P(k)$ - subalgebras B and C of $P(m)$ are equivalent iff they share the same characteristic data.

If the distributive lattice M is not a Boolean algebra, say $\binom{P(\text{rk } M)}{M} = \{M_0,\ldots,M_{\chi-1}\}$ where $\chi > 1$, then by observation (3.3) and theorem (2.5) to each type $\nu < \chi$ there belongs a certain k-canonical family $(\xi_i^\nu,\rho_i^\nu,F_i^\nu)_{i\leq \ell^\nu}$.
Now two M - sublattices of the same type ν are colored the same iff they share the same characteristic data.

But what happens with M - sublattices $\hat{M}$ and $\hat{M}'$ of different type? Note that even if the k-canonical families associated with typ $\hat{M}$ resp. with typ $\hat{M}'$ are different, the characteristic data of $\hat{M}$ and $\hat{M}'$ can be the same. Thus we can color $\hat{M}$ and $\hat{M}'$ with the same color iff they have the same characteristic data, but of course we need not. The theorem states that precisely one of these two possibilities occurs: either $\hat{M}$ and $\hat{M}'$ are colored the same iff they have the same characteristic data (i.e. typ $\hat{M} \approx$ typ $\hat{M}'$ (mod π)) or $\hat{M}$ and $\hat{M}'$ are colored differently in spite of the fact that they could have the same characteristic data (i.e. typ $\hat{M} \not\approx$ typ $\hat{M}'$ (mod π)) .

Finally, from the preceding remarks it should be obvious, that none of the equivalence relations mentioned in theorem (3.7) may be omitted without violating the assertion of (3.7) .

4. Proof of theorem (3.7)

For the remainder of this section let $M \in \mathbb{D}$ be a fixed distributive lattice. Let $\chi = |\mathbb{D}\binom{P(k)}{M}|$ be the number of M - sublattices of $P(k)$, say $\mathbb{D}\binom{P(k)}{M} = \{\hat{M}_0,\ldots,\hat{M}_{\chi-1}\}$, where $k = \text{rk } M$.

(4.1) Lemma: Let $\nu < \chi$. For every m there exists an n such that for every coloring $\Delta : \mathbb{D}\binom{P(n)}{M} \to \omega$ there exists a $P(m)$ - subalgebra $A \in \mathbb{B}_1\binom{n}{m}$

and there exists a $(\mathrm{rk}\ M)$ - canonical family $(\xi_i, \rho_i, F_i)_{i \leq \ell}$ such that each two M - sublattices $\hat{M}, \hat{M}' \in \mathbb{D}\binom{P(m)}{M}$ of type ν are colored the same (i.e. $\Delta(A \cdot \hat{M}) = \Delta(A \cdot \hat{M}'))$ iff

$$(\mathrm{Env}\ \hat{M})^{\xi_i} \cdot F_i = (\mathrm{Env}\ \hat{M}')^{\xi_i} \cdot F_i \text{ for every } i \leq \ell .$$

<u>Proof:</u> This is a straightforward application of theorem (2.4) . Choose n according to $k = \mathrm{rk}\ M$ and m . Given the coloring $\Delta : \mathbb{D}\binom{P(n)}{M} \to \omega$ consider the coloring $\Delta^* : \mathbb{B}_1\binom{n}{\mathrm{rk}\ M} \to \omega$ which is defined as $\Delta^*(A) = \Delta(A \cdot \hat{M}_\nu)$. □

Applying Lemma (4.1) for every $i < \chi$ yields the following corollary:

(4.2) <u>Corollary:</u> For every m there exists an n such that for every coloring $\Delta : \mathbb{D}\binom{P(n)}{M} \to \omega$ there exists a $P(m)$ - subalgebra $A \in \mathbb{B}_1\binom{n}{m}$ and for every $\nu < \chi$ there exists a $(\mathrm{rk}\ M)$ - canonical family $(\xi_i^\nu, \rho_i^\nu, F_i^\nu)_{i \leq \ell^\nu}$ such that each two M - sublattices $\hat{M}, \hat{M}' \in \mathbb{D}\binom{P(m)}{M}$ of type ν are colored the same (i.e. $\Delta(A \cdot \hat{M}) = \Delta(A \cdot \hat{M}'))$ iff

$$(\mathrm{Env}\ \hat{M})^{\xi_i^\nu} \cdot F_i^\nu = (\mathrm{Env}\ \hat{M}')^{\xi_i^\nu} \cdot F_i^\nu \text{ for every } i \leq \ell^\nu .$$

Let $\Delta : \mathbb{B}_1\binom{n}{k} \to \omega$ be a coloring and let $(\xi_i, \rho_i, F_i)_{i \leq \ell}$ be a k-canonical family. We say that Δ is of fibre-type $(\xi_i, \rho_i, F_i)_{i \leq \ell}$ provided that each two $P(k)$ - subalgebras $B, C \in \mathbb{B}_1\binom{n}{k}$ are colored the same iff $B^{\xi_i} \cdot F_i = C^{\xi_i} \cdot F_i$ for every $i \leq \ell$.

