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SELECTED TOPICS OF MATROID THEORY AND ITS APPLICATIONS

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1/ Some "classical" algorithms
2/ Matroid oracles
3/ Submodular functions
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5/ On the engineering applications of matroids

The present paper summarizes some results in matroid theory. The authors have chosen the topics in a subjective way, nevertheless, the algorithmic aspects dominate throughout. Basic concepts and results of matroid theory are supposed; Chapters 1 and 4 of [Welsh] are sufficient for most parts of the paper.

1/ Some "classical" algorithms

The classical algorithms of [Borůvka] and [Kruskal] to find a spanning forest of maximal weight in a graph can be generalized as follows:

Input: $(S,\mathcal{M})$ is a matroid, where S is the underlying set, $\mathcal{M} \subseteq 2^S$ is the collection of independent subsets. $w: S \rightarrow \mathbb{R}_0^+$ is a weight function, which associates nonnegative real weight $w(s)$ to every $s \in S$.
Output: A base $B \subseteq S$ of the matroid, with maximum weight $w(B) = \sum \{w(x),\ x \in B\}$ among all the possible bases.
Description: Start from the empty set, i.e. let $B = \emptyset$. In every step put $B = B \cup \{x_0\}$ where x_0 has maximal weight among those elements x for which $B \cup \{x\} \in \mathcal{M}$. If no such x exists, stop.

The algorithm is called greedy since it increases the weight of the independent set with the maximal possible amount in every step.

Theorem 1 [Rado] The greedy algorithm gives a maximum weight base for every matroid and for an arbitrary weight-function. On the other hand, if $\mathcal{M} \subset 2^S$ does not satisfy the exchange axiom of matroid theory, one can always find a weight-function $w_{\mathcal{M}}$ so that the greedy algorithm gives a wrong answer (i.e. a subset whose weight is not maximum).

Sketch of proof: a/ Suppose that the greedy algorithm gives $B = \{x_1, x_2, \dots, x_k\}$ where the elements were given in this order, and suppose there were a base B_0 with $w(B_0) > w(B)$. If there were several bases with maximum weight, choose one with $|B \cap B_0|$ maximal. Let x be the maximum weight element of $B_0 - B$ and consider $B \cup \{x\}$. This contains a unique circuit C. Let x_i be an element of $C - \{x\}$ with maximal subscript. Since $(B \cup \{x\}) - \{x_i\}$ is a base, $w(x) > w(x_i)$ would contradict the choice of x_i during the greedy algorithm, while $w(x_i) \geq w(x)$ would contradict the choice of B_0 since $(B_0 - \{x\}) \cup \{x_i\}$ is also a base.

b/ If $\mathcal{M}$ violates the exchange axiom, there are subsets X, Y so that $X \in \mathcal{M}$, $Y \in \mathcal{M}$, $|X| > |Y|$ yet $Y \cup \{x\} \notin \mathcal{M}$ for every $x \in X$. Let us define a weight function w so that $w(y) = 1$ for $y \in Y$, $w(x) = 1 - \varepsilon$ for $x \in X - Y$ and $w(z) = 0$ for $z \notin X \cup Y$. Then the greedy algorithm leads to Y (plus perhaps some elements from $S - (X \cup Y)$) which is certainly not of maximum weight if $\varepsilon < 1 - (|Y| / |X|)$.

Another basic tool of matroid theory is the matroid partition algorithm [Edmonds 1]. First we present an essentially equivalent alternative, the matroid intersection algorithm.

Input: $(S, \mathcal{M}_1)$ and $(S, \mathcal{M}_2)$ are two matroids on the same underlying set.
Output: A subset $X \subseteq S$ with maximum cardinality satisfying $X \in \mathcal{M}_1 \cap \mathcal{M}_2$.
Description: Start with $X = \emptyset$. In every step define a directed graph G with vertex set S as follows. For every $x \notin X$ and $X \cup \{x\} \notin \mathcal{M}_1$ draw an edge (x, y) if y belongs to the unique circuit of $\mathcal{M}_1$, contained in $X \cup \{x\}$. Furthermore, for every $x \notin X$ and $X \cup \{x\} \notin \mathcal{M}_2$ draw an edge (y, x) if y belongs to the unique circuit of $\mathcal{M}_2$ contained in $X \cup \{x\}$.

