Radko Mesiar

Martingale theorems in the ergodic theory

In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [187]--[191].

Persistent URL: http://dml.cz/dmlcz/701273

MARTINGALE THEOREMS IN THE ERGODIC THEORY

Radko Mesiar

It was felt for a long time that martingales and ergodic theory, beeing essentially theories of integration in infinitely many variables, should be obtainable from a single structure. In fact there are many similarities in form as in proofs of the main theorems in both cases, e. g. maximal theorems, limit convergence theorems, c. f. see e. g. [2, p. 342] , [5, p. 135] . Several authors have tried to solve this problem, c. f. see [4], [7] , [8] , [9] , [10] . However, the hope to find such a single structure has not yet been completely realized.

In this paper we look at this problem from a different point of wiew. If the hypotesis of existence of a single structure for both martingales and ergodic theories is true, another analogies of martingale and conditional expectation theorems should exist in ergodic theory. In this way we obtain some conjectures in ergodic theory. Some of them have been proved, the others, as I know, are not proved yet. But no conjecture was proved to be false.

Throughout this paper let $(\Omega, \mathcal{S}, P)$ be a probability triple, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ a monotone sequence of sub-σ-algebras, T a measure preserving transformation on $(\Omega, \mathcal{S}, P)$.

<u>Theorem 1.</u> (c. f. see [2]) Let $0 \leq X.\log^+|X| \in \mathcal{L}_1$ for $X \in \mathcal{L}_1$. Then $\sup_n \{E(X/\mathcal{F}_n)\} \in \mathcal{L}_1$ ∎

<u>Conjecture 1.</u> Let $0 \leq X.\log^+|X| \in \mathcal{L}_1$ for $X \in \mathcal{L}_1$. Then $\sup_n \{\frac{1}{n}.\sum_{i=1}^{n} X \circ T^i\} \in \mathcal{L}_1$ ∎

Conjecture 1 is true. It was proved e. g. in [3, Theorem VIII. 6. 8.] .

<u>Theorem 2.</u> (c. f. see [1]). If $X \in \mathcal{L}_1$, $X \geq 0$, $X.\log^+ X \notin \mathcal{L}_1$, there are, on a suitable probability space, a random variable Y with the same distribution as X and a monotone sequence $\{\mathcal{B}_n\}_{n=1}^{\infty}$

of sub-σ-algebras, which can be chosen either increasing or decreasing, for which $\sup_n \{ E(Y/\mathcal{B}_n)\} \notin \mathcal{L}_1$ ∎

Theorem 2 shows that the condition $X.\log^+|X| \in \mathcal{L}_1$ in Theorem 1 is best possible.

<u>Conjecture 2.</u> If $X \in \mathcal{L}_1$, $X \geq 0$, $X.\log^+ X \notin \mathcal{L}_1$, there are, on a suitable probability space, a random variable Y with the same distribution as X and a measure preserving transformation T, for which $\sup_n \{ \frac{1}{n}. \sum_{i=1}^{n} Y \circ T^i \} \notin \mathcal{L}_1$. ∎

We are unable to prove Conjecture 2. However, Example 1 shows that condition $X \in \mathcal{L}_1$ is not sufficient for $\sup_n \{\frac{1}{n}. \sum_{i=1}^{n} X \circ T^i\} \in \mathcal{L}_1$.

<u>Example 1.</u> Let $(\Omega, \mathcal{A}, P) = (\langle 0, 1), \mathcal{B}, \lambda)^N$, where $\mathcal{B}$ is a Borel-σ-algebra, λ is a Lebesque measure, N is a set of positive integers. Let T be a shift. Denote $Y_k = a_k . \chi_{A_k}$, where $a_k = \exp(k^3).k^{-2}$, A_k depends only on the first coordinate, $P(A_k) = \exp(-k^3)$. Let $X = \sum_{k=1}^{\infty} Y_k$. Then $X \in \mathcal{L}_1$, $X \geq 0$, but $\sup_n \{\frac{1}{n}. \sum_{i=1}^{n} X \circ T^i\} \notin \mathcal{L}_1$ ∎