(4.3) <u>Lemma:</u> Let $(\xi_i, \rho_i, F_i)_{i \leq \ell}$ and $(\hat{\xi}_i, \hat{\rho}_i, \hat{F}_i)_{i \leq \hat{\ell}}$ be two k-canonical families, and let $m > 2k$ be a positive integer. Then there exists a positive integer n such that for every two colorings

$$\Delta_1 : \mathbb{B}_1\binom{n}{k} \to \omega \quad \text{of fibre-type } (\xi_i, \rho_i, F_i)_{i \leq \ell}$$

and

$$\Delta_2 : \mathbb{B}_1\binom{n}{k} \to \omega \quad \text{of fibre-type } (\hat{\xi}_i, \hat{\rho}_i, \hat{F}_i)_{i \leq \hat{\ell}}$$

there exists an $A \in \mathbb{B}_1\binom{n}{m}$ such that

$H = H_M = \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid \Delta_1(AMB) = \Delta_2(AMC)\}$

is independent of M , i.e. $H_M = H_{\hat{M}}$ for every $M, \hat{M} \in \mathbb{B}_1\binom{m}{2k}$.

Additionally H satisfies:

Either $H = \emptyset$ or

$H = \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid (B^{\xi_i} \cdot F_i)_{i \leq \ell} = (C^{\hat{\xi}_i} \cdot \hat{F}_i)_{i \leq \hat{\ell}}\}$.

Proof: Applying (1.7) we may restrict our considerations to colorings

$$\Delta_1 : \mathbb{B}_1\binom{m}{k} \to \omega \quad \text{of fibre-type } (\xi_i, \rho_i, F_i)_{i \leq \ell}$$

and

$$\Delta_2 : \mathbb{B}_1\binom{m}{k} \to \omega \quad \text{of fibre-type } (\hat{\xi}_i, \hat{\rho}_i, \hat{F}_i)_{i \leq \hat{\ell}}$$

such that

(4.3.1) $H = H_M = H_{\hat{M}}$ for every $M, \hat{M} \in \mathbb{B}_1\binom{m}{2k}$,
where $H_M = \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid \Delta_1(M \cdot B) = \Delta_2(M \cdot C)\}$

and such that either

(4.3.2) for every $B \in \mathbb{B}_1\binom{2k}{k}$ there exists a $C \in \mathbb{B}_1\binom{2k}{k}$ with $(B,C) \in H$

or

(4.3.3) for every $B \in \mathbb{B}_1\binom{2k}{k}$ holds $(B,C) \notin H$ for every $C \in \mathbb{B}_1\binom{2k}{k}$.

If (4.3.3) is valid then obviously $H = \emptyset$. Thus let us assume that (4.3.1) and (4.3.2) are valid. First we show that

$$H \subseteq \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid (B^{\xi_i} \cdot F_i)_{i \leq \ell} = (C^{\hat{\xi}_i} \cdot \hat{F}_i)_{i \leq \hat{\ell}}\} .$$

Assume to the contrary that

(4.3.4) $(B^{\xi_i} \cdot F_i)_{i \leq \ell} \neq (C^{\hat{\xi}_i} \cdot \hat{F}_i)_{i \leq \hat{\ell}}$ for some $(B,C) \in H$.

Let $i \leq \min(\ell, \hat{\ell})$ be maximal such that $B^{\xi_\nu} \cdot F_\nu = C^{\hat{\xi}_\nu} \cdot \hat{F}_\nu$ for every $\nu < i$. Say that $\mu^B(\xi_i) \leq \mu^C(\hat{\xi}_i)$. By (4.3.4) and (2.1.6) then one of the following three alternatives (4.3.5) , (4.3.6) or (4.3.7) is valid:

(4.3.5) $(B^{\xi_i} \cdot F_i)(\xi) \neq (C^{\hat{\xi}_i} \cdot \hat{F}_i)(\xi)$ for some $\xi < \mu^B(\xi_i)$, (4.3.6) $\mu^B(\xi_i) < \mu^C(\hat{\xi}_i)$ and $(C^{\hat{\xi}_i} \cdot \hat{F}_i)\ (\mu^B(\xi_i)) \neq e(\rho_i)$,

(4.3.7) $(B^{\xi_{i+1}} \cdot F_{i+1})\ (\xi) \neq (C^{\hat{\xi}_i} \cdot \hat{F}_i)\ (\xi)$ for some $\xi \leq \mu^B(\xi_i)$.