If G has no sink (i.e. no vertex with outdegree zero) then X is a base of $\mathcal{M}_1$ and we can stop. If G has no source (i.e. no vertex with indegree zero) then X is a base of $\mathcal{M}_2$ and we can stop. Similarly, we can stop if there is no directed path in G from a source to a sink: then X is of maximum cardinality in $\mathcal{M}_1 \cap \mathcal{M}_2$.

If there are source-to-sink directed paths in G, consider a minimal one $(v_1, v_2, \dots, v_t)$, i.e. one with no shortcuts. Observe that if $t = 1$, i.e. when $v_1 \in S - X$ is isolated in G (which is certainly a minimal source-sink path) then $X \cup \{v_1\} \in \mathcal{M}_1 \cap \mathcal{M}_2$. The source v_1 is certainly not in X and in general $v_i \in X$ if and only if i is even. In particular, the sink v_t is not in X and t is odd. Then $(X - \{v_2, v_4, v_6, \ldots\}) \cup \{v_1, v_3, v_5, \ldots\}$ is taken for X and start the procedure from the beginning, with a new graph.

Theorem 2 [Krogdahl, Lawler] This algorithm always solves the matroid intersection problem.

We do not give detailed proof here. The minimality of the source-sink path should be applied and induction is used on the length of such a (so called augmenting) path. On the other hand observe that if $V_1 \subseteq S$ is the set of those vertices which can be reached via a directed path form a source then $r_1(V_1) = |V_1 \cap X|$ and $r_2(S - V_1) = |X - V_1|$. Hence, when the algorithm stops, we have a subset $X \in \mathcal{M}_1 \cap \mathcal{M}_2$ with cardinality $r_1(V_1) + r_2(S - V_1)$ which proves the non-trivial part of the following theorem:

Theorem 3 [Edmonds 1] $\max\{|B|;\ B \in \mathcal{M}_1 \cap \mathcal{M}_2\} = \min\{r_1(V) + r_2(S - V);\ V \subseteq S\}$.

Once we have these results, a number of equivalent statements can easily be obtained.

Theorem 4 [Edmonds 1] The union $(S, \mathcal{M}_1) \vee (S, \mathcal{M}_2)$ of two matroids $(S, \mathcal{M}_1)$ and $(S, \mathcal{M}_2)$ on the same set S equals the free matroid, i.e. S can be partitioned into $S_1 \cup S_2$ so that $S_1 \in \mathcal{M}_1$ and $S_2 \in \mathcal{M}_2$, if and only if $r_1(X) + r_2(X) \geq |X|$ for every $X \subseteq S$.

Proof: The "only if" part is trivial. If the condition holds, i.e. if $\min\{r_1(X) + r_2(X) - |X|;\ X \subseteq S\} \geq 0$ then $\min\{r_1(X) + r_2^*(S - X);\ X \subseteq S\} \geq$ $\geq r_2^*(S)$, by the well known rank function formula $r_2^*(S - X) = |S - X| +$ $+ r_2(X) - r_2(S)$ for dual matroids. This latter minimum is $\max\{|B|;\ B \in \mathcal{M}_1 \cap \mathcal{M}_2^*\}$ by the previous theorem, and is therefore obviously at most $r_2^*(S)$. The equality means that the maximum is attained at such a B_0 which is a base of $\mathcal{M}_2^*$. Then $B_0 \in \mathcal{M}_1$ and $S - B_0 \in \mathcal{M}_2$.

Theorem 5 [Nash-Williams] Let $\varphi : S \to T$ and for a matroid $(S, \mathcal{M})$, let $\mathcal{M}_\varphi = \{\varphi(A);\ A \in \mathcal{M}\}$. Then $(\varphi(S), \mathcal{M}_\varphi)$ is a matroid with rank function $r_\varphi(X) = \min\{r(\varphi^{-1}(Y)) + |X - Y|;\ Y \subseteq X\}$.

Sketch of proof: Consider the partition Π_φ on S defined by φ, i.e. s_1 and s_2 are in the same subset of Π_φ if and only if $\varphi(s_1) = \varphi(s_2)$. Define a partition matroid $(S, \mathcal{P})$ so that a subset of S is independent in $\mathcal{P}$ if and only if it intersects any subset of Π_φ in at most one element. Then the independent subsets of $\mathcal{M}_\varphi$ are just those corresponding to the common independent subsets of $\mathcal{M}$ and $\mathcal{P}$. Since the rank of such a subset W in the matroid $\mathcal{P}$ is just $|\varphi(W)|$, the nontrivial part of the statement follows from Theorem 3.