<u>Proof.</u> $\{Y_k \circ T^n\}_{n=1}^{\infty}$ forms a sequence of independent random variables for k = 1, 2,... Then

$$E(\sup_n\{\frac{1}{n}. \sum_{i=1}^{n} Y_k \circ T^i\}) \geq E(Y_k \circ T) + \frac{1}{2}.E(Y_k \circ T^2 . \chi_{\{Y_k \circ T = 0\}}) +$$

$$+ \ldots + \frac{1}{n}.E\{Y_k \circ T^n . \chi_{\{Y_k \circ T = Y_k \circ T^2 = \ldots = Y_k \circ T^{n-1} = 0\}}) + \ldots =$$

$$= k.(1 - \exp(-k^3))^{-1} > k \qquad \text{for } k = 1, 2, \ldots$$

So we have $E(\sup_n\{\frac{1}{n}. \sum_{i=1}^{n} X \circ T^i\}) \geq \sup_k\{E(\sup_n\{\frac{1}{n}. \sum_{i=1}^{n} Y_k \circ T^i\})\} = \infty$,

so that $\sup_n\{\frac{1}{n}. \sum_{i=1}^{n} X \circ T^i\} \in \mathcal{L}_1$.

<u>Theorem 3.</u> Let X, X_n, n = 1, 2,... be integrable random variables of $\mathcal{L}_1$, $\sup_n\{|X_n|\} \in \mathcal{L}_1$, $X_n \longrightarrow X$ a. e. Then $E(X_n/\mathcal{F}_n) \longrightarrow E(X/\mathcal{F}_\infty)$ a. e., L_1, where $E(X/\mathcal{F}_\infty)$ is a limit of martingale $\{E(X/\mathcal{F}_n)\}_{n=1}^{\infty}$ ∎

Theorem 3 is an easy consequence of Doob's dominated convergence theorem and martingale convergence theorem.

<u>Conjecture 3.</u> Let X, X_n, n = 1, 2,... be integrable random variables of $\mathcal{L}_1$, $\sup_n\{|X_n|\} \in \mathcal{L}_1$, $X_n \longrightarrow X$ a. e. Then $\frac{1}{n}. \sum_{i=1}^{n} X_i \circ T^i \longrightarrow \lim_n \frac{1}{n}. \sum_{i=1}^{n} X \circ T^i$ a. e., L_1, where $\lim_n \frac{1}{n}. \sum_{i=1}^{n} X \circ T^i$

is an ergodic limit of X ∎

Conjecture 3 is true. We have succeeded to prove it in [6].

Theorem 4. (c. f. see [1]) If $X \in \mathcal{L}_1$, $X_n \in \mathcal{L}_1$, $n = 1, 2, \ldots$, $X_n \geq 0$, $n = 1, 2, \ldots$, $X_n \longrightarrow X$ a. e. and $\sup\{X_n\} \notin \mathcal{L}_1$, there are, on a suitable probability space, random variables $\{Y_n, n = 1, 2, \ldots\}$, Y and a sub-σ-algebra $\mathcal{C}$ such that $Y, Y_1, Y_2, \ldots$ have the same joint distribution as $X, X_1, X_2, \ldots$, and $P(\{E(Y_n/\mathcal{C}) \longrightarrow E(Y/\mathcal{C})\}) = 0$ ∎

Theorems 4 shows that condition $\sup_n\{|X_n|\} \in \mathcal{L}_1$ in Theorem 3 is best possible.

Conjecture 4. The condition $\sup_n\{|X_n|\} \in \mathcal{L}_1$ in Conjecture 3 (which is true) is best possible ∎

Again we are not able to prove Conjecture 4. It is clear that if $X_n \circ T^n \longrightarrow 0$ (if X in Conjecture 3 is 0) almost everywhere, the condition $\sup_n\{|X_n|\} \in \mathcal{L}_1$ is superfluous (due to Cesaro convergence of the sequence $\{X_n \circ T^n\}_{n=1}^{\infty}$). The condition $X_n \circ T^n \longrightarrow 0$ a. e. is fulfiled e. g. if $X_n \rightrightarrows 0$ (uniform convergence) a. e., or if $\sum_{n=1}^{\infty} P(\{X_n \neq 0\}) < \infty$. This all leads to the following form of Conjecture 4.

Conjecture 4a. If $X \in \mathcal{L}_1$, $X_n \in \mathcal{L}_1$, $X_n \geq 0$, $n = 1, 2, \ldots$, $X_n \longrightarrow X$ a. e., $\inf_{Z \in \mathcal{L}_1}\{\sum_{n=1}^{\infty} P(\{X_n > Z\})\} = \infty$, there are, on a suitable probability space, random variables $\{Y_n, n = 1, 2, \ldots\}$, Y and a measure preserving transformation T such that $Y, Y_1, Y_2, \ldots$ have the same joint distribution as $X, X_1, X_2, \ldots$ and

$$P(\{\tfrac{1}{n}.\sum_{i=1}^{n} Y_i \circ T^i \longrightarrow \lim_n \tfrac{1}{n}.\sum_{i=1}^{n} Y \circ T^i\}) = 0$$ ∎

Example 2 shows that condition $X_n \longrightarrow X$ a. e., L_1, is not sufficient for $\frac{1}{n}.\sum_{i=1}^{n} X_i \circ T^i \longrightarrow \lim_n \frac{1}{n}.\sum_{i=1}^{n} X \circ T^i$ a. e.