We show that each of these three cases yields a contradiction.

For technical convenience let us assume that all matrices have a (-1)-st row, namely "0". Analogously let $e(-1) = 0$.

Let us consider first (4.3.5):

Let ξ be minimal satisfying (4.3.5) and say that

$$\Theta = \min(B^{\xi_i} \cdot F_i)^{-1}\ ((B^{\xi_i} \cdot F_i)\ (\xi)) < \xi \ \text{, where } \Theta = -1 \text{ if } (B^{\xi_i} \cdot F_i)\ (\xi) = 0 \ .$$

The case $\min(C^{\hat{\xi}_i} \cdot \hat{F}_i)^{-1}\ ((C^{\hat{\xi}_i} \cdot \hat{F}_i)\ (\xi)) < \xi$ can be handled analogously.

Let

$$M = (e(0),e(1),\ldots,e(\mu^B(\xi_i) - 1),e(\xi),e(\mu^B(\xi_i)),e(\mu^B(\xi_i) + 1),\ldots,e(2k), e(2k),\ldots,e(2k)) \in \mathbb{B}_1\binom{m}{2k}$$

and let

$$\hat{M} = (e(0),e(1),\ldots,e(\mu^B(\xi_i) - 1),e(\Theta),e(\mu^B(\xi_i)),e(\mu^B(\xi_i) + 1),\ldots,e(2k), e(2k),\ldots,e(2k)) \in \mathbb{B}_1\binom{m}{2k} \ .$$

As $(B,C) \in H$ it follows that

(4.3.8) $\Delta_1(M \cdot B) = \Delta_2(M \cdot C)$ and $\Delta_1(\hat{M} \cdot B) = \Delta_2(\hat{M} \cdot C)$.

As $(B^{\xi_i} \cdot F_i)\ (\xi) = (B^{\xi_i} \cdot F_i)\ (\Theta)$, but $(C^{\hat{\xi}_i} \cdot \hat{F}_i)\ (\xi) \neq (C^{\hat{\xi}_i} \cdot \hat{F}_i)\ (\Theta)$ it follows that

$$(M \cdot B)^{\xi_\nu} \cdot F_\nu = (\hat{M} \cdot B)^{\xi_\nu} \cdot F_\nu \quad \text{for every } \nu \leq \ell \ \text{, but}$$
$$(M \cdot C)^{\hat{\xi}_i} \cdot \hat{F}_i \neq (\hat{M} \cdot C)^{\hat{\xi}_i} \cdot \hat{F}_i \ \text{, viz. } ((M \cdot C)^{\hat{\xi}_i} \cdot F_i)\ (\mu^B(\xi_i)) \neq ((\hat{M} \cdot C)^{\hat{\xi}_i}\ (\mu^B(\xi_i)) \ .$$

Since Δ_1 is of fibre-type $(\xi_i,\rho_i,F_i)_{i \leq \ell}$ and Δ_2 is of fibre-type $(\hat{\xi}_i,\hat{\rho}_i,\hat{F}_i)_{i \leq \hat{\ell}}$ it follows that

(4.3.9) $\Delta_1(M \cdot B) = \Delta_1(\hat{M} \cdot B)$, but $\Delta_2(M \cdot C) \neq \Delta_2(\hat{M} \cdot C)$

contradicting (4.3.8) .

Next let us consider (4.3.6):

We can assume that

(4.3.10) $(B^{\xi_i} \cdot F_i)(\xi) = (C^{\hat{\xi}_i} \cdot \hat{F}_i)(\xi)$ for every $\xi < \mu^B(\xi_i)$.

Let $\Theta = \min(C^{\hat{\xi}_i} \cdot \hat{F}_i)^{-1}((C^{\hat{\xi}_i} \cdot \hat{F}_i)(\mu^B(\xi_i)))$.

From (4.3.10) it follows that $\Theta < \mu^B(\xi_i)$.

Let

$$M = (e(0), e(1), \ldots, e(\mu^B(\xi_i)-1), e(\mu^B(\xi_i)), e(\mu^B(\xi_i)), e(\mu^B(\xi_i)+1), \ldots, e(2k), e(2k), \ldots, e(2k)) \in \mathbb{B}_1\binom{m}{2k}$$

and let

$$\hat{M} = (e(0), e(1), \ldots, e(\mu^B(\xi_i)-1), e(\Theta), e(\mu^B(\xi_i)), e(\mu^B(\xi_i)+1), \ldots, e(2k), e(2k), \ldots, e(2k)) \in \mathbb{B}_1\binom{m}{2k} .$$

Again from $(B,C) \in H$ it follows that

(4.3.11) $\Delta_1(M \cdot B) = \Delta_2(M \cdot C)$ and $\Delta_1(\hat{M} \cdot B) = \Delta_2(\hat{M} \cdot C)$.