Theorem 6 [Nash-Williams] The rank function R of the union of the matroids $(S, \mathcal{M}_i)$, $i = 1, 2, \ldots, k$ is $R(X) = \min\{\sum_{i=1}^{k} r_i(Y) + |X - Y|;\ Y \subseteq X\}$.

Proof: Let $\mathcal{M}$ be the direct sum of the matroids $(S, \mathcal{M}_i)$, constructed on k disjoint copies of S, and let φ be the natural homomorphism, identifying the k copies of each element of S. Apply Theorem 5.

2/ Matroid oracles

Both of the algorithms in Section 1 are usually very effective if implemented for various practical purposes. If, for example, the input matroids are graphic and are actually represented by graphs then the number of steps for these algorithms is a polynomial of the number of vertices of the graphs. Similarly, if the matroids are represented by matrices (as columnspace matroids), the complexity of the algorithms is again a polynomial function of the size of the input. But how is a matroid stored "in general"?

Since the number of different matroids on an n-element set is almost 2^{2^n}, any "general" description would require exponentially large storage space. Hence, instead of the usual requirement of "being polynomial in the size of the input" we would prefer being polynomial in n. A usual way to formalize this is to assume that our matroid is described by an oracle (subroutine) which somehow can tell us whether a given subset is independent or not. (We are not interested in how this oracle is realized by a program.) Then a lower bound on efficiency can be obtained from the number of questions posed to this oracle. (Roughly speaking, the calls of a certain subroutine are counted as single steps, no matter how complex the interior structure of the subroutine may be.)

This oracle is called an independence-oracle. Some other, more or less usual oracles are the following.

Name	Input	Output
Base-oracle	a subset X	"yes" if X is a base and "no" otherwise
Circuit-oracle	a subset X	"yes" if X is a circuit and "no" otherwise
Rank-oracle	a subset X	the rank $r(X)$ of X
Girth-oracle	a subset X	the length of the shortest circuit contained in X (and, say, ∞ if X is independent)

Unlike in case of graphs, where the various storages (incidence matrix, adjacency matrix, adjacency lists etc) are in a sense equivalent (no matter which one is used, the complexity of a certain algorithm is either always polynomial - the exponent may vary, of course - or never polynomial), in case of matroids the complexity highly depends on the actual oracle. The interested reader is referred to [Hausmann-Korte] for a detailed analysis. In what follows only some typical results are presented.

Theorem 7 The rank-oracle and the independence-oracle are polynomially equivalent.

Proof: If one has a rank-oracle O_1, it can be used to build an independence-oracle O_2 as follows: If X is the input of O_2, one simply inputs X to O_1; if the output $r(X)$ of O_1 equals $|X|$ then one outputs "yes" for O_2, while if $r(X) < |X|$ then the output for O_2 is "no".

On the other hand, $r(X)$ can be determined applying O_2 only, since this is exactly what the greedy algorithm does, with a constant weight function.

Theorem 8 The base-oracle and the circuit-oracle are less powerful then the independence oracle.

Proof: Let $(S,\mathcal{M}_1)$ be defined on an $|S| = 2n$ -element set so that the only base of $\mathcal{M}_1$ is a certain n-element subset $X \subset S$. Any algorithm, using the base-oracle only in $O(n^k)$ times, might get a "no" answer for every question. Since the number of n-element subsets of S grows exponentially, several n-element subsets were not asked at all, and we cannot deduce which one of them is the base. Thus even the independence of the singletons cannot be determined. For a similar proof of the "weakness" of the circuit-oracle apply $(S,\mathcal{M}_2)$ where the above set X is the only circuit. On the other hand, the independence oracle is at least as powerful as these latter two, as can be proved in a straightforward way.

However, the reader should verify that the base- and the independence-oracles are polynomially equivalent if one knows one base in advance.

Theorem 9 The girth-oracle is more powerful than the other four oracles.

