G. Jetschke

# Different approaches to stochastic parabolic differential equations

In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [161]--169.

Persistent URL: http://dml.cz/dmlcz/701271

DIFFERENT APPROACHES TO STOCHASTIC PARABOLIC DIFFERENTIAL EQUATIONS

G. Jetschke

Different approaches to stochastic parabolic differential equations with a multiparameter "white noise" on the right hand side are reviewed, which turn out to be equivalent. Some important properties of the solutions, especially the distribution in function space, are given.

## 1. Motivation

Two concepts more and more enter into modern physics, namely nonlinearity and stochasticity. Especially the theory of selforganization in non-equilibrium systems is fully governed by the counterplay of these two aspects of matter /7/. Nonlinear systems may have several stationary states being stable or unstable against small perturbations. In macroscopic systems such perturbations occur in form of random fluctuations which not only enable the system to leave an unstable state but may determine essentially the further evolution. So a stochastic description is inevitable.
In the following we will consider the nonlinear stochastic partial differential equation of parabolic type

$$\partial u(t,x)/\partial t = D\cdot(\partial^2 u/\partial x^2) + f(u) + c\cdot\xi(t,x) \quad , \quad D>0 \ ,$$
$$t \in [0,T]=\mathcal{T} \ , \ x \in [0,L]=\mathcal{D} \ , \quad u(t,0) = u(t,L) = 0, \quad u(0,x) = u_0(x) \ . \qquad (1)$$

This is a stochastic reaction-diffusion equation and describes the interaction between local production of a chemical substance (with the nonlinear rate function f, usually polynomial) and its spatial transport by (linear) diffusion.
The internal fluctuations are modelled very globally by adding a random source term which is assumed to be a spatio-temporal "white" Gaussian noise, i.e. a Gaussian random field with

$$E\xi_{tx} = 0 \quad , \quad E\xi_{tx}\xi_{t'x'} = \delta(t-t')\cdot\delta(x-x') \quad , \tag{2}$$

the constant c controlling the strength of the noise. The Gaussian assumption is suggested by the central limit theorem and the "white" character idealizes the fact, that the correlation time and length of the noise are extremely small compared with typical values of the macroscopic system. (In physics equations of type (1) are called LANGEVIN equations.)

The problem is to give a unique sense to such a (heuristic) equation and to develop a mathematical calculus to treat it.

There are (at least) two different approaches leading to different techniques:

(i) $\{u_{tx}\}$ is a real number-valued field with two parameters $t,x$ ,

(ii) $\{u_t(\cdot)\}$ is a function-valued process with one parameter $t$ .

(Note that for convenience we often write $u_{tx}$ instead of $u(t,x)$, $F_t$ instead of $F(t)$ and so on.)

The aim of this lecture is to give a short review on the above mentioned subject, i.e. to describe the starting point and to state the main results. Proofs are omitted or only sketched and details can be found in the references. One important thing is to point out the equivalence of these two approaches which are found independently in the literature.

One remark should be added. At the beginning the case of white noise requires a certain mathematical investment (i.e. a stochastic calculus), but then very useful results are the outcome (f.e. the Markov property of the solutions, moment equations) as they have been obtained already by physicists with less rigorous methods. In more realistic (but also more complicated) models $\xi$ should be substituted by a "coloured" (or still more general) noise $\zeta$ with $E\zeta_{tx} = 0$ and (piecewise) continuous paths (f.e. a Gaussian random field with a suitable covariance $E\zeta_{tx}\zeta_{t'x'} = R(t-t',x,x')$). Then equation (1) will be understood as a family of equations indexed by all individual realizations of the noise.

Shall in the following for the time being be $f \equiv 0$ (the nonlinear case is treated in section 6). For real noise the integration of (1) can be done pathwise,

$$u^{(\zeta)}_{tx} = \int_0^L G(t,x,y)u_0(y)dy + c\cdot\int_0^t\int_0^L G(t-s,x,y)\zeta(s,y)dyds \quad , \tag{3}$$

where G is the Green's function of the deterministic problem. This formula gives some hint how to treat the white noise case.