As $\mu^{M \cdot B}(\mu^B(\xi_i)) \neq \mu^{\hat{M} \cdot B}(\mu^B(\xi_i))$ but $(C^{\hat{\xi}_i} \cdot \hat{F}_i)(\Theta) = (C^{\hat{\xi}_i} \cdot \hat{F}_i)(\mu^B(\xi_i))$, it follows that

(4.3.12) $\Delta_1(M \cdot B) \neq \Delta_1(\hat{M} \cdot B)$, but $\Delta_2(M \cdot C) = \Delta_2(\hat{M} \cdot C)$

which again contradicts (4.3.8) .

Finally we consider (4.3.7):

We can assume that (4.3.10) holds and that

$$(C^{\hat{\xi}_i} \cdot \hat{F}_i)(\mu^B(\xi_i)) = e(\rho_i) .$$

Let ξ be minimal satisfying (4.3.7) . Let

$$\Theta = \min(B^{\xi_{i+1}} \cdot F_{i+1})^{-1}((B^{\xi_{i+1}} \cdot F_{i+1})(\xi)) .$$

From (2.1.5) it follows particularly that $\Theta < \xi$.

Let

$$M = (e(0),e(1),\dots,e(\mu^B(\xi_i)),e(\xi),e(\mu^B(\xi_i)+1),\dots,e(2k),\\ e(2k),\dots,e(2k)) \in \mathbb{B}_1\binom{m}{2k}$$

and let

$$\hat{M} = (e(0),e(1),\dots,e(\mu^B(\xi_i)),e(\Theta),e(\mu^B(\xi_i)+1),\dots,e(2k),\\ e(2k),\dots,e(2k)) \in \mathbb{B}_1\binom{m}{2k} \quad .$$

Again it follows easily that

$$\Delta_1(M\cdot B) = \Delta_2(M\cdot C) \quad \text{and} \quad \Delta_1(\hat{M}\cdot B) = \Delta_2(\hat{M}\cdot C) \quad , \quad \text{but}$$

$$\Delta_1(M\cdot B) = \Delta_1(\hat{M}\cdot B) \quad \text{and} \quad \Delta_2(M\cdot C) \neq \Delta_2(\hat{M}\cdot C)$$

which is a contradiction.

Finally from (4.3.2) it follows then that

$$\{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid (B^{\xi_i}\cdot F_i)_{i\leq \ell} = (C^{\hat{\xi}_i}\cdot \hat{F}_i)_{i\leq \hat{\ell}}\} \subseteq H \quad . \qquad \square$$

Applying Lemma (4.3) $\binom{\chi}{2}$ - times yields the following corollary:

(4.4) <u>Corollary:</u> For every $m \geq 2k$ there exists an n such that for every family $((\xi_i^\nu,\rho_i^\nu,F_i^\nu)_{i\leq \ell^\nu} \mid \nu<\chi)$ of k-canonical families and every family

$$(\Delta_\nu : \mathbb{B}_1\binom{n}{k} \to \omega \mid \nu<\chi)$$

of colorings, where Δ_ν is of fibre-type $(\xi_i^\nu,\rho_i^\nu,F_i^\nu)_{i\leq \ell^\nu}$, there exists an $A \in \mathbb{B}_1\binom{n}{m}$ such that for every $\nu < \nu' < \chi$ the sets

$$H(\nu,\nu') = H_M(\nu,\nu') = \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \mid \Delta_\nu(AMB) = \Delta_{\nu'}(AMC)\} \quad ,$$

where $M \in \mathbb{B}_1\binom{m}{2k}$, are independent of M and satisfy:

either $H_M(\nu,\nu') = \emptyset$ or

$$H_M(\nu,\nu') = \{(B,C) \in \mathbb{B}_1\binom{2k}{k} \times \mathbb{B}_1\binom{2k}{k} \mid (B^{\xi_i^\nu}\cdot F_i^\nu)_{i\leq \ell^\nu} = (C^{\xi_i^{\nu'}}\cdot F_i^{\nu'})_{i\leq \ell'}\} .$$

Now theorem (3.7) follows from (4.2) and (4.4) using the fact that for $m \geq 2k$ each two $P(k)$ - subalgebras of $P(m)$ are contained in some $P(2k)$ - subalgebra of $P(m)$.

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