Proof: It is enough to prove that the girth-oracle cannot be realized using the independence-oracle a polynomial number of times only. Define $(S, \mathcal{M}_3)$ and $(S, \mathcal{M}_4)$ on the same set as above ($|S| = 2n$) and let every subset of cardinality n be a base in $\mathcal{M}_3$ and all except one be bases in $\mathcal{M}_4$. Of course, the girth of $\mathcal{M}_3$ is $n+1$ while that of $\mathcal{M}_4$ is n. But one cannot tell the difference using the independence-oracle, unless asking the independence of all the n-element subsets of S.

3/ Submodular functions

A function $b : 2^S \to \mathbb{R}$ is submodular if, for any pair X, Y of subsets of S, the relation $b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ holds. If the relation always holds with equality, the function is modular. The sum of two submodular functions (or that of a submodular and a modular function) is submodular again. The rank function of a matroid is a special submodular function.

Theorem 10 [Grötschel, Lovász, Schijver] Suppose that a submodular function b is given by an oracle, which gives $b(x)$ for every $X \subseteq S$. Then b can be minimized on the underlying set by a polynomial algorithm.

The proof applies the celebrated ellipsoid method [Khachiyan], [Shor] and is therefore of a significantly different character. However, some special cases have "traditional" solution (without real arithmetic with approximative results etc), e.g. if the submodular function is $r_1(V)+r_2(S-V)$ or $[\sum_{i=1}^{k} r_i(V)] - |V|$, where r_i are rank functions of matroids (see Theorems 3 and 6 respectively). Similarly, the max-flow-min-cut theorem of Ford and Fulkerson can be considered as such a special case.

If a non-negative submodular function $b : 2^S \to \mathbb{R}$ is integer-valued, with $b(\emptyset) = 0$, then the only reason why b is not the rank function of a matroid can be that there are subsets $X \subseteq S$ with $b(X) > |X|$. This emphasizes the usefulness of the following theorem.

Theorem 11 [Edmonds 3] Those subsets $Y \subseteq S$ for which $\min\{b(X) - |X|;\ X \subseteq Y\} \geq 0$, form the independent subsets of a matroid.

If b happens to be the rank function of a matroid then this "new" matroid is just the original one. The reader should verify by Theorem 4 that if b is the sum of the rank functions of some matroids $\mathcal{M}_1, \mathcal{M}_2, \dots$ then this "new" matroid is just their union $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \dots$.

We close this section with a more recent result. A function $p : 2^S \to \mathbb{R}$ is called supermodular if $-p$ is submodular.

Theorem 12 [Frank] Let p and b be supermodular and submodular functions, respectively, both integer valued, on the same set S. Let $p \leq b$ hold for every subset of S. Then there always exists an integer valued modular function m so that $p \leq m \leq b$.

As an application let us deduce Theorem 3 from Theorem 12. Suppose that $k = \min\{r_1(X) + r_2(S - X);\ X \subseteq S\}$ and we have to show that $\mathcal{M}_1 \cap \mathcal{M}_2$ contains a k-element subset. We may assume that $r_1(S) = r_2(S) = k$ by truncating $\mathcal{M}_1$ and $\mathcal{M}_2$ if necessary. By the definition of k we have $k - r_2(S - X) \leq r_1(X)$ for every $X \subseteq S$. Since $k - r_2(S - X)$ is supermodular and $r_1(X)$ is submodular, there exists a modular function $m(X)$ between them. Since $m(\{x\}) = 0$ or 1 for every $x \in S$, it is very easy to show that the set $B = \{x;\ m(\{x\}) = 1\}$ is just the requested k-element subset.

4/ Some further results

Let $(S, \mathcal{M})$ be a matroid and $\{a_1, b_1\}, \{a_2, b_2\}, \dots$ be disjoint pairs from the elements of S. Find a maximal number of such pairs $\{a_i, b_i\}$ so that their union be independent in $(S, \mathcal{M})$. This is the matroid parity or matroid matching problem. The corresponding problem for disjoint n-tuples $(n \geq 3)$ is known to be NP-hard, while several advanced results of the class P are shown (e.g. [Lawler]) to be special cases of this problem, e.g. finding a maximal matching in a (not necessarily bipartite) graph, or the whole first section of the present paper.