Example 2. Let $(\Omega, \mathcal{A}, P)$ be the probability triple from Example 1. Let T be a shift. Let $X_n = n.\chi_{A_n}$, $A_n = \{\omega, \omega_1 \in \langle 0, \frac{1}{n.\log(n+10)}\rangle\}$. Then $X_n \longrightarrow 0$ a. e., L_1, but

$$P(\{\tfrac{1}{n}.\sum_{i=1}^{n} X_i \circ T^i \longrightarrow 0\}) = 0$$ ∎

Proof. The events $\{X_n \circ T^n \neq 0\}_{n=1}^{\infty}$ are independent. Since $\sum_{n=1}^{\infty} P(\{X_n \circ T^n \neq 0\}) = \infty$, almost all ω belong to infinitely many sets $\{X_n \circ T^n \neq 0\}$ (Borel-Cantelli). Hence for almost all ω, $\limsup_n\{\frac{1}{n}.\sum_{i=1}^{n} X_i \circ T^i\} \geq \limsup_n\{\frac{1}{n}.n\} = 1$, which is the required result.

I should like to thank Prof. H. von Weizsäcker for a very helpful discussion as well as for the Example 1.

REFERENCES

[1] BLACKWELL D. , DUBINS L. E. "A converse to the dominated convergence theorem", Illinois J. Math. , 7 1963 , 508-514.

[2] DOOB J. L. "Stochastic processes", Wiley, New York 1953 .

[3] DUNFORD N. , SCHWARTZ J. T. "Linear operators part I ", Interscience, New York 1958 .

[4] JERISON M. "Martingale formulation of ergodic theorems", Trans. Amer. Math. Soc. 10 1959 , 531-539.

[5] KAKUTANI S. "Ergodic theory", Int. Congr. Math., Proc 2 1950 , 128-142.

[6] MESIAR R. "A generalization of the individual ergodic theorem", Math. Slovaca 30 1980 , 327-330.

[7] NEVEU J. "Relations entre la théorie des martingales et la théorie ergodique", Ann. Inst. Fourier. Grenoble 15 1965 , 31-42.

[8] RAO M. M. "Abstract martingales and ergodic theory", J. Multivariate Anal. 3 1973 , 45-60.

[9] ROTA G.-C. "Une théorie unifiée martingales et des moyennes ergodique", C. R. Acad. Sci. Paris, Ser. A - B 252 1961 , 2064-2066.

[10] TULCEA A. I. , TULCEA C. I. "Abstract ergodic theorems", Trans. Amer. Math. Soc. 107 1963 , 107-124.

[11] v. WEIZSÄCKER H. "Sublineare abbildungen und ein Konvergenzsatz von Banach", Math. Ann. 212 1974 , 165-171.

RADKO MESIAR
KAT. MATEMATIKY STAV. FAK. SVŠT
RADLINSKÉHO 11
813 68 BRATISLAVA
ČSSR - CZECHOSLOVAKIA

APPENDIX TO "MARTINGALE THEOREMS IN THE ERGODIC THEORY"

Radko Mesiar

Conjecture 2 is true. I should like to thank Prof. Kellerer and Prof.von Weizsäcker for announcement of verification of Conjecture 2.

Sketch of the proof. If $X \geq 0$, $X \in \mathcal{L}_1$, there are, on a suitable probability space, a random variable Y and a measure preserving transformation T, such that $\{Y \circ T^n\}_{n=1}^{\infty}$ forms a sequence of i.i.d. random variables. Then it holds

$$\sup_n \frac{Y \circ T^n}{n} \in \mathcal{L}_1 \quad \text{iff} \quad Y.\log^+ Y \in \mathcal{L}_1 \quad .$$

As $\sup_n \frac{1}{n}.\sum_{i=1}^{n} Y \circ T^i \geq \sup_n \frac{Y \circ T^n}{n}$, then if $X.\log^+ X \notin \mathcal{L}_1$, i. e. $Y.\log^+ Y \notin \mathcal{L}_1$, it holds $\sup_n \frac{1}{n}.\sum_{i=1}^{n} Y \circ T^i \notin \mathcal{L}_1$.

The condition $X.\log^+ X \in \mathcal{L}_1$ of Conjecture 1 is really best possible.