## 2. Real-valued field approach

Shall be given a probability space $(\Omega,\mathcal{F},P)$. Basic object is the two-parameter Wiener field $\{W_{tx}\}$, $(t,x)\in\mathcal{T}\times\mathcal{D}$ (or more general $(t,x)\in\mathbb{R}^2_+$ ). It is Gaussian with the moments

$$EW_{tx} = 0 \quad , \quad EW_{tx}W_{t'x'} = (t\wedge t')\cdot(x\wedge x')$$

and has P-a.s. continuous sample paths /4/,/17/.
Now one can introduce a stochastic integral with respect to the Wiener field W. Because c is constant and G is deterministic it is sufficient to consider only non-random integrands. The integral

$$\int_M g_{sy}\,dW_{sy} \qquad , \qquad M\in\mathcal{L}^2\cap(\mathcal{T}\times\mathcal{D}) \quad ,$$

can be understood as an integral with respect to a random probability measure $\nu_W$ , derived from the Wiener field,

$$\nu_W([s,t)\times[y,x)) = W_{tx}-W_{ty}-W_{sx}+W_{sy} \qquad .$$

It is defined at first for simple functions, constant on rectangles, and then extended in the standard way to arbitrary functions $g\in\mathcal{L}^2(\mathcal{T}\times\mathcal{D})$. (For more details see /15/, if g is allowed to be random see /3/,/16/.) If we choose $M = [0,t]\times[0,x]$ and vary t and x we get a two-parameter random field. Because g is non-random and the definition is linear, it is Gaussian with vanishing expectation and covariance

$$E\left\langle \int_{[0,t]\times[0,x]} g\,dW \cdot \int_{[0,t']\times[0,x']} g\,dW \right\rangle = \int_0^{t\wedge t'}\int_0^{x\wedge x'} g^2(s,y)\,ds\,dy . \tag{4}$$

Now we can define: The (mild) solution of equation (1) shall be

$$u_{tx} = \int_0^L G(t,x,y)u_0(y)\,dy + c\cdot\int_{[0,t]\times\mathcal{D}} G(t-s,x,y)\,dW_{sy} \quad . \tag{5}$$

This is motivated by the following assertion:
If $\{\zeta^{(n)}_{tx}\}_{n=1}^{\infty}$ is a sequence of processes allowing pathwise solutions $\{u^{(\zeta^{(n)})}_{tx}\}_{n=1}^{\infty}$ of the form (3) and

$$\int_0^t\int_0^x \zeta^{(n)}_{sy}\,ds\,dy \xrightarrow[n\to\infty]{} W_{tx}$$

in the sense of the convergence of all finite-dimensional distributions, then (under some technical condition) $u^{(\zeta^{(n)})}_{tx}$ converges to the solution $u_{tx}$ in (5) in the same sense /10/. (For a smaller class of approximating noise one can even get weak convergence. Another special approximation is given in /15/.)
By definition, the solution (5) fulfills boundary and initial conditions of (1). Using standard criteria for Gaussian fields one can proof that $u_{tx}$ has P-a.s. continuous paths /15/.

## 3. Function-valued process approach

We want to consider the solution of equation (1) as a random process in $t\in\mathcal{T}$ with values in a space of functions of the spatial coordinate $x\in\mathcal{D}$. The choice of a suitable function space is of great importance, but for simplicity we will choose the Hilbert space $H=\mathcal{L}^2(\mathcal{D})$.

Shall be given a probability space $(\Omega,\mathfrak{F},P)$. Basic object is now the Wiener cylindrical process $\{z_t\}, t\in\mathcal{T}$,

$$z_t=\sum_{i=1}^{\infty} b_i(t)\cdot e_i \quad ,$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in H and $\{b_i(t)\}_{i=1}^{\infty}$, $t\in\mathcal{T}$, are independent standard Wiener processes in $\mathbb{R}^1$. This definition looks rather formally, but it means that $z_t$ is an object all projections on finite dimensional subspaces of H being Gaussian distributed and determined by

$$E(z_t,e_i)_H(z_t,e_j)_H = t\cdot\delta_{ij} \qquad \forall\, t\in\mathcal{T} \quad .$$

Regarded on $H^{\mathcal{T}}$ $\{z_t\}$ only induces a Gaussian cylinder measure with expectation 0 and covariance operator

$$\mathrm{Cov}(z_t,z_{t'})=(t\wedge t')\cdot I \quad ,$$

I being the identity operator.

Another possibility to introduce $z_t$ is to consider it as a random linear functional in $D'(\mathcal{D})$ /12/.

Now we will define a stochastic integral of non-random integrands F with respect to z for $F\in\mathcal{L}^2(\mathcal{T};\mathcal{L}_{(2)}(H))$, i.e. $F_t$ is a Hilbert-Schmidt operator for all $t\in\mathcal{T}$ and

$$\int_0^T \|F_t\|^2_{\mathcal{L}_{(2)}(H)}dt<\infty \quad .$$

Then we put

$$\int_0^t F_s dz_s = \sum_{i=1}^{\infty}\int_0^t F_s e_i db_i(s) \quad , \tag{6}$$

the integrals on the right hand side being understood in the sense of Bochner, so that this definition is reduced to usual stochastic integrals with respect to one-dimensional Wiener processes. The stochastic integral (6) is then an H-valued random process in t with vanishing expectation and covariance operator

$$\mathrm{Cov}\left(\int_0^t F_s dz_s\,,\int_0^{t'} F_s dz_s\right) = \int_0^{t\wedge t'} F_s F_s^* ds$$

having finite trace (cf. /12/).

Two remarks seem to be necessary: One can extend $z_t$ on a larger space (of generalized functions) where it induces a $\sigma$-additive measure and $\{z_t\}$ becomes a function-valued process. For example one can choose the Sobolev space $H_{-1}$ , i.e. the completion of H with respect to the weaker norm

$$\|\cdot\|_{H_{-1}} = \|B^{-1}(\cdot)\|_H \quad , \quad B \equiv \sqrt{-D(d^2/dx^2)} \quad .$$

If one takes the eigenfunctions $\{e_i\}_{i=1}^{\infty}$ of B as the basis in H (normalized on H) then an orthonormal basis in $H_{-1}$ is given by $\{\tilde{e}_i\}_{i=1}^{\infty}$ , $\tilde{e}_i = e_i/\lambda_i$ , where $\lambda_i = L/i\pi\sqrt{D}$ , $i=1,\dots,\infty$ , are the corresponding eigenvalues of B. Then we have

$$z_t = \sum_{i=1}^{\infty} \lambda_i b_i(t)\cdot e_i \quad , \quad \sum_{i=1}^{\infty} \lambda_i^2 < \infty ,$$

and $\{z_t\}_{i=1}^{\infty}$ is an $H_{-1}$-valued Wiener process with covariance operator

$$\mathrm{Cov}(z_t, z_{t'}) = \sum_{i=1}^{\infty} \lambda_i \; e_i \underset{H_{-1}}{\circ} e_i$$

belonging to trace class. Because H is dense and continuous in $H_{-1}$ $F_t$ can be extended to a linear bounded operator from $H_{-1}$ to H. Now, because $F_t \in \mathcal{L}(H_{-1}, H)$ , the definition of a stochastic integral with respect to a Hilbert space-valued Wiener process /6/,/1/,/2/ can be applied and gives the same result (6).

The other remark shall state that the most general approach uses stochastic integrals with respect to a Wiener process with values in a Banach space which is given on an abstract Wiener space $(i,H,\mathfrak{B})$. These integrals are defined for integrands $F \in \mathcal{L}(\mathcal{T};\mathcal{L}_{(2)}(\mathfrak{B},K))$ , K another Hilbert space, using an approximation of F by simple functions /9/,/5/. If $\mathfrak{B}$ is a Hilbert space itself (and K=H) then this definition is equivalent to (6). (The same result holds for random integrands, see /8/.)

Within the function-valued process approach we define: The (mild) solution of equation (1) shall be

$$u_t(\cdot) = T_t u_o + c\cdot \int_0^t T_{t-s} dz_s \quad , \tag{7}$$

where $\{T_t\}_{t\geq 0}$ is the semigroup generated by $A \equiv D(d^2/dx^2)$, i.e.

$$(T_t h)(x) = \int_0^L G(t,x,y)h(y)dy \quad .$$

It turns out that $u_t(\cdot)$ has a.s. continuous sample paths /12/,/1/.

## 4. Connection of both approaches and properties of the solutions

The most important results are the following ones /8/:

(i) The approaches discussed in 2. and 3. are essentially the same (although introduced independently in the literature) because

$$\int_{[0,t]\times\mathcal{D}} G(t-s,\cdot,y)dW_{sy} \quad \text{and} \quad \int_0^t T_{t-s}dz_s$$

have the same distribution (on a suitable function space). If we choose $(\Omega,\mathcal{F},P) = (C_o(\mathcal{T}\times\mathcal{D}),\mathcal{B}_{C_o},P_W)$, $C_o$ being the continuous functions vanishing on the axes and $P_W$ being the Wiener measure (obtained f.e. by restriction of $P_{\{W_{tx}\}}$ on $C_o$) we can put

$$z_t(x,\omega) = \frac{\partial}{\partial x} W_{tx}(\omega) \qquad (\text{and } \xi_{tx}(\omega) = \frac{\partial^2}{\partial t\partial x} W_{tx}(\omega) )$$

in the sense of generalized functions (i.e. the noise $z_t$ is only spatially "white"), but the support of $z_t$ is the smaller space $H_{-1}(\mathcal{D}) \subset D'(\mathcal{D})$. In this case we even have

$$\int_{[0,t]\times\mathcal{D}} G(t,\cdot,y)dW_{sy} = \int_0^t T_{t-s}dz_s \qquad \text{P-a.s.}$$

on H and, consequently, $u_{t(\cdot)} = u_t(\cdot)$ P-a.s. for the solutions (5) and (7), which justifies the notation. To prove this one has to use

$$\int_{[0,t]\times\mathcal{D}} g(s)h(x)dW_{sx} = \int_0^t g(s)dW_s$$

for $g\in\mathcal{L}^2(\mathcal{T})$, $h\in\mathcal{L}^2(\mathcal{D})$, $\|h\|_{\mathcal{L}^2} = 1$, and $\{W_s\}_{s\geq 0}$ being a standard Wiener process /15/.

(ii) The solution $\{u_{tx}\}$, $(t,x)\in\mathcal{T}\times\mathcal{D}$ is a Gaussian random field with

$$Eu_{tx} = \int_0^L G(t,x,y)u_o(y)dy \quad ,$$

$$Cov(u_{tx},u_{t'x'}) = c^2 \cdot \int_0^{t\wedge t'}\int_0^L G(t-s,x,y)G(t'-s,x',y)dyds$$

and P-a.s. continuous paths. Equivalently, the solution $\{u_t(\cdot)\}$, $t\in\mathcal{T}$, is an H-valued process (which actually can be considered on $C(\mathcal{D})$ ) with $Eu_t = T_t u_o$ and covariance operator

$$Cov(u_t,u_{t'}) = c^2 \cdot \int_0^{t\wedge t'} T_{t-s}T_{t'-s}ds$$

and has P-a.s. continuous paths.

(iii) Up to here the initial value was fixed. Now shall be $u_o(\cdot)$ an H-valued random variable, but independent of the noise $\{z_t(\cdot)\}_{t\in[0,\infty)}$. Let us decompose $u = (u_o - v) + v$, where v is a Gaussian variable with

$$Ev = 0 \quad , \quad Cov(v) = -\frac{c^2}{2}\cdot A^{-1} \quad , \quad A = D(d^2/dx^2) \quad .$$

Then $\lim T_t(u_o-v) = 0$ P-a.s., and $T_t v + c\cdot\int_0^t T_{t-s}dz_s$ is a stationary process (in t), i.e. $\{u_t\}_{t\geq 0}$ is asymptotically stationary. The first part follows from the property of $T_t$ to smear out any (finite) initial condition, the second part is a consequence of the fact that v is chosen to have the distribution of $\lim\limits_{t\to\infty} \int_0^t T_{t-s} dz_s$ .

## 5. Markov property

The solution $\{u_t(\cdot)\}$, $t \geq 0$, is a homogeneous Markov process (see /12/) with a transition probability

$$Q_{t-s}(h,M) = P\{u_t \in M \mid u_s = h\} , \quad 0 \leq s \leq t ,$$

which is given by the distribution of

$$T_{t-s}h + c\cdot \int_s^t T_{t-r} dz_r .$$

The unique invariant measure $\mu_G$ is a centered Gaussian with the covariance $(-c^2/2)\cdot A^{-1}$. The essential reason is that $u_s$ and $\{z_r\}$, $s \leq r \leq t$ , are independent and $u_t$ is fully determined by these values,

$$u_t = T_t u_s + c\cdot \int_s^t T_{t-r} dz_r .$$

The following comment should be given: Within the two-parameter field approach several mathematical definitions of a Markov-like property are possible /14/,/3/,/16/,/13/, but they all have no direct physical interpretation. For function-valued processes with one (time) parameter, however, the Markov property is defined mathematically as usually and reflects the causality principle in physics. The price for this advantage (to apply the powerful techniques developed in Markov process theory) is to work in function spaces and to do some hard functional analysis.

## 6. Nonlinear Equation

Let us shortly consider the nonlinear case where in (1) $f(u) \not\equiv 0$ . Generalizing (7) we define that the mild solution of (1) shall fulfill the integral equation

$$u_t = T_t u_0 + \int_0^t T_{t-s} f(u_s) ds + c\cdot \int_0^t T_{t-s} dz_s . \quad (8)$$

If f fulfills a certain Lipschitz condition then the solution of (8) exists uniquely and is a homogeneous Markov process with P-a.s. continuous sample paths. This result has been proven in /1/ for a Hilbert space-valued Wiener process (but more general integrands) and can be extended to our case of a Wiener cylindrical process z. In the case that f is monotone (in some sense) (f.e. a polynomial function with negative derivative) the invariant measure $\mu_I$ can be given explicitly /11/: It is absolutely continuous with respect to the invariant measure $\mu_G$ of the linear problem, $\mu_i \ll \mu_G$ , and has the (unnormalized) Radon-Nikodym derivative

$$\left(\frac{d\mu_i}{d\mu_G}\right)(u) = \text{const} \cdot \exp\left\langle \frac{2}{c^2} \cdot \int_0^L V(u(x))dx \right\rangle , \quad \frac{dV(u)}{du} \equiv f(u) \quad .$$

The proof uses a sequence of approximations of $u_t$ given by the projection onto subspaces of H spanned by the first n eigenvectors of $A \equiv D(d^2/dx^2)$ .

## 7. Final remarks

Obviously similar results are valid if the operator $D(d^2/dx^2)$ is replaced by an arbitrary negative operator A when $A^{-1}$ is of Hilbert-Schmidt type.
Also the second result of 6. is expected to be true if f is non-monotone (f.e. an arbitrary polynomial function).
Moreover the results are easily extended to the case **where the** strength of the noise $c = c(t)$ is a given time-dependent function.
The essential generalization will be the case of state-dependent strength of the noise $c = c(u)$ , or eqivalently, the equation

$$\partial u/\partial t = D\cdot(\partial^2 u/\partial x^2) + f(u) + c\cdot g(u)\cdot \xi(t,x) \quad .$$

This needs a more refined definition of stochastic integrals (for example of Ito or Stratonovich type), because the integrands become random functions, and the chose of the "right" definition is a question of modelling.

## REFERENCES

/1/ ARNOLD,L.,R.F.CURTAIN, P.KOTELENEZ,"Nonlinear stochastic evolution equations in Hilbert space", Universität Bremen, Report Nr. 17 (1980)

/2/ ARNOLD,L.,R.F.CURTAIN,P.KOTELENEZ, "Linear stochastic evolution equation models for chemical reactions", In: Stochastic Nonlinear Systems in Physics, Chemistry and Biology, ed. by L.ARNOLD and R.LEFEVER, Springer Series in Synergetics, vol.8, Berlin-Heidelberg-New York 1981

/3/ CAIROLI,R.,J.B.WALSH, "Stochastic integrals in the plane", Acta Math. 134 (1975), 111-181

/4/ CHENTSOV,N.N., "Wiener random fields depending on several parameters", Dokl.Akad.Nauk SSSR 106 (1956), 607-609 (in Russian)

/5/ CHOW,P.-L., "Stochastic Partial Differential Equations in Turbulence Related Problems", In: Probabilistic Analysis and Related Topics, vol.1, ed. by A.BHARUCHA-REID, New York 1978, 1-43

/6/ CURTAIN,R.F., "Stochastic Differential Equations with General White Noise Disturbance", J.Math.Anal.Appl. 60 (1977), 570-595

/7/ HAKEN,H., Synergetics.An Introduction, Springer Series in Synergetics, vol.1, Berlin-Heidelberg-New York 1978

/8/ JETSCHKE,G., "On the equivalence of different approaches and results to stochastic partial differential equations", preprint Universität Jena (1982)

/9/ KUO,H.H., "Gaussian Measures in Banach Spaces", Lecture Notes in Mathematics, vol.463, Berlin-Heidelberg-New York 1975

/10/ MANTHEY,R., "Über die Lösung und eine Approximation der homogenen Wärmeleitungsgleichung mit zweiparametrigem weißem Gaußschen Rauschen", preprint Universität Jena (1982)

/11/ MARCUS,R., "Parabolic Ito Equations with Monotone Nonlinearities", J.Funct.Anal. 29 (1978), 275-286

/12/ MIYAHARA,Y., "Infinite Dimensional Langevin Equation and Fokker Planck Equation", Nagoya Math.J. 81 (1981), 177-223

/13/ NUALART,D.,M.SANZ, "A Markov Property for Two Parameter Gaussian Processes", Stochastica 3 (1979), 1-15

/14/ ROZANOV,Y.A., "Markovian Random Fields", Moscow 1981 (in Russ.)

/15/ WALSH,J.B., "A Stochastic Model of Neural Response", Adv. Appl.Prob. 13 (1981), 231-281

/16/ WONG,E.,M.ZAKAI, "Martingales and Stochastic Integrals for Processes with a Multidimensional Time Parameter", Z.Wahrsch. u.v.G. 29 (1974), 109-122

/17/ YEH,J., "Wiener Measure in a Space of Functions of Two Variables", Trans.Amer.Math. Soc. 95 (1960), 433-450

G.Jetschke, Sektion Mathematik der Friedrich-Schiller-Universität, DDR-6900 Jena, Universitätshochhaus