The matroid parity problem is of exponential complexity [Jensen-Korte], [Lovász 2] but a very important special case is polynomially solvable [Lovász 1], namely when $\mathcal{M}$ is linear, i.e. represented over the field of the reals. In this case the above pairs can be imagined as a set H of lines in the real projective space $\mathcal{P}$ and one should find k lines so that their union spans a subspace of dimension 2k. Their maximal number ν can be expressed by the following formula:

Theorem 13 [Lovász 1,2] $\nu = \min \left\{ r(A) + \sum_{i=1}^{k} \left\lfloor \frac{r(H_i + A) - r(A)}{2} \right\rfloor \right\}$ where A ranges over all flats of $\mathcal{P}$ and $\{H_1, H_2, \dots, H_k\}$ over all partitions of H.

Example: Consider the set $H = \{e_1, e_2, e_3\}$ of lines where $e_1 = \{1, 2\}$, $e_2 = \{3, 4\}$ and $e_3 = \{5, 6\}$ in the matroid $\mathcal{M}$, shown by its affine representation on Fig. 1. Since the rank of $\mathcal{M}$ is four, one cannot find more than two lines with the required properties but one set $\{e_1, e_3\}$ of two lines will be appropriate. Hence $\nu = 2$ and the above minimum can really be attained by $A = \{3\}$ and by almost any partition of H.

Another important result in the past few years was a new characterization of regular matroids [Seymour 1]. We recollect that [Tutte] has already given an excluded minor type characterization of them. This imples that non-regularity is an NP-property, i.e. one could prove (in polynomial time) that the matroid is not regular, provided he/she has already found a forbidden minor (after a no matter how long search). But a proof of regularity (in polynomial time) was an open problem.

Let us refer to direct sum as 1-sum and introduce the concepts of 2- and 3-sums. These are amalgamations of two matroids along a common element and along a common circuit of length 3, respectively. Instead of formal definitions we offer the intuitive drawings (Fig. 2) for graphic matroids. These operations preserve regularity.

Theorem 14 [Seymour 1] Any regular matroid can be obtained by 1-, 2- and 3-sums from graphic matroids, cographic matroids, and from several copies of a further regular matroid R_{10} which can be described in the simplest way by the following binary representation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Before finishing the pure mathematical part of the paper, it might be instructive to recollect some famous problems of matroid theory from the point of view of computational complexity.

Polynomially solvable problems
Base of maximal weight in a matroid (Theorem 1)
Subset of maximal cardinality, simultaneously independent in two matroids (Theorem 3)
Subset of maximal weight, simultaneously independent in two matroids [Edmonds 5], [Lawler]
Rank of a subset in the union of matroids (Theorem 6)
Minimum of an arbitrary submodular function (Theorem 10)
The matroid parity problem for linearly represented matroids [Lovász 1]
Test of graphicity of a matroid [Seymour 2]
Test of regularity of a matroid [Seymour 1]

Non-polynomial problems
Length of a shortest circuit of the matroid (Theorem 9)
Length of a longest circuit of the matroid
Maximum of an arbitrary submodular function
The matroid matching problem for arbitrary matroids [Jensen-Korte], [Lovász 2]
Test of binarity of a matroid [Seymour 2]
Test of linearity of a matroid

NP-hard problems
Find a maximal cardinality subset, which is simultaneously independent in more than two graphic matroids
Generalize the matroid matching problem (with n-tuples, $n \geq 3$), but restricted for linearly represented matroids (or even to graphic matroids)
Find the length of a shortest/longest circuit of a lineary represented matroid

5/ On the engineering applications of matroids

Since finiteness and linearity are perhaps the most usual assumptions when modelling physical phenomena, matroids can certainly be very well applied to decide qualitative problems in science, engineering, operations research etc. Examples for such problems are

a/ Decide whether a linear electric network is uniquely solvable.

b/ Decide whether a framework (constructed from rods and joints) is rigid.

c/ Decide whether a 2-dimensional drawing correctly represents a polyhedron.

All these problem could theoretically be solved by classical methods of linear algebra but round-off errors in arithmetic operations among real numbers (which are represented by decimals of a finite length in a computer) can cause qualitative mistakes, especially in case of large systems.

Let us associate a matroid (simply the column space matroid of a matrix) to the objects under consideration. E.g. if an electric device is modelled as a multiport, say an ideal transformer (Fig. 3a) by the system of equations

$$\begin{bmatrix} -1 & k & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or if a framework is described by the usual rigidity conditions, e.g. the four-rods planar framework (Fig. 3c) by the system

$$\begin{bmatrix} x_1-x_2 & x_2-x_1 & 0 & 0 & y_1-y_2 & y_2-y_1 & 0 & 0 \\ 0 & x_2-x_3 & x_3-x_2 & 0 & 0 & y_2-y_3 & y_3-y_2 & 0 \\ 0 & 0 & x_3-x_4 & x_4-x_3 & 0 & 0 & y_3-y_4 & y_4-y_3 \\ x_1-x_4 & 0 & 0 & x_4-x_1 & y_1-y_4 & 0 & 0 & y_4-y_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

then the circuit matroid of the graph of Fig. 3b represents the matroidal model of the above 2-port and the affine representation on Fig. 3d visualizes the matroidal model of the above framework.

Now, the answer to some simple questions can directly be obtained from these matroidal models. E.g. if one terminates the first port of the transformer by a voltage source and the second port by a current source (Fig. 4a) then the network is uniquely solvable (the other voltages and currents can uniquely be expressed by them) while the network is singular if both ports are terminated by voltage sources (Fig. 4b). These answers can also be obtained by checking that the set $\{i_1, u_2\}$ is independent while $\{i_1, i_2\}$ is dependent in the matroid of the transformer (cf. Fig. 3b). Similarly, if we pin down the first and the third joints of the planar framework of Fig. 3c then the whole system becomes rigid, which is certainly not the case if the first and second joints are pinned down (see Figs. 4c. and 4d respectively). These answers can also be obtained by checking that the set $\{\dot{x}_2, \dot{x}_4, \dot{y}_2, \dot{y}_4\}$ is independent while $\{\dot{x}_3, \dot{x}_4, \dot{y}_3, \dot{y}_4\}$ is dependent in the matroid of the framework (cf. Fig. 3d).

However, if one wishes to answer some more complicated qualitative problems, some of the advanced matroidal results of the previous sections are required. For example, an n-port has a hybrid immittance description if its ports can be terminated by voltage and current sources to ensure unique solvability.

Theorem 15 [Iri-Tomizawa] An n-port has (at least one) hybrid immittance description if and only if its matroid has a common base with the partition matroid $\mathcal{B}$, defined so that $X \subseteq \{u_1, u_2, \ldots, u_n, i_1, i_2, \ldots, i_n\}$ is a base of $\mathcal{B}$ if and only if $|X \cap \{u_j, i_j\}| = 1$ for every $j = 1, 2, \ldots, n$.

Theorem 16 [Laman] If a planar framework with n joints and $e = 2n - 3$ rods is rigid then $e' \leq 2n' - 3$ for every "subgraph" of the framework with n' joints and e' rods. This condition is also sufficient for rigidity if the framework is generic, i.e. if its joints are in general position.

Theorem 17 [Lovász-Yemini] Generic rigidity of the planar framework with graph G is equivalent to the condition that $\mathcal{M}_e(G) \vee \mathcal{M}_e(G)$ is the free matroid for every $e \in E(G)$, where $\mathcal{M}_e(G)$ is the circuit matroid of the graph, obtained from G by doubling the edge e.

Theorem 18 [Rosenberg], [Recski 3], [White-Whiteley] The subdeterminants of the describing matrix of the frameworks can be expressed as sums over the possible decompositions of the graph of the framework into two trees.

This result is analogous to the so called topological formulae of linear active networks. In case of certain frameworks (the so called simple trusses [Timoshenko and Young], see e.g. the frameworks on Fig. 5) an electric network model can directly be established [Recski 3].

Theorem 19 [Recski 2, 3] If some genericity-type condition is prescribed, the matroidal model of the interconnection of several multiports or several frameworks can be obtained from the matroids of the original objects by the union of the matroids.

The basic tool in these investigations is a result of [Edmonds 2].

Essentially in the same way, matroid partition algorithms can be used for checking the solvability of complex interconnected electric networks [Iri-Tomizawa], [Recski 1], [Petersen].

Theorem 20 [Lovász 1] The minimal number of pins required to fix a framework to the plane can be obtained by the matroid matching algorithm.

Finally we present a somewhat less recent (!) result which relates problem c/ (see the beginning of this section) to the rigidity problem.

Theorem 21 [Maxwell] A framework with n joints and $e = 2n - 3$ rods is rigid in the plane if and only if it does not contain the projection of a 3-dimensional polyhedron.

For example, it is intuitively clear that only the first framework on Fig. 6 is rigid. For further results related to problem c/ the reader is referred to [Sugihara].

REFERENCES

BORŮVKA O. "O jistém problému minimálnim", Práce Mor. Přirodeved. Spol. v Brne, 3(1926) 37-58.

EDMONDS J. 1. "Minimum partition of a matroid into independent sets", J. Res. Nat. Bureau Stand. 69B (1965) 67-72.

2. "Systems of distinct representatives and linear algebra", J. Res. Nat. Bureau Stand. 71B (1967) 241-245.

3. "Submodular functions, matroids and certain polyhedra", Combin. Structures and their Appl., Gordon and Breach, N.Y. 1970. 69-87.

4. "Matroids and the greedy algorithm", Math. Prog. 1 (1971) 127-136.

5. "Matroid intersection", Annals of Discrete Math. 4 (1979) 39-49.

FRANK A. "An algorithm for submodular functions on graphs", Annals of Discrete Math. 16 (1982) 189-212.

GRÖTSCHEL M. - LOVÁSZ L. - SCHRIJVER L. "The ellipsoid method and its consequences in combinatorial optimization", Combinatorica 1 (1981) 169-197.

HAUSMANN D. - KORTE B. "Algorithmic versus axiomatic definitions of matroids", M Programming Study 14 (1981) 98-111.

IRI M. - TOMIZAWA N. "A unifying approach to fundamental problems in network theory by means of matroids", Trans. IECEJ 58-A (1975) 33-40.

JENSEN P.M. - KORTE B. "Complexity of matroid property algorithms", SIAM J. on Computing

KHACHIYAN L.G. "A polynomial algorithm in linear programming", Doklady Akademii Nauk SSSR 244 (1979) 1093-1096; English translation: Soviet Math. Dokl. 20 191-194.

KROGDAHL S. "A combinatorial base for some optimal matroid intersection algorithms", Report Stan-CS-74-468, Stanford University, 1974.

KRUSKAL J.B. "On the shortest spanning subgraph of a graph and the travelling salesman problem, Proc. AMS. 7 (1956) 48-49.

LAMAN G. "On graphs and rigidity of plane skeletal structures", J. Engineering Mat. 4 (1970) 331-340.

LAWLER E.L. "Combinatorial optimization: Networks and matroids", Holt, Rinehart and Winston, New York, 1976.

LOVÁSZ L. 1. "Polymatroid matching and some applications", J. Combinatorial Theory B 28 (1980) 208-236.

2. "Selecting independent lines from a family of lines in a space". Acta Sci. Math. Szeged 42 (1980) 121-131.

LOVÁSZ L. - YEMINI Y. "On generic rigidity in the plane", SIAM J. Algebraic and Discrete Methods,

MAXWELL J.C. "On reciprocal figures and diagrams of forces" Phil. Mag. 4, 27 (1864) 250-261.

NASH - WILLIAMS C.St.J.A. "On applications of matroids to graph theory", Theory of graphs, I.C.C. Rome, 1967 (Dunod, 1968), 263-265.

PETERSEN B. "Investigating solvability and complexity of linear active networks by means of matroids", IEEE Trans. Circuits and Systems CAS-26 (1979) 330-342.

RADO R. "A note on independence functions", Proc. London Math. Soc. 7 (1957) 300-320.

RECSKI A. 1. "Contributions to the n-port interconnection problem by means of matroids", Coll. Math. Soc. J. Bolyai 18 (1978) II. 877-892.

2. "Matroids and network synthesis", Proc. 1980 European Conference on Circuit Theory and Design, 192-197.

3. "A network theory approach to the rigidity of skeletal strucutes" Parts I-II submitted to Discrete Applied Math., Part III to appear.

ROSENBERG I. "Structural rigidity in the plane", preprint CRM-510, Université de Montréal, 1975.

SEYMOUR P. 1. "Decomposition of regular matroids", J. Comb. Theory B 28 (1980) 305-359.

2. "Recognizing graphic matroids", Combinatorica 1 (1981) 75-78.

ŚHOR N.Z. "Convergence rate of the gradient descent method with dilatation of the space", Kibernetika 2 (1970) 80-85; English translation: Cybernetics 6 (1970) 102-108.

SUGIHARA K. "A step toward man-machine communication by means of line drawings of polyhedra, Bull. Electrotechn. Lab. of Japan, 42 (1978) 20-43.

TIMOSHENKO S. - YOUNG D.H. "Theory of structures", McGraw Hill, N.Y. 1945.

TUTTE W.T. "Lectures on matroids", J. Res. Nat. Bureau Stand. 69 B (1965) 1-47.
WELSH D.J.A. "Matroid theory", Academic Press, London, 1976.
WHITE N.L. - WHITELEY W. "The algebraic geometry of stresses in frameworks", preprint

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Fig. 1

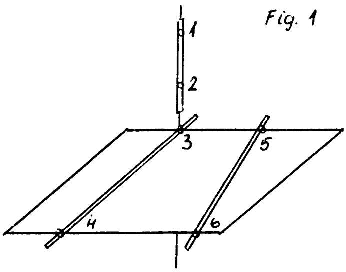


Fig. 2

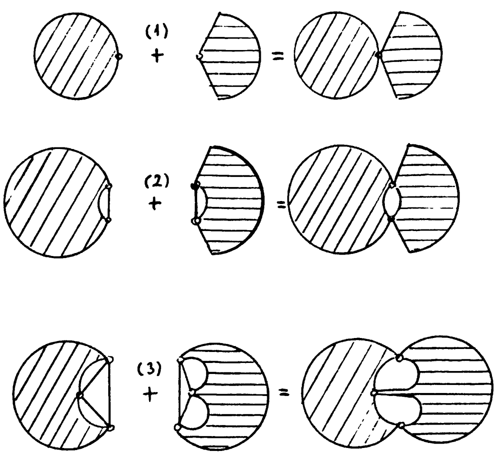
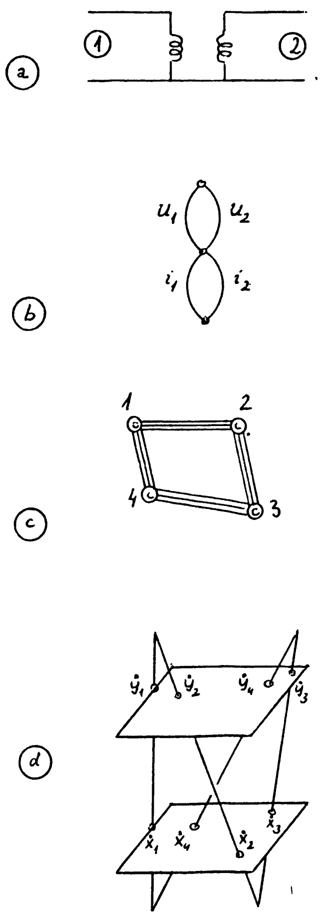


Fig. 3



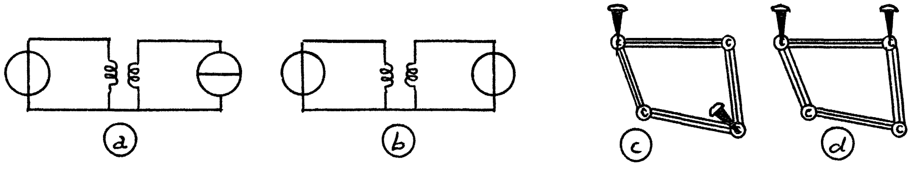


Fig. 4.

Fig. 5

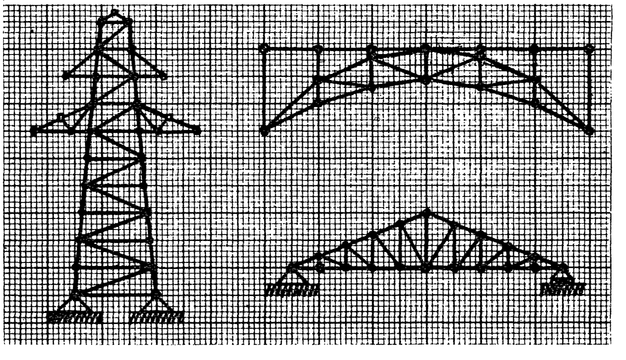


Fig